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TABLE OF CONTENTS

VOLUME 8 1907

PAGES

BLICHFELDT, H. F., of Stanford University, Cal. On modular groups isomorphic with a given linear group 30– 32

BLISS, GILBERT AMES, of Princeton, N. J. A new form of the simplest problem of the calculus of variations 405–414

BOLZA, OSKAR, of Chicago, Ill. Existence proof for a field of extremals tangent to a given curve 399–404

DICKSON, LEONARD EUGENE, of Chicago, Ill. Invariants of binary forms under modular transformations 205–232

——— Modular theory of group-matrices 389–398

EISENHART, LUTHER PFAHLER, of Princeton, N. J. Applicable surfaces with asymptotic lines of one surface corresponding to a conjugate system of another 113–134

FITE, WILLIAM BENJAMIN, of Ithaca, N. Y. Irreducible linear homogeneous groups whose orders are powers of a prime . . 107–112

FRÉCHET, MAURICE, of Paris, France. Sur les opérations linéaires (troisième note) 433–446

GREENHILL, ALFRED GEORGE, of Woolwich, England. The elliptic integral in electromagnetic theory 447–534

HEDRICK, E. R., of Columbia, Mo. On derivatives over assemblages 345–353

HUTCHINSON, J. I., of Ithaca, N. Y. A method for constructing the fundamental region of a discontinuous group of linear transformations 261–269

KASNER, EDWARD, of New York, N. Y. Dynamical trajectories: the motion of a particle in an arbitrary field of force . . . 135–158

LEVI, BEPPO, of Turin, Italy. Geometrie proiettive di congruenza e geometrie proiettive finite 354–365

LONGLEY, WILLIAM RAYMOND, of New Haven, Conn. A class of periodic orbits of an infinitesimal body subject to the attraction of n finite bodies 159–188

MACLAGAN–WEDDERBURN, J. H., of Edinburgh, Scotland. (With O. VEBLEN.) Non-desarguesian and non-pascalian geometries 379–388

MASON, MAX, of New Haven, Conn. The expansion of a function in terms of normal functions 427–432

MILLER, G. A., of Urbana, Ill. Generalization of the groups of genus zero 1– 13

Kenyon 475 Bindery 24 Mar 42 1.42

3

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PAGES

MILLER, G. A., of Urbana, Ill. The groups in which every subgroup is either abelian or hamiltonian 25– 29

MOORE, CHARLES N., of Cambridge, Mass. On the introduction of convergence factors into summable series and summable integrals 299–330

MOORE, R. L., of Princeton, N. J. Geometry in which the sum of the angles of every triangle is two right angles 369–378

MORLEY, F., of Baltimore, Md. On reflexive geometry 14– 24

RANUM, ARTHUR, of Ithaca, N. Y. The groups of classes of congruent matrices, with application to the group of isomorphisms of any abelian group 71– 91

SMITH, CLARA E., of Northford, Conn. A theorem of Abel and its application to the development of a function in terms of Bessel's functions 92–106

STORY, WILLIAM EDWARD, of Worcester, Mass. Denumerants of double differentiants 33– 70

VAN VLECK, EDWARD B., of Madison, Wis. A proof of some theorems on pointwise discontinuous functions 189–204

VEBLEN, O., of Princeton, N. J. Collineations in a finite projective geometry 366–368

———— (with J. H. MACLAGAN–WEDDERBURN). Non-desarguesian and non-pascalian geometries 379–388

WILCZYNSKI, E. J., of Berkeley, Cal. Projective differential geometry of curved surfaces (first memoir) 233–260

WILSON, EDWIN BIDWELL, of New Haven, Conn. Oblique reflections and unimodular strains 270–298

YOUNG, ARCHER EVERETT, of Lafayette, Ind. On certain isothermic surfaces 415–426

YOUNG, JOHN WESLEY, of Princeton, N. J. General theory of approximation by functions involving a given number of arbitrary parameters 331–344

Notes and errata: volume 8 535–536

GENERALIZATION OF THE GROUPS OF GENUS ZERO*

BY

G. A. MILLER

One of the most important systems of groups is that which is composed of all the groups which may be defined by the orders of two generators (s_1, s_2) and the order of their product. It is known that these three orders determine a finite group only when one of them is unity, when two of them are equal to 2, or when one is 2 while the other two are one of the following three pairs of numbers: 3, 3; 3, 4; 3, 5.† The first of these sets of orders define a cyclic group and may be regarded as trivial. When the order of two among the three operators $s_1, s_2, s_1 s_2$ is equal to 2, the group $\{s_1, s_2\}$, generated by s_1, s_2, is the dihedral rotation group whose order is twice that of the third operator. This useful category has recently been generalized under the heading, *The groups generated by two operators which have a common square.*‡ This generalized category could also be defined as the set of groups generated by two operators such that the square of one of them is equal to the square of their product.

The object of the present paper is to complete this kind of generalization for the dihedral rotation group and to extend it to the other groups of genus zero. The resulting groups have a two-fold interest in view of their close contact with the important system of groups of genus zero and their elementary structures. It is believed that a complete list of these groups will prove very useful in many investigations. Our object is to study all the groups which result when one of the three conditions which may be used to define a group of genus zero is preserved while the other two are replaced by a single one of an elementary type. For instance, the dihedral rotation groups are defined by the conditions §

$$s_1^2 = 1, \qquad s_2^2 = 1, \qquad (s_1 s_2^{-1})^n = 1.$$

In the generalization mentioned above these conditions are replaced by

* Presented to the Society at the New Haven meeting September 3, 1906. Received for publication July 11, 1906.

† American Journal of Mathematics, vol. 24 (1902), p. 96.

‡ Archiv der Mathematik und Physik, vol. 9 (1905), p. 6.

§ Throughout the paper care has been taken to insure that, when one of the given generational relations of a group is $s^n = 1$, the operator s shall be of order n.

$$s_1^2 = s_2^2, \qquad (s_1 s_2^{-1})^n = 1.$$

A similar generalization is given by the equations

$$s_1^2 = s_2^n, \qquad (s_1 s_2^{-1})^2 = 1 \qquad (n > 2).$$

We proceed to consider the groups whose generators satisfy the last equation. From $(s_1 s_2^{-1})^2 = 1$ it results that $s_1 s_2^{-1} s_1 = s_2$. Hence $s_1^{-1} s_2 s_1 = s_2^{n-1}$ and $s_1^{-1} s_2^n s_1 = s_2^{n^2-n}$ Since s_2^n is invariant under $\{s_1, s_2\}$ it is necessary that $n^2 - n \equiv n \pmod{m}$, m being the order of s_2. That is, m must divide $(n-1)^2 - 1$. Since $\{s_1, s_2\}$ is generated by s_2 and $s_1 s_2^{-1}$ and since the latter transforms the former into its $(n-1)$-th power, the order of $\{s_1, s_2\}$ is $2m$. These results may be expressed as follows: *If the two generators of a group satisfy the conditions* $s_1^2 = s_2^n$, $(s_1 s_2^{-1})^2 = 1$, $n > 2$, *it is non-abelian, involves a cyclic subgroup of half its order, and an additional operator of order* 2 *which transforms each operator of this cyclic subgroup into its* $(n-1)$*-th power.*

For $m = n$ this category of groups reduces to the dihedral rotation groups. It is clear that we may assume that $m \geqq n$; for, if m were less than n, the number n could be replaced by $n - km$, where k is arbitrary. The invariant operators of $\{s_1, s_2\}$ constitute a subgroup whose order is the highest common factor of m and $n - 2$. With respect to this subgroup the quotient group of $\{s_1, s_2\}$ is a dihedral rotation group. If H represents any Sylow subgroup of odd order in $\{s_1, s_2\}$, then either s_1 is commutative with each operator of H or it transforms each of these operators into its inverse. In the former case the order of s_1 is divisible by that of H. In the latter case the orders of s_1 and H are relatively prime. The numbers n and m cannot be relatively prime since $\{s_1, s_2\}$ is non-cyclic. For the same reason s_1 must be of even order, as is also otherwise evident.

It may be added that if n is replaced by $n' = m - n$, the given equations become

$$t_1^2 = t_2^{n'}, \qquad (t_1 t_2)^2 = 1 \qquad (n' < m - 2).$$

Hence these equations define the same system of groups as those given above. Similar remarks apply to the system defined by the equations $t_1^2 = t_2^2$, $(t_1 t_2)^n = 1$. Since the dihedral rotation groups constitute an infinite system it is necessary that the generalization of the conditions satisfied by its generators are also satisfied by the generators of an infinite system of groups. The other groups of genus zero are defined as individuals, and the corresponding generalizations are satisfied by a very small number of groups.

§ 1. *Generalization of the tetrahedral group.*

It is known that any two operators of order 3 whose product is of order 2

generate the tetrahedral group. That is, this group is completely defined by two operators s_1, s_2 which satisfy either of the following sets of conditions:

$$s_1^3 = s_2^3 = 1,\ (s_1 s_2)^2 = 1; \qquad s_1^2 = s_2^3 = 1,\ (s_1 s_2)^3 = 1.$$

We proceed to consider the groups which satisfy the more general equations

$$s_1^3 = s_2^3,\ (s_1 s_2)^2 = 1; \qquad s_1^2 = s_2^3,\ (s_1 s_2)^3 = 1.$$

We begin with the first of these two cases; that is, we consider all the groups which are generated by two operators having a common cube and a product of order 2.

The three operators $s_1 s_2$, $s_2 s_1$, $s_1^{-1} s_2 s_1^2$ are conjugate under s_1 and the product of the last two is $s_2^2 s_1^2 = s_1 s_2 \cdot s_1^6$ since $s_1 s_2 = s_2^{-1} s_1^{-1}$. As $s_2 s_1$ transforms this product into its inverse and is commutative with s_1^6 the order of s_1 must divide 24. As it is a multiple of 3 it has one of the three valuees 24, 12, 6, 3. If this order is 24 the two operators of order two $s_2 s_1$, $s_1^{-1} s_2 s_1^2$ generate the octic group since $s_2^2 s_1^2$ is of order 4. This octic group and s_1^6 generate an invariant subgroup of order 16 since it is transformed into itself by s_1 and also by s_2. As this invariant subgroup contains the quaternion group as a characteristic subgroup, this quaternion group and s_1^8 generate the group of order 24 which does not involve a subgroup of order 12. Hence the given group of order 16 and s_1^8 generate the group of order 48 known as G_{50}.* As this involves s_1^6 and s_1^8 it also involves s_1^2. If we extend it by means of s_1^3 there results a group of order 96 which involves both s_1 and s_2 and hence is $\{s_1, s_2\}$. As s_1^3 is invariant under $\{s_1, s_2\}$ there is only one such group.† It may be represented as a substitution group of degree 32 by means of

$$s_1 = ac'ce'eg'ga' \cdot bm'nf'oj'hk'ld'mp'fi'jb'kn'do'ph'il',$$

$$s_2 = am'je'on'gk'pc'ml'ei'na'kj'co'lg'ip' \cdot bd'df'fh'hb'.$$

When s_1 is of order 12 the two operators $s_2 s_1$, $s_1^{-1} s_2 s_1^2$ are commutative since their product is of order 2. Hence $\{s_2 s_1, s_1^{-1} s_2 s_1^2\}$ is the four-group. This four-group and s_1^6 generate the group of order 8 which involves seven operators of order 2. This is invariant under $\{s_1, s_2\}$. Since the group of isomorphisms of this group of order 8 is known, it is known that s_1^4 and one of its subgroups of order 4 generate the tetrahedral group. Hence $\{s_1, s_2\}$ is the direct product of this tetrahedral group and s_1^3. When s_1 is of order 6 the four-group $\{s_2 s_1, s_1^{-1} s_2 s_1^2\}$ involves $s_1 s_2$, and together with s_1^2 generates the tetrahedral group. Hence $\{s_1, s_2\}$, in this case, is the direct product of the tetrahedral group and the group of order 2. These results may be expressed as follows:

*Quarterly Journal of Mathematics, vol. 30 (1898), p. 258.

†Bulletin of the American Mathematical Society, vol. 3 (1897), p. 218.

There are exactly four groups whose two generators s_1, s_2 *satisfy the relations* $s_1^3 = s_2^3$, $(s_1 s_2)^2 = 1$. *They are the tetrahedral group; the direct products of the tetrahedrl group and the cyclic group of order* 4 *or the group of order* 2; *and the group of order* 96 *considered above.*

The second generalization of the tetrahedral group relates to the groups whose two generators satisfy the equations

$$s_1^2 = s_2^3, \qquad (s_1 s_2)^3 = 1.$$

The three conjugates of s_1 under s_2 are

$$s_1, \qquad s_2^{-1} s_1 s_2, \qquad s_2^{-2} s_1 s_2^2.$$

As these conjugates have a common square we are concerned with a generalization of the dihedral rotation group. Multiplying the second into the inverse of the third, we get

$$s_2^{-1} s_1 s_2 \cdot s_2^{-2} s_1^{-1} s_2^2 = s_2^{-1} s_1 s_2^{-1} s_1 s_2^2 \cdot s_1^{-2} = s_1^5.$$

Since s_1^5 is transformed into its inverse by $s_2^{-1} s_1 s_2$ and s_1^2 is invariant under $\{s_1, s_2\}$ it follows that the order of s_1 divides 20. If the order of s_1 is 20 we consider the group generated by $s_1^{25} = t_1$, $s_2^{25} = t_2$. From the fact that $(s_1^{25} s_2^{25})^3 = s_1^{120} (s_1 s_2)^3 = 1$, it results that $t_1^2 = t_2^3$ and $(t_1 t_2)^3 = 1$. Hence $\{t_1, t_2^{-1} t_1 t_2, t_2^{-2} t_1 t_2^2\}$ is the quaternion group, which is invariant under $\{t_1, t_2\}$ since it includes t_1 and is transformed into itself by t_2. This quaternion group and t_2 generate the group of order 24 which does not contain a subgroup of order 12. Hence this is $\{t_1, t_2\}$, while $\{s_1, s_2\}$ is the direct product of $\{t_1, t_2\}$ and the cyclic group of order 5 generated by s_1^4.

When the order of s_1 is 10, it follows from the preceding paragraph that $\{s_1^{25}, s_2^{25}\}$ is the tetrahedral group and that $\{s_1, s_2\}$ is the direct product of this group and the cyclic group of order 5. These results may be expressed as follows: *There are exactly four groups whose two generators satisfy the conditions* $s_1^2 = s_2^3$, $(s_1 s_2)^3 = 1$. *They are the tetrahedral group, the group of order* 24 *which does not contain a subgroup of order* 12, *and the direct products of these groups and the cyclic group of order* 5. As only one of these groups occurs among the four given above we have seven distinct groups resulting from one or the other of the given generalizations of the defining equations of the tetrahedral group.

It may be of interest to note how a slight change in these generalizations may affect the system. Instead of the equations $s_1^3 = s_2^3$, $(s_1 s_2)^2 = 1$, we proceed to consider $s_1^3 = s_2^3$, $(s_1 s_2^{-1})^2 = 1$. The three conjugates $s_1 s_2^{-1}$, $s_2^{-1} s_1$, $s_1^{-1} s_2^{-1} s_1^2$ are commutative and constitute, together with the identity, the four-group since $s_2^{-2} s_1^2 = (s_1 s_2^{-1})^{-1} = s_1 s_2^{-1}$. This four-group is invariant under $\{s_1, s_2\}$ and has

only the identity in common with the cyclic group generated by s_1^3. Hence the theorem:

*Two operators s_1, s_2 which satisfy the conditions $s_1^3 = s_2^3$ and $(s_1 s_2^{-1})^2 = 1$ generate the group which may be obtained by establishing a $(4, h)$ isomorphism between the tetrahedral group and a cyclic group of order $3h$, where $3h$ is the order of s_1.**

Since there is one and only one such group for every value of h, a group is completely defined by the value of h and the given equations. In particular, when $h = 1$, $\{s_1, s_2\}$ is the tetrahedral group. When h is prime to 3, $\{s_1, s_2\}$ is the direct product of the tetrahedral group and the cyclic group of order h.†

Similarly, the second set of equations given at the beginning of this section may be replaced by $s_1^3 = s_2^2$ and $(s_1^{-1} s_2)^3 = 1$. The three conjugates of s_2 under s_1 are

$$s_2, \quad s_1^{-1} s_2 s_1, \quad s_1^{-2} s_2 s_1^2.$$

The product of the second into the inverse of the third is $s_1^{-1} s_2 s_1^{-1} s_1^2 = s_2^{-1}$. Since $s_1^{-1} s_2 s_1$ transforms s_2 into its inverse and is commutative with s_2^2 the order of s_2 is either 2 or 4. In the former case $\{s_1, s_2\}$ is the tetrahedral group. In the latter case, $\{s_2, s_1^{-1} s_2 s_1, s_1^{-2} s_2 s_1^2\}$ is the quaternion group. As this is invariant under $\{s_1, s_2\}$ we have the result: *If the two generators of a group satisfy the conditions $s_1^3 = s_2^2$, $(s_1^{-1} s_2)^3 = 1$, it is either the tetrahedral group or the group of order 24 which does not contain a subgroup of the order 12.*

§ 2. *Generalization of the octahedral group.*

The octahedral group is completely defined by the fact that it may be generated by two operators of orders 2 and 4, respectively, such that their product is of order 3. That is, s_1, s_2 define the octahedral group provided they satisfy the equations $s_1^2 = s_2^4 = 1$, $(s_1 s_2)^3 = 1$, and the orders of s_1, s_2 are 2 and 4 respectively. We shall now consider the generalization of the octahedral group given by the equations $s_1^2 = s_2^4$, $(s_1 s_2)^3 = 1$.

It will be convenient to consider the three conjugates of s_2^2 under $s_1 s_2$. These are

$$s_2^2, \qquad s_2^{-1} s_1^{-1} s_2^2 s_1 s_2, \qquad s_1 s_2 s_2^2 s_2^{-1} s_1^{-1} = s_1 s_2^2 s_1^{-1}.$$

The form of the last one of these three conjugates results from the fact that $(s_1 s_2)^{-2} = s_1 s_2$ since $(s_1 s_2)^3 = 1$. These conjugates have a common square since s_2^4 is invariant under $\{s_1, s_2\}$. The group generated by any two of them is therefore a generalized dihedral rotation group. We shall consider the group

*Bulletin of the American Mathematical Society, vol. 1 (1897), p. 218; cf. MANNING, Transactions of the American Mathematical Society, vol. 7 (1906), p. 233.

†Transactions of the American Mathematical Society, vol. 1 (1900), p. 67.

generated by the last two. The product of the third into the inverse of the second is $s_1 s_2 s_2^2 s_1 s_2 s_2^{-2} s_1 s_2 = s_1 s_2^{-1} s_1 s_2^{-1} s_1 s_2^5 = s_2 s_1^5 s_1 s_2^5 = s_2^{18}$; for, from the given conditions we have $(s_2^{-1} s_1^{-1})^3 = (s_1^{-1} s_2^{-1})^3 = 1$, and therefore

$$(s_1 s_2^{-1})^3 = (s_1^2 s_1^{-1} s_2^{-1})^3 = s_1^6 \qquad \text{or} \qquad (s_1 s_2^{-1})^2 = s_2 s_1^5.$$

Since $s_1 s_2^2 s_1^{-1}$ transforms s_2^{18} into its inverse and has a common square with s_2^2, the order of s_2 is a divisor of 72. The group generated by $s_2^{-1} s_1^{-1} s_2^2 s_1 s_2$ and $s_1 s_2^2 s_1^{-1}$ contains s_2^2 since it contains both s_2^4 and s_2^{18}. This group is transformed into itself by s_2, since this operator transforms the latter generator into the former, as $s_1 s_2^2 s_1^{-1} = s_1^{-1} s_2^2 s_1$, and also transforms their product into itself. Hence it is also transformed into itself by s_1, as it is transformed into itself by $s_1 s_2$. That is, s_2^2, $s_2^{-1} s_1^{-1} s_2^2 s_1 s_2$, $s_1 s_2^2 s_1^{-1}$ generate a subgroup which is invariant under $\{s_1, s_2\}$. When this subgroup is extended by means of $s_1 s_2$ there results another invariant subgroup since $s_2 s_1 = (s_1^{-1} s_2^{-1})^{-1}$, $s_1 s_2 = s_1^2 s_1^{-1} s_2^{-1} s_2^2$, and s_1^2, s_2^2 are in the preceding invariant subgroup. The order of $\{s_1, s_2\}$ must therefore divide the product obtained by multiplying the order of $\{s_2^{-1} s_1^{-1} s_2^2 s_1 s_2, s_1 s_2^2 s_1^{-1}\}$ by 6. That is, this order divides $6 \cdot 4 \cdot 18 = 432$.

If the order of s_2 is 72, the group generated by s_2^2, $s_2^{-1} s_1^{-1} s_2^2 s_1 s_2$, $s_1 s_2^2 s_1^{-1}$ is of order 72, and its invariant operators constitute the cyclic group of order 18. It may be defined by the fact that it is non-abelian and contains more than one cyclic subgroup of order 36. Its three cyclic subgroups of order 36 are conjugate under $s_1 s_2$ and its cyclic subgroup of order 18 is composed of the invariant operators of $\{s_1, s_2\}$. By extending this group of order 72 by means of $s_1 s_2$ there results the direct product of the cyclic group of order 9 and the group of order 24 which does not contain a subgroup of order 12. As this direct product contains just three subgroups of order 24, one of them is transformed into itself by s_2 and includes s_2^{18}. This subgroup and s_2^2 generate a group of order 48, known* as G_{52}, and hence $\{s_1, s_2\}$ is the direct product of the latter and the cyclic group of order 9. That is, *if* s_1, s_2 *satisfy the conditions* $s_1^2 = s_2^4$, $(s_1 s_2)^3 = 1$, *then either they generate the direct product of the cyclic group of order* 9 *and the group of order* 48 *known as* G_{52} *or they generate a quotient group of this direct product.*

If the order of s_2 is 36, the three operators s_2^2, $s_2^{-1} s_1^{-1} s_2^2 s_1 s_2$, $s_1 s_2^2 s_1^{-1}$ generate a group of order 36. As this is abelian and contains three cyclic subgroups of order 18 it is the direct product of the four-group and the cyclic group of order 9. By extending it by means of $s_1 s_2$ we obtain the direct product of the alternating group of order 12 and the cyclic group of order 9. Hence $\{s_1, s_2\}$ is the direct product of the symmetric group of order 24 and the cyclic group of order 9 whenever the order of s_2 is 36. If the order of s_2 were not divisible by 4,

*Quarterly Journal of Mathematics, vol. 30 (1898), p. 258.

s_2^4 would be equal to the square of a generator of s_2 and hence we would be dealing with a group generated by two operators having a common square. Hence it is only necessary to consider the cases when the order of s_2 has one of the following values: 4, 12, 8, 24. In the first two cases, we arrive at the symmetric group of order 24, or the direct product of this and the group of order 3. In the last two cases we arrive at the group of order 48 considered in the preceding paragraph or the direct product of this and the group of order 3. As the considerations are similar to those given above, it seems unnecessary to enter into details, especially since the quotient groups mentioned at the end of the preceding paragraph are so well known from the fact that G_{52} has a $(2, 1)$ isomorphism with the symmetric group of order 24.

A similar generalization of the definition of the octahedral group is given by the equations

$$s_1^2 = s_2^3, \qquad (s_1 s_2)^4 = 1.$$

To find an upper limit for the order of s_2 we may consider the following product of two conjugate operators of order 2:

$$(s_2 s_1 s_2 s_1 \cdot s_1 s_2 s_1 s_2)^2 = (s_1^2 s_2 s_1 s_2^2 s_1 s_2)^2 = s_1^4 s_2 s_1 s_2^2 s_1 s_2^2 s_1 s_2^2 s_1 s_2^2 s_2^{-1} = s_2^{30}.$$

As s_2^{30} is both commutative with $(s_2 s_1)^2$ and transformed into its inverse by this operator, its order cannot exceed 2 and hence the order of s_2 divides 60. We shall first consider the group generated by s_1^5, s_2^5 when the order of s is 60. It is clear that these two operators satisfy the given conditions since

$$(s_1^5 s_2^5)^4 = (s_1^4 s_2^3 s_1 s_2^2)^4 = s_1^{16} s_2^{12} (s_1 s_2^2)^4 = s_2^{60}.$$

It will appear that $\{s_1, s_2\}$ is the direct product of $\{s_1^5, s_2^5\}$ and the cyclic group generated by s_2^{12}. From the equation given above it follows that $\{s_2 s_1 s_2 s_1, s_1 s_2 s_1 s_2\}$ is the octic group. Extending this by means of s_2^{15}, we obtain a group of order 16 which is invariant under $\{s_1, s_2\}$ since s_2 transforms one generator of this octic group into the other, and the product of the two into $s_1^3 s_2^2 s_1 s_2^2 = (s_2 s_1 s_2 s_1)^{-1} \cdot s_2^{15}$. By extending this group of order 16 by means of s_2^5, we get a group of order 48 which is invariant under $\{s_1, s_2\}$ since $s_1^{-1} s_2^5 s_1 = (s_2 s_1 s_2 s_1)^{-1} \cdot s_2^{10}$. Finally, if we extend this by means of s_1^5 we obtain the group of order 96 in question. It is easy to verify that this group may be generated by the following substitutions:

$$s_1 = aa'gg'ee'cc' \cdot bo'hm'fk'di' \cdot id'ob'mh'kf' \cdot jn'pl'nj'lp',$$

$$s_2 = amlgkjeipcon \cdot bdfh \cdot a'm'l'g'k'j'e'i'p'c'o'n' \cdot b'd'f'h'.$$

When the order of s_2 is 60 the invariant subgroup generated by s_2^{12} has only the identity in common with the G_{96} of the preceding paragraph. *Hence*

$\{s_2^{12}, s_1^5, s_2^5\} \equiv \{s_1, s_2\}$ *is the direct product of* G_{96} *and the cyclic group of order* 5. When the order of s_2 is 12 it follows directly from the preceding paragraph that $\{s_1, s_2\}$ is G_{96}. Hence it remains to consider the cases when the order of s_2 is not divisible by 4. If this order is 30 we again consider $\{s_1^5, s_2^5\}$. Hence $\{s_2 s_1 s_2 s_1, s_1 s_2 s_1 s_2\}$ is either the four-group or the group of order 2. In the former case it does not contain s_2^{15}, since one of its two generators is transformed into the other by s_2 and their product is transformed into itself multiplied by s_2^{15} by the same operator. Hence this four-group and s_2^{15} generate the group of order 8 which contains 7 operators of order 2. Just as in the preceding paragraph, we observe that this group of order 8 is invariant under $\{s_1, s_2\}$.

As s_2^5 transforms this group of order 8 into itself and is commutative with only one of its operators of order 2, it transforms one of its subgroups of order 4 into itself and this subgroup and s_2^{10} generate the alternating group of order 12.* That is, s_2^5 and this group of order 8 generate the direct product of the alternating group of order 12 and an operator of order 2. By extending this group of order 24 by s_1^5 we obtain the group of order 48 which has 24 additional operators of order 4 of which 12 have the same square as s_1^5 while the other 12 have a different common square. This elementary group of order 48 is $\{s_1^5, s_2^5\}$ and has the symmetric group of order 24 for its group of cogredient isomorphisms. When s_2 is of order 6, $\{s_1, s_2\}$ is this group of order 48, and when s_2 is of order 30 it is the direct product of this group and the cyclic group of order 5.

It remains only to consider the cases when s_2 is of order 3 or 15. In the former case $\{s_1, s_2\}$ is clearly the octahedron group, while in the latter it is the direct product of this and the cyclic group of order 5, since s_1^5, s_2^5 generate the octahedron group. The proof of this fact follows directly from the preceding paragraphs. *Hence there are just six groups whose two generators satisfy the conditions* $s_1^2 = s_2^3$, $(s_1 s_2)^4 = 1$. *Three of these are of orders* 24, 48 *and* 96, *respectively; while the other three are the direct products of these groups and the cyclic group of order* 5.

It remains to consider the generalization of the octahedral group whose two generators satisfy the equations

$$s_1^3 = s_2^4, \qquad (s_1 s_2)^2 = 1.$$

The superior limit of the order of s_2 may be found as follows:

$$(s_2^{-2} s_1^{-2} s_2^2 s_1^2)^2 = s_2^{-16}(s_2^2 s_1 s_2^2 s_1^2)^2 = s_2^{-24}(s_2 s_1^2)^4 = s_2^{-28},$$

since

$$s_1 s_2^2 = s_2^{-1} s_1^{-1} s_2, \qquad (s_2 s_1^2)^n = s_1^{-1} s_2^{-n} s_1.$$

* Cf. MOORE, Bulletin of the American Mathematical Society, vol. 1 (1894), p. 61.

As s_2^{-2} and $s_1^{-2}s_2^2s_1^2$ have a common square, s_2^{-28} is transformed into its inverse and into itself by s_2^3. That is, the order of s_2 divides 56. The two operators s_1^{49}, s_2^{49} also satisfy the given conditions since $(s_1^{49}s_2^{49})^2 = s_2^{224}(s_1s_2)^2 = 1$. If the order of s_2 is 56 the orders of these two operators are 6 and 8 respectively. We proceed to consider the group generated by them.

From the given equations it follows that s_2^{98}, $s_1^{-98}s_2^{98}s_1^{98}$ generate the quaternion group. That this is invariant under $\{s_1, s_2\}$ follows from the proof that $\{s_2^2, s_1^{-2}s_2^2s_1^2\}$ is invariant under $\{s_1, s_2\}$. That this group contains

$$s_1^{-1}s_2^2s_1 = s_2^{-4}\cdot s_1^2s_2^2s_1 = s_2^{-4}s_1s_2^{-1}s_1^{-1}s_2s_1 = s_2^{-4}s_1s_2^{-1}s_1^{-2}s_2^{-1} = s_2^{-8}s_1s_2^{-2}s_1^{-1}s_2^{-2} = s_2^{-20}s_1s_2^2s_1^2s_2^2$$

follows from the given form of the product $s_2^{-2}\cdot s_1^{-2}s_2^2s_1^2$. It is transformed into itself by s_2 since s_2^3 transforms $s_2^2s_1s_2^2s_1^2$ into $s_2s_1s_2^2s_1^2s_2 = s_1^3\cdot s_1^{-1}s_2^2s_1s_2^2$, which is in this group. The quaternion group $\{s_2^{98}, s_1^{-98}s_2^{98}s_1^{98}\}$ and s_1^{49} generate the group of order 24 which does not contain a subgroup of order 12. As this is clearly invariant under $\{s_1, s_2\}$ it results that $\{s_1^{49}, s_2^{49}\}$ is the group of order 48 known* as G_{51}, which is very closely related to the group of this order G_{52} considered at the beginning of this section. When the order of s_2 is 8, $\{s_1, s_2\}$ is therefore G_{51}, and when this order is 56, $\{s_1, s_2\}$ is the direct product of G_{51} and the cyclic group of order 7, since the group generated by s_2^8 is invariant under $\{s_1^{49}, s_2^{49}\}$ and has only the identity in common with it.

When the order of s_2 is 28, $\{s_1^{49}, s_2^{49}\}$ is the symmetric group of order 24 and $\{s_1, s_2\}$ is the direct product of it and the cyclic group of order 7. Hence *there are exactly four groups whose two generating operators satisfy the conditions* $s_1^3 = s_2^4$, $(s_1s_2)^2 = 1$. *Two of these are of orders* 24 *and* 48 *respectively while the other two are the direct products of these and the cyclic group of order* 7. It may be added that the three groups of order 48 which have been considered in this section are distinct.

§ 3. *Generalization of the icosahedral group.*

If two operators s_1, s_2 satisfy the equations

$$s_1^2 = 1, \qquad s_2^5 = 1, \qquad (s_1s_2)^3 = 1,$$

they generate the simple group of order 60. This fact was observed by HAMILTON as early as 1856†, and it is of especial interest since there are only a few finite groups which may be defined in such an elementary manner. It is well known that the 60 operators of this group may be written as follows:

$$s_2^m,\ s_2^m s_1 s_2^n,\ s_2^m s_1 s_2^2 s_1 s_2^n,\ s_2^m s_1 s_2^2 s_1 s_2^3 s_1 \quad (m, n = 1, 2, \cdots, 5).$$

*Quarterly Journal of Mathematics, vol. 30 (1898), p. 258.

† HAMILTON, Philosophical Magazine, vol. 12 (1856), p. 446.

In the following generalization it will be convenient to use a group which is defined by a very closely related set of equations, viz.,

$$s_1^4 = 1, \qquad s_2^5 = s_1^2, \qquad (s_1 s_2)^3 = 1.$$

Since the icosahedral group does not involve any operators of order 4 the order of the present group is not less than 120. That it is exactly of this order follows from the fact that each of its operators may be written in one of the following forms:

$$s_2^m,\ s_2^m s_1 s_2^n,\ s_2^m s_1 s_2^2 s_1 s_2^n,\ s_2^m s_1 s_2^2 s_1 s_2^3 s_1 \quad (m=1, 2, \cdots, 10;\ n=1, 2, \cdots, 5).$$

The reductions to these forms follow directly from those of the icosahedral group, since any product involving powers of s_1 and s_2 may be written as a power of s_2 multiplied by a product in which only the first power of s_1 and the first four powers of s_3 occur; for s_1^2 and s_2^5 are invariant under $\{s_1, s_2\}$.

This group (G_{120}) of order 120 is well known as an imprimitive group of degree 24 * and it seems therefore unnecessary to give an existence proof here. It may be observed that it does not include the icosahedral group, but this group is its group of cogredient isomorphisms. It will be found that the groups whose two generating operators satisfy the generalization of the icosahedral group given by the equations

$$s_1^2 = s_2^5, \qquad (s_1 s_2)^3 = 1$$

are merely direct products of a cyclic group whose order divides 21 and one of the two groups given above. As a first step in this proof, we consider the product of two operators of order two which have a common square, viz., s_1 and $s_2^{-4} s_1 s_2^4$. The following equations result directly from the two given relations between s_1 and s_2:

$$(s_2^{-4} s_1 s_2^4 s_1^{-1})^2 = (s_2^2 s_1^{-1} s_2^{11})^2 = s_2^2 s_1^{-1} s_2^{13} s_1^{-1} s_2^{11},$$
$$(s_2^{-4} s_1 s_2^4 s_1^{-1})^5 = s_2^2 (s_1^{-1} s_2^{13})^5 s_2^{-2} = s_2^2 (s_1 s_2^3)^5 s_2^{23} = s_2^{105}.$$

Since s_1 transforms s_2^{105} into its inverse † and is commutative with it, the order of s_2 cannot exceed 210. If this order is 210 the two operators s_1^{21}, s_2^{21} generate the given group of order 120 since $(s_1^{21} s_2^{21})^3 = s_2^{210}(s_1 s_2)^3 = 1$. Hence $\{s_1, s_2\}$ contains the group generated by s_2^{10} and this group of order 120 as invariant subgroups having only the identity in common. It is therefore the direct product of these two subgroups, and each of its operators is represented once and only once by one of the following expressions:

$$s_2^m,\ s_2^m s_1 s_2^n,\ s_2^m s_1 s_2^2 s_1 s_2^n,\ s_2^m s_1 s_2^2 s_1 s_2^3 s_1 \quad (m=1, 2, \cdots, 210;\ n=1, 2, \cdots, 5).$$

* It is also the compound perfect group of lowest possible order. Cf. American Journal of Mathematics, vol. 20 (1898), p. 277.

† Archiv der Mathematik und Physik, vol. 9 (1895), p. 6.

The cyclic group generated by s_2^5 is composed of the invariant operators of this group and the corresponding quotient group is the icosahedral group. This group of order 2520 may be regarded as the generalization of the icosahedral group which is defined by the equations $s_1^2 = s_2^5$ and $(s_1 s_2)^3 = 1$. The other groups whose two generators satisfy these conditions must also satisfy an additional condition, viz., that the order of s_2 is less than 210. If the order of s_2 is either 70 or 30 it still follows that $(s_2^{21} s_2^{21})^3 = 1$ and hence the corresponding $\{s_1, s_2\}$ is the direct product of the given group of order 120 and either the cyclic group of order 7 or the cyclic group of order 3.

Finally when the order of s_2 is 105, 35 or 15, the two operators s_1^{21}, s_2^{21} generate the icosahedral group and $\{s_1, s_2\}$ is the direct product of this group and the cyclic groups of orders 21, 7, 3, respectively. If we include the icosahedral group and the given group of order 120, *there are exactly eight groups whose two generators satisfy the equations* $s_1^2 = s_2^5$, $(s_1 s_2)^3 = 1$. In each case the operator of the group is represented once and only once by one of the following forms:

$$s_2^m, \quad s_2^m s_1 s_2^n, \quad s_2^m s_1 s_2^2 s_1 s_2^n, \quad s_2^m s_1 s_2^2 s_1 s_2^3 s_1 \qquad (m = 1, 2, \cdots, \text{order of } s_2;\ n = 1, 2, \cdots, 5).$$

It may be added that the existence of the given direct products could have been seen directly, since, in the definition of the icosahedron group by means of the equations $s_1^2 = 1$, $s_2^5 = 1$, $(s_1 s_2)^3 = 1$, it is clearly possible to select an independent operator of order 21 and multiply s_1, s_2, respectively, by its first and its thirteenth powers. The resulting products will satisfy the equations of the generalized icosahedral group. The above proof may therefore be regarded as merely establishing that there are no other such groups.

The two other possible similar generalizations of the icosahedron group can readily be reduced to the preceding one. We shall first consider the case when

$$s_1^3 = s_2^5, \qquad (s_1 s_2)^2 = 1.$$

To exhibit the connection with the preceding case we let $t_1 = s_1 s_2$ and $t_2 = s_2^{-1}$. Hence we have to consider $\{t_1, t_2\}$ where

$$t_1^2 = 1, \qquad t_2^5 = (t_1 t_2)^{-3}.$$

To obtain an upper limit for the order of t_2 we may use the product $t_2^{-4} t_1 t_2^4 t_1$. Since t_2^5 is invariant under $\{t_1, t_2\} \equiv \{s_1, s_2\}$, it is possible to write the given product in a more compact form as follows: From $t_2^{-1} t_1 t_2^{-1} t_1 t_2^{-1} t_1 = t_2^5$ it results that $t_2^{-4} t_1 t_2^4 t_1 = t_2^2 t_1 t_2 \cdot t_2^5$. Hence it results that $(t_2^{-4} t_1 t_2^4 t_1)^n = t_2^2 (t_1 t_2^3)^n t_2^{5n-2}$. If we write the equation $t_2^{-1} t_1 t_2^{-1} t_1 t_2^{-1} t_1 = t_2^5$ in the form $t_1 t_2^3 = t_2 t_1 t_2 t_1 t_2^4 \cdot t_2^5$ it is clear that $(t_1 t_2^3)^n = t_2 t_1 t_2^n t_1 t_2^{-1} \cdot t^{10n}$. Letting $n = 5$ in these equations we arrive at the equation

$$(t_2^{-4} t_1 t_2^4 t_1)^5 = t_2^{80}.$$

As t_1 is commutative with t_2^{80} and also transforms it into its inverse, the order of t_2 is a divisor of 160. We shall now prove that it cannot be 160. If t_2 were of order 160 $t_1 t_2$ would be of order 96. The group generated by t_2^{16} and $(t_1 t_2)^{16} = t_1 t_2 \cdot t_2^{-25}$ is the same as the one generated by t_2^{16} and $t_1 t_2 \cdot t_2^{-25} \cdot t_2^{80} = t_1 t_2 \cdot t_2^{55}$. As the last operator is of order 3 and as $t_1 t_2^{56} \cdot t_2^{-16} = t_1 t_2^{40}$ is of order 4 we are dealing with a group whose two generators s_1, s_2 satisfy the equations $s_1^2 = s_2^5$, $(s_1 s_2)^3 = 1$, $s_1^4 = 1$. This is the group of order 120 whose properties were considered above. The cyclic group generated by t_2^{15} is invariant under this group and with it generates $\{t_1, t_2\}$. This is impossible, since t_1 would correspond to an operator of order 4 in the $(k, 1)$ isomorphism between $\{t_1, t_2\}$ and the given group of order 120. The order of s_2 must therefore divide 80.

It is now easy to prove that *there are exactly five groups whose two generators s_1, s_2 satisfy the equations $s_1^3 = s_2^5$, $(s_1 s_2)^2 = 1$. They are the direct products of the icosahedron group and the cyclic group of order 2^α, $\alpha = 0, 1, 2, 3, 4$.* The proof of this theorem follows very readily from the fact that the order of s_2 divides 80. Hence $\{s_1^{16}, s_2^{16}\}$ is the icosahedron group, since $(s_1^{16} s_2^{16})^2 = (s_1 s_2)^2 = 1$ and s_1^{16}, s_2^{16} are of orders 3 and 5 respectively. As $\{s_1, s_2\}$ contains the cyclic group generated by s_2^5 and the icosahedron group as invariant subgroups having only the identity in common and the order of $\{s_1, s_2\}$ cannot exceed the product of the orders of this invariant subgroup, the proof is complete.

It remains only to consider the groups whose two generators satisfy the equations

$$s_1^2 = s_2^3, \qquad (s_1 s_2)^5 = 1.$$

It will be convenient to use the auxiliary symbols $t_1 = s_1$, $t_2 = s_1 s_2$. Proceeding as before we have

$$(t_2^{-4} t_1 t_2^4 t_1^{-1})^n = t_2^2 (t_1 t_2^3)^n t_2^3 s_2^{-12n}, \qquad (t_1 t_2^3)^n = t_2 t_1 t_2^n t_1 t_2^4 \cdot s_2^{-6n-3}.$$

Hence

$$(t_2^{-4} t_1 t_2^4 t_1^{-1})^5 = s_2^{-90}.$$

Moreover, $(t_2^{-2} t_1 t_2^2 t_1^{-1})^3 = s_2^{42} (t_2^2 t_1)^5 t_2^{-8} = s_2^{75}$. Since t_1 transforms both s_2^{-90} and s_2^{75} into their inverses and is commutative with each of them, the order of s_2 divides 30.

If the order of s_2 is 30 that of s_1 is 20. In this case s_1^5, s_2^{10} generate the group G_{120} considered above; for $(s_1^5 s_2^{10})^5 = s_2^{15}$ and the orders of s_1^5, s_2^{10} are 4 and 3 respectively. As s_1^4 is invariant under this group and has only the identity in common with it and as $\{s_1^4, s_1^5, s_2^{10}\} \equiv \{s_1, s_2\}$ it follows that the latter is the direct product of G_{120} and the cyclic group of order 5. When the order of s_2 is 15 that of s_1 is 10 and s_1^5, s_2^{10} generate the icosahedron group. In this case $s_1^5 s_2^{10} = s_2^{15} s_1 s_2 = s_1 s_2$. When the order of s_2 is either 6 or 3, $\{s_1, s_2\}$ is either the given group of order 120 or the icosahedron group. Hence the theorem:

There are exactly four groups whose two generators s_1, s_2 *satisfy the equations* $s_1^2 = s_2^3$, $(s_1 s_2)^5 = 1$. *They are the icosahedron group,* G_{120}, *and the direct products of these groups and the cyclic group of order* 5.

Combining the results of this section, we conclude that there are fourteen distinct groups which are generated by two operators satisfying one of the three immediate generalizations of the defining relations of the icosahedron group. The generators of the icosahedron group satisfy each of these generalized definitions, those of G_{120} satisfy two of them, while those of each of the other twelve groups determined above satisfy only one of these sets of conditions. The simple defining relations and the close contact with the simple group of lowest composite order make these fourteen groups of peculiar interest.

ON REFLEXIVE GEOMETRY*

BY

F. MORLEY

This memoir is in continuation of two in these Transactions (vol. I, p. 97 and vol. IV, p. 1). It follows especially § 3 of the first memoir. The object of the three is to illustrate an algebraic method of handling planar displacements, which applies more immediately to elementary geometry than does the usual coördinate geometry. While the latter heads directly for projective geometry, the present method heads for inversive geometry. It may perhaps be called *reflexive geometry.*

§ 1. *Outline of the method.*

This is not the place to expound the beginnings of the method, but an outline may be given. On analysing displacements of a plane on a plane, on the hypothesis that translations exist, we find easily that they amount to

(1) reflexion in a line chosen once for all,

(2) rotations about a point, chosen once for all, and

(3) translations.

We take the point on the line and call them base-point and base-line.

Let now the displaced object have symmetry; and passing at once to the extreme case let it be a point. Displacements previously different become the the same, and equations arise.

The subsequent details are merely an enlargement of the planimetric interpretation of the fundamental operations of algebra, as given in works on the theory of functions and in some works on algebra.

The notation used is a superposed bar for reflexion in the base-line (thus $\bar{x}$ is the image in the base-line, or "conjugate" of x) and the letter t or τ for rotation about the base-point. The symbol 0 denotes that the moving particle is at the base-point, the symbol 1 that it has undergone a unit translation along the base-line, the symbol t that it is somewhere on the circle with center 0 and radius the unit of length. This circle is called the *base-circle*, and the special complex number t is called a *turn.*

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§ 2. *The cyclogen.*

Consider the series of self-conjugate equations:

$$a + \bar{a}t = 0,$$

$$a + 2\mu t + \bar{a}t^2 = 0,$$

$$a + bt + \bar{b}t^2 + \bar{a}t^3 = 0,$$

$$a + bt + 2\mu t^2 + \bar{b}t^3 + \bar{a}t^4 = 0,$$

$$\cdots\cdots\cdots\cdots$$

where μ is real. Such an equation is of the same generality as the equation with real coefficients usually studied in the theory of equations. In fact by mapping the base-line on the base-circle they are interchanged. To the real roots of the latter correspond the turn roots of the former, to conjugate roots of the latter correspond inverse points as to the base-circle.

Thus every equation of our series, whose degree is odd, has one turn as a root. We take as the standard case an equation all of whose roots are turns.

Now regard in the above equations a pair of conjugate coefficients as variable. We have then for given t a line, for varying t a curve of lines.

If we denote by n the class of the curve and by m the number of lines of the curve in a given direction, then the curve may be denoted by C^n_m. The simpler curves are:

C^1_1, a point;	C^3_3, a cardioid;
C^2_0, a segment;	C^4_0, a segment*;
C^2_2, a circle;	C^4_2, a parastroid;
C^3_1, a deltoid;	C^4_4, a paranephroid.

These curves may be called *cyclogens.*

We are concerned here solely with the curves C^n_n for which the end coefficients are variables. If we mean by the *aspect* of a curve the number of parallel tangents, then these curves are cyclogens of full aspect. They might be called cardioids, or better ennacardioids. Thus C^4_4 is the tetracardioid. Denoting the curve C^n_n now simply by C^n, the series of ennacardioids begins with the point C^1, the circle C^2, the cardioid C^3. We call the equation by which a cyclogen was defined its line-equation, having no other line-equation in the context. It comes under what Laguerre (*Works*, vol. 2, p. 190), called the "équation mixte."

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* Described with superposed harmonic motions.

§ 3. *The ennacardioid.*

Taking now the case of C^n, we apply to it STUDY's theory of osculants. We write the line equation in abridged form

$$(\alpha t)^n = 0, \tag{1}$$

where $(\alpha t) = \alpha_1 + \alpha_2 t$, $\alpha_1^r \alpha_2^{n-r}$ is a coefficient conjugate to $\alpha_1^{n-r}\alpha_2^r$ and $x = \alpha_1^n$.

The equation

$$(\alpha t_1)(\alpha t)^{n-1} = 0$$

defines a (first) osculant; it is a curve C^{n-1}. An osculant of this is a second osculant of C^n. And so on till we come to the completely polarized form

$$(\alpha t_1)(\alpha t_2)\cdots(\alpha t_n) = 0, \tag{2}$$

which shows that the n osculant points, formed from $n-1$ of the n t's, lie on a *line.*

The relation of x to t, deduced from

$$(\alpha t)^n = 0$$

is

$$\alpha_1(\alpha t)^{n-1} = 0. \tag{3}$$

This is the map-equation of C^n. Its polars are map-equations of the osculants.

If in a map-equation we calculate $D_t x$ and make it zero, we obtain values of t which, when turns, give cusps. The cusps of $(\alpha t)^n$ are then given by

$$\alpha_1\alpha_2(\alpha t)^{n-2} = 0, \tag{4}$$

a self-conjugate equation. Thus C^n can have $n-2$ cusps. To avoid periphrasis I will use the projective view and say that there are always $n-2$ cusps, real or imaginary, though, strictly, imaginary points have no foothold in the reflexive geometry, which is a chapter of mathematics preceding projective geometry.

The cusps themselves are

$$\alpha_1(\alpha t)^{n-1} = 0$$

where

$$\alpha_1\alpha_2(\alpha t)^{n-2} = 0,$$

or are

$$\alpha_1^2(\alpha t)^{n-2} = 0. \tag{5}$$

The map equation of an osculant is

$$\alpha_1(\alpha t_1)(\alpha t)^{n-2} = 0$$

and this equation holds when (4) and (5) do. That is: *The osculant is on the cusps of* C^n.

The osculants are thus defined by the facts of touching the curve and being on the $n-2$ cusps.

§ 4. *The envelopes of osculants.*

Denoting the osculant

$$(\alpha t_1)^r(\alpha t)^{n-r} = 0$$

by O_r, it has an envelope and a cusp-locus. The envelope contains the original curve C^n and the curve χ_r. It is to be shown that *the cusp-locus of* O_r *is* χ_{r+1}. The map-equation of O_r is

$$\alpha_1(\alpha t_1)^r(\alpha t)^{n-r-1} = 0.$$

Now to obtain the envelope of

$$x = f(t, t_1)$$

the rule is that

(6) $$D_t x/t : D_{t_1} x/t_1, \text{ is real.}$$

Here then

$$\alpha_1\alpha_2(\alpha t_1)^r(\alpha t)^{n-r-2}/t : \alpha_1\alpha_2(\alpha t_1)^{r-1}(\alpha t)^{n-r-1}/t_1, \text{ is rea}$$

But the equation

$$\alpha_1\alpha_2(\alpha t_1)^r(\alpha t)^{n-r-2} = 0$$

is self-conjugate. Hence either $t = t_1$, which gives C^n, or

$$\alpha_1\alpha_2(\alpha t_1)^r(\alpha t)^{n-r-2} = 0,$$

which manifestly gives the cusp-locus, or

$$\alpha_1\alpha_2(\alpha t_1)^{r-1}(\alpha t)^{n-r-1} = 0,$$

which gives χ_r. Thus χ_{r+1} is given by

(7) $$\begin{cases} \alpha_1(\alpha t_1)^{r+1}(\alpha t)^{n-r-2} = 0, \\ \alpha_1\alpha_2(\alpha t_1)^r(\alpha t)^{n-r-2} = 0. \end{cases}$$

The former equation may be replaced by

$$\alpha_1^2(\alpha t_1)^r(\alpha t)^{n-r-2} = 0,$$

and this again by

$$\alpha_1(\alpha t_1)^r(\alpha t)^{n-r-1} = 0.$$

This proves the theorem.

There are then associated with C^n a series of curves

χ_1 the cusps,

χ_2 the cusp locus of O_1 and envelope of O_2,

χ_3 the cusp locus of O_2 and envelope of O_3;

and writing again the equations for χ_r:

$$\alpha_1\alpha_2(\alpha t_1)^{r-1}(\alpha t)^{n-r-1} = 0,$$

$$\alpha_1^2(\alpha t_1)^{r-1}(\alpha t)^{n-r-1} = 0,$$

we see that since we may interchange t and t_1 the curves χ_r and $\chi_{r'}$ are the same if

$$r + r' = n.$$

Thus for the curve C^4 there are

χ_1 the cusps,

χ_2 a circle or arc of a circle (according as the cusps are not real or real) which is the cusp locus of osculant cardioids and the envelope of osculant circles;

and for the curve C^5, which has as osculants C^4, C^3 and C^2,

χ_1 the cusps,

χ_2 which is at once the cusp locus of C^4, the envelope of C^3, the cusp locus of C^3, and the envelope of C^2.

§ 5. *Note on the rational plane curve in general.*

For rational plane curves in general, given for convenience of statement in lines, it is equally true that the cusp locus of O_r is the envelope of O_{r+1}. We know from STUDY's theory that the cusps of the given curve are on O_1, and the cusps of O_r on O_{r+1}. Let O_r be given by

$$\xi = (\alpha s)^r (\alpha t)^{n-r},$$

$$\eta = (\beta s)^r (\beta t)^{n-r},$$

$$\zeta = (\gamma s)^r (\gamma t)^{n-r},$$

where

$$\alpha t = \alpha_1 + \alpha_2 t.$$

The cusp parameters are given by

$$|\xi \quad \eta \quad \zeta| = 0,$$

that is by

$$\begin{vmatrix} (\alpha s)^r (\alpha t)^{n-r} & \cdots \\ \alpha_2 (\alpha s)^r (\alpha t)^{n-r-1} & \cdots \\ \alpha_2^2 (\alpha s)^r (\alpha t)^{n-r-2} & \cdots \end{vmatrix} = 0$$

or by

$$\begin{vmatrix} \alpha_1^2 & \beta_1^2 & \gamma_1^2 \\ \alpha_2^2 & \beta_2^2 & \gamma_2^2 \\ \alpha_1\alpha_2 & \beta_1\beta_2 & \gamma_1\gamma_2 \end{vmatrix} (\alpha s)^r (\alpha t)^{n-r-2} \cdots = 0.$$

The envelope of O_{r+1}, or

$$\xi = (\alpha s)^{r+1} (\alpha t)^{n-r-1}, \text{ etc.},$$

is given by

$$|\xi \quad D_s\eta \quad D_t\zeta| = 0,$$

or by

$$\begin{vmatrix} & (\alpha s)^{r+1}(\alpha t)^{n-r-1}\cdots \\ \alpha_2(\alpha s)^r & (\alpha t)^{n-r-1}\cdots \\ \alpha_2(\alpha s)^{r+1} & (\alpha t)^{n-r-2}\cdots \end{vmatrix} = 0;$$

and this on throwing out the factor $s - t$ gives the same result as before.

Thus for example with the rational curve of class 4 is associated a curve χ_2 on which lie the cusps of osculant cubics, and which has as tritangent conics all osculant conics.

§ 6. *Construction of the ennacardioid.*

The number of coördinates specifying a C^n is $n+1$. For the number of arbitrary real constants in $(\alpha t)^n$ is $n-1$, and the base-point is here a definite point as to the curve — say its center.

Hence when $2(n-2) = n+1$, or $n = 5$, the cusps determine a C^n, though not perhaps uniquely. Where $n > 5$ the cusps are not independent.

Two osculants, say

$$(\alpha t_1)(\alpha t)^{n-1} \qquad \text{and} \qquad (\alpha t_2)(\alpha t)^{n-1},$$

have a common osculant

$$(\alpha t_1)(\alpha t_2)(\alpha t)^{n-2}.$$

But conversely if two C^{n-1}'s have a common osculant, they are osculants of a C^n. For an osculant of one C^{n-1} can touch the other; if in addition it is on the cusps, this is $n-3$ conditions and the coördinates of the two and

$$2n - (n-3) \qquad \text{or} \qquad n+3.$$

But this is also the specification of C^n and two osculants.

The tangent at t of the osculant at t_1, C^{n-1}, is

$$(\alpha t_1)(\alpha t)^{n-1} = 0.$$

This is satisfied when $\alpha_1(\alpha t)^{n-1} = 0$, and this gives the point t of C^n. Varying t, we obtain a rigid pencil of lines, each touching some osculant C^{n-1}. Conversely, consider any two C^{n-1}'s. Let an angle slide around them. The locus of its vertex is easily seen to be a polynomic or curve of the form

$$x = Pt + Q(1/t)$$

where P and Q are polynomials. And in fact any polynomic can be so constructed.

But when the two C^{n-1}'s have a common osculant C^{n-2}, and are therefore osculants of a C^n, the locus of the vertex can be this C^n.

Thus granting that we can construct one and therefore two C^n's of which an osculant C^{n-1} is given, then we mark on each line of C^{n-1} the corresponding

points of the two C^n's, and the lines of these C^n's at these points meet on a C^{n+1}. Thus the curves can be drawn in succession; the first step is to take two intersecting circles, and draw lines on a common point; the tangents of the circles at the other intersections meet on a cardioid.

§ 7. *The C^{n-1} of an n-line.*

It was shown in these Transactions (vol. 1, p. 102) that (with the notation there used) the circles of 3 from 4 lines are osculants of a C^3,

$$x = a_1 - 2a_2 t + a_3 t^2,$$

where $|a_2| = |a_3|$.

Calling this the cardioid of the 4-line we have now the theorem:

The cardioids of 4 *from* 5 *lines are osculants of a* C^4*; and in particular meet at two points (the cusps).*

For the curve for 5 lines

$$x = a_1 - 3a_2 t + 3a_3 t^2 - a_4 t^3$$

is a C^4.

And so in general:

The C^{n-1}'s of $n-1$ from n lines are osculants of the C^n of the n lines, and in particular are on the $n-2$ cusps.

Conversely, a C^n and $n+1$ osculants determine at once $n+1$ lines, for a C^n and n osculants determine the line

$$(\alpha t_1)(\alpha t_2)\cdots(\alpha t_n) = 0.$$

This line may be found thus. We are given a C^n and n osculant C^{n-1}'s. Every two C^{n-1}'s have a common osculant C^{n-2}. Hence a selected C^{n-1} has $n-1$ osculant C^{n-2}'s; and so finally we come to a C^3 with three osculant C^2's whose common osculants are points on the line in question.

There is then for an n-line a curve C^{n-1}. There is also a definite circle called (vol. 1, p. 99) the centre-circle. I will call it now the centric circle and its center the centric of the n-line. I proceed to give a general meaning to these two curves.

§ 8. *The images of a point in 4 lines.*

To connect by a curve the images of a point x_0 in four given lines we employ the process of interpolation. This is not a definite process, but employed as follows what results is *the conic on x_0 and its images.*

Write the lines

$$x - a_2 s_1 + a_3 s_2 - \bar{x} s_3 = 0,$$

where the 3 turns whose symmetric functions are s_1, s_2, s_3 are selected from 4 whose product is -1.

Then the image of x_0 is

$$x = a_2 s_1 - a_3 s_2 + \bar{x}_0 s_3 .$$

We have to symmetrize this.

Assume two turns τ, τ_1 such that

$$x = A\tau^3 + A_1\tau_1^3 - A\prod^3(\tau - t_i) - A_1\prod^3(\tau_1 - t_i), \tag{8}$$

where

$$A\tau^2 + A_1\tau_1^2 = a_2,$$

$$A\tau + A_1\tau_1 = a_3,$$

$$A + A_1 = \bar{x}_0 .$$

Then

$$\bar{x}_0\tau\tau_1 - a_3(\tau + \tau_1) + a_2 = 0,$$

or

$$x_0 - a_2(\tau + \tau_1) + a_3\tau\tau_1 = 0 .$$

Thus x_0 is any point inside the C^3

$$x_1 - 2a_2\tau + a_3\tau^2 = 0 .$$

We have then, symmetrizing (8),

$$x = A\tau^3 + A_1\tau_1^3 - A\frac{\prod^4(\tau - t_i)}{\tau - t_i} - A_1\frac{\prod^4(\tau_1 - t_i)}{\tau_1 - t_i}, \tag{9}$$

where A and A_1 are known.

This is the map-equation of a hyperbola. It evidently is on the 4 images of x_0, but further it is on x_0 itself.

For eliminating A and A_1 the equation is

$$\begin{vmatrix} x & \tau^3 - \dfrac{\Pi}{\tau - t} & \tau_1^3 - \dfrac{\Pi_1}{\tau_1 - t} \\ a_2 & \tau^2 & \tau_1^2 \\ a_3 & \tau & \tau_1 \end{vmatrix} = 0 . \tag{10}$$

The terms independent of t give x_0. But the terms in t are

$$\Delta = \begin{vmatrix} 0 & -\dfrac{\Pi}{\tau - t} & -\dfrac{\Pi_1}{\tau_1 - t} \\ a_2 & \tau^2 & \tau_1^2 \\ a_3 & \tau & \tau_1 \end{vmatrix},$$

and the conjugate of this is

$$\Delta t/\tau^3\tau_1^3 .$$

Hence $\Delta = 0$ is a self-conjugate equation, giving a value of t which is a turn. That is, we have the hyperbola on x_1 and its images.

The clinants of the asymptotes are found at once to be $1/\tau$ and $1/\tau_1$.

There are now two special conics:

1) the rectangular hyperbola, when

$$\tau + \tau_1 = 0,$$

the point x_1 is then given by

$$x + a_3 \tau\tau_1$$

and is on the centric circle;

2) the parabola, when

$$\tau = \tau_1,$$

the point is then on the cardioid. Thus the conic on a point x_0 and its images in 4 given lines is a rectangular hyperbola when x_0 is on a circle and a parabola when x_0 is on a cardioid.

§ 9. *The images of a point in n lines.*

The generalization is now immediate. Thus if 5 lines be written

$$x - a_2 s_1 + a_3 s_2 - a_4 s_3 + \bar{x} s_4 = 0,$$

where the s_i are symmetric functions of 4 from 5 turns whose product is 1, the images of of x_0 in the 5 are on

$$\begin{vmatrix} x & \tau^4 - \dfrac{\Pi}{\tau - t} & \tau_1^4 - \dfrac{\Pi_1}{\tau_1 - t} & \tau_2^4 - \dfrac{\Pi_2}{\tau_2 - t} \\ a_2 & \tau^3 & \tau_1^3 & \tau_2^3 \\ a_3 & \tau^2 & \tau_1^2 & \tau_2^2 \\ a_4 & \tau & \tau_1 & \tau_2 \end{vmatrix} = 0$$

where

$$x_0 - a_2(\tau + \tau_1 + \tau_2) + a_3(\tau_1\tau_2 + \tau_2\tau + \tau\tau_1) - a_4\tau\tau_1\tau_2 = 0.$$

We have then a pencil of cubic curves J^3 passing through the images of x_0, and twice through x_0.

If we make $\tau = \omega\tau_1 = \omega^2\tau_2$ where $\omega^2 + \omega + 1 = 0$, then

$$x_0 - a_4\tau^3 = 0,$$

that is, for points x_0 on the centric circle the cubic has equispaced asymptotes.

If we make $\tau = \tau_1 = \tau_2$ the point x_0 is on the C^4 of the 5 lines and the J^3 meets infinity (regarding infinity as a line, i. e., speaking projectively) at one point only.

And so in general, calling a curve of order m with a multiple point of order $m - 1$ a JONQUIÈRES *curve* J^m:

Let a J^{n-2} be drawn with multiple point at x_0 and on the images of x_0 in n given lines. If the J^{n-2} have equispaced asymptotes then x_0 is on the centric circle of the n-line; and if the J^{n-2} have coincident points at infinity then x_0 is on the C^{n-1} of the n-line.

There are naturally other special cases. Important among these as giving the CLIFFORD theorem is the one where the system of J^{n-2}'s (for n odd) reduces to a $J^{(n-1)/2}$. This involves the vanishing of minors of the elements in t in the determinant and brings us back to the analysis given in these Transactions, loc. cit., p. 104.

The statement of CLIFFORD's theorem in terms of C^{n-1} is: The curve C^{2m-1} has one osculant C^m which reduces to a point; the parameters are the canonizant of the cusp-parameters. For the curve C^{2m} the locus of points which are degenerate osculant C^m's is a circle.

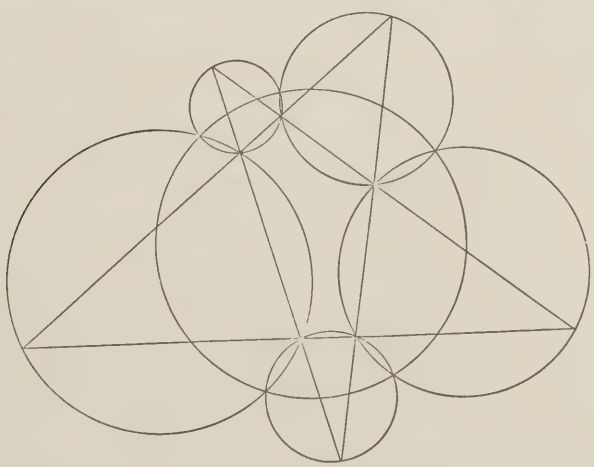
§ 10. *The case when the C^{n-1} of an n-line is a circle.*

We have seen that the image-curve for a point on the centric circle is an equispaced J^{n-2}. Let the point be the centric of $n-1$ of the lines. The remaining line and therefore the image of x_0 in it becomes arbitrary. Hence the centric of $n-1$ lines is the multiple point of not one but a pencil of equispaced J^{n-2}'s on the images of x_0. But the pencil meets again in $(n-2)^2-(n-3)^2-(n-1)$ or $n-4$ points. Hence the $n-1$ lines belong to a symmetric system of $2n-5$ lines. That is, *m lines belong to a set of $2m-3$ lines.*

Any m of these have the same centric, and therefore the centric circle of $m+1$ is a point. The analytic condition is then, since the centric circle is

$$x = a_1 - a_2 t,$$

that a_2 formed for any $m+1$ lines in 0. This requires for the $2m-3$ lines



that $a_2 = a_3 = \cdots = a_{m-2} = 0$. The radius of the centric circle of any m lines is $|a_{m-1}|$, and therefore any m have the same centric circle. Since all a's except a_{m-1} vanish, the conditions are summed up by saying that the C^{2m-4} of the lines is a circle.

Thus in particular 4 lines determine a fifth; the 5 forming say a *pentacle*, such that the centres of the 10 3-lines are on a circle. The figure is drawn simply by placing a ring of 5 circles with centres on a given circle and each intersecting the next on this circle. The 5 other intersections of the adjacent circles being joined in order form the pentacle, and the salient thing is that *the intersections of non-adjacent sides are also on the respective* 5 *circles.*

CRATER CLUB, ESSEX, N. Y.,
July, 1906.

THE GROUPS IN WHICH EVERY SUBGROUP IS EITHER ABELIAN OR HAMILTONIAN*

BY

G. A. MILLER

The non-abelian groups which contain only abelian subgroups have been determined and many of their fundamental properties have been noted. † The present paper is devoted to a determination and a study of the non-abelian groups which involve only abelian and hamiltonian subgroups. It will be found that there are only two exceptions to the following theorem: If a group contains at least one hamiltonian subgroup and if all its subgroups are either abelian or hamiltonian the group itself is hamiltonian. The two exceptional groups are of orders 16 and 24 respectively, and are completely determined in the following paragraphs.

Let G represent any group in which every subgroup is either abelian or hamiltonian and suppose that it is represented as a transitive substitution group of the lowest possible degree. If it is imprimitive its systems of imprimitivity are permuted according to some primitive group. To prove that G is solvable it is only necessary to prove that this primitive group is composite since it may be simply isomorphic with G and since every subgroup of G is solvable. The primitive group in question is of class $n-1$, n being its degree, since two of its maximal subgroups of degree $n-1$ can have only the identity in common. ‡ As every group of class $n-1$ and degree n is composite G is solvable.

In the preceding paragraph it is not assumed that G necessarily contains a hamiltonian subgroup. This assumption may, however, be made since the non-abelian groups in which every subgroup is abelian are known. Hence we shall assume in what follows that G contains at least one hamiltonian subgroup. Its order can therefore not be the power of an odd prime number. We shall consider all the possible G's under two headings as their orders are a power of 2 or involve more than one prime factor.

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† MILLER and MORENO, Transactions of the American Mathematical Society, vol. 4 (1903), p. 398.

‡ Loc. cit.

§ 1. *The order of G is 2^m.*

Since G involves at least one hamiltonian subgroup it must involve a hamiltonian subgroup (H) which is composed of half the operators of G. Since H is hamiltonian, three-fourths of its operators are non-invariant, of order four, and have a common square.* The remaining one-fourth are the invariant operators of H and they constitute an abelian subgroup (K) of type $(1, 1, 1, \cdots)$. With respect to K, H is isomorphic to the four-group. There is at least one subgroup (H') in H which involves K and one-third of the operators of order four contained in H and is invariant under G. The subgroup H' is abelian and of type $(2, 1, 1, \cdots)$.

As H does not contain any operator whose order exceeds 4 there can be no operator whose order exceeds 8 in G. Moreover, *every operator of order* 2 *is invariant under* G. This results from the following considerations. Any operator of G together with K generate a subgroup which is either abelian or hamiltonian since this operator and K do not generate G. As every operator of order 2 in a hamiltonian group is invariant this proves that every operator of K is invariant under G. If any other operator of order 2 were not commutative with some operator of order 4 in H the latter operator could not be in H' since a hamiltonian group involves no non-invariant operator of order 2. That is, each operator of H' is commutative with every operator of order 2 contained in G. If an operator of order 2 were not commutative with any other operator of order 4 in H it would transform this operator into itself multiplied by an operator of order 2 contained in H. Hence the order of G could not exceed 16. As the groups of order 16 are so well known it seems unnecessary to go into further details to establish the theorem in question.

We shall now prove that there is one and only one G which involves operators of order 8. As an operator (t) of order 8 in G is commutative with all the operators of K and has its square in H' it must be commutative with every operator of H'. It cannot be commutative with every operator of H since t^2 transforms some of these operators into their inverses. As t transforms an operator of order 4 in H into itself multiplied by an operator of order 4 which is commutative with t†, the group generated by t and any operator (s) of order 4 in H which is not commutative with t is of order 16 or 32. This group must coincide with G since it is neither abelian nor hamiltonian.

We proceed to prove that the order of the group $\{s, t\}$ generated by s and t cannot be 32. If this order were 32, t^2 would not be in the group of order 8 generated by s and the commutative subgroup of $\{s, t\}$. This group and t^2 would therefore generate the hamiltonian group of order 16 and G would involve the abelian group of type $(3, 1)$. This abelian subgroup would involve all the

* Bulletin of the American Mathematical Society, vol. 4 (1898), p. 510.

† Bulletin of the American Mathematical Society, vol. 7 (1901), p. 350.

operators of order 8 contained in G and two of the remaining operators of order 4 would generate a non-abelian group of order 16 involving a commutator subgroup of order 4. As such a subgroup could not occur in G it has been proved that the order of G cannot exceed 16. It may be remarked that this result could be directly obtained from the properties of the known groups of order 32.* As the groups of order 16 are known we may state the result as follows: *There is one and only one group of order 2^m which involves operators whose orders exceed four and satisfies the additional conditions that every subgroup is either abelian or hamiltonian and that at least one subgroup is hamiltonian.* This is the group of order 16 which contains a cyclic subgroup of order 8 while the remaining operators are of order 4 and transform the operators of this cyclic subgroup into their inverses.

Half of the operators of G which are not also in H transform the operators of order 4 in H' into their inverses while the other half are commutative with these operators. Hence H' is contained in two hamiltonian and one abelian subgroup whose order is half the order of G. We proceed to prove that G contains operators of order 2 which are not in H whenever its order exceeds 16. Let t_1 represent an operator of G which is not in H and which transforms into their inverses the operators of order 4 in H'. The operators of order 4 in H and t_1 have a common square. The group generated by t_1 and an operator of H which is not contained in H is at most of order 16. It could not be of this order since two operators of a hamiltonian group cannot generate a group of order 16. From this it follows that t_1 is commutative with half the operators of H and hence with operators of order 4 in H. As the product of t_1 into such an operator is of order 2, G is the direct product of H and an operator of order 2. That is, *If every subgroup of a group of order 2^m, $m > 4$, is either abelian or hamiltonian and if there is at least one hamiltonian subgroup, the entire group is hamiltonian.*

It remains to consider the groups of order 16. If such a group contains a hamiltonian subgroup it must be the quaternion group. As every group of order 16 contains an abelian subgroup of order 8 and as the groups under consideration do not involve any operators of order 8, it follows that G would have to contain operators of order 2 in addition to the one contained in the quaternion subgroup. As such an operator has to be commutative with each operator in the quaternion subgroup G itself is the hamiltonian group of order 16, and it results that the group of order 16 which was considered above is the only non-hamiltonian group of order 2^m in which every subgroup is either abelian or hamiltonian and in which at least one subgroup is hamiltonian. With this single exception the hamiltonian groups are the only ones involving hamiltonian subgroups, but no other non-abelian subgroups.

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*Quarterly Journal of Mathematics, vol. 28 (1896), p. 233.

§ 2. *The order of G is divisible by at least two different prime numbers.*

Since G involves at least one hamiltonian subgroup its order is divisible by 8. It contains an invariant subgroup H of prime index (p) as it is solvable. This subgroup is the direct product of its own Sylow subgroups since it is either hamiltonian or abelian, and the order of every operator of G which is not also in H is divisible by p. Let t represent one of these operators, so that t^p is in H but t is not in H. Each of the Sylow subgroups of H is transformed into itself by t.

When $p = 2$, H involves at least two different Sylow subgroups and t is commutative with all the operators of odd order contained in H. The group generated by t and the Sylow subgroup of order 2^m contained in H is hamiltonian since it is a Sylow subgroup of G. As all of its operators are commutative with every operator of odd order in H, *G is the direct product of its Sylow subgroups whenever it contains an invariant subgroup of half its order.* The Sylow subgroup of order 2^{m+1} is hamiltonian while all the others are abelian. Moreover, any such direct product is a group of the required kind. It remains to consider the cases when p is odd.

When $p > 2$ the Sylow subgroup of order 2^m contained in H is hamiltonian. When H involves more than one Sylow subgroup, t is commutative with every operator of H since it is commutative with all the operators in its Sylow subgroups. Hence *G is the direct product of its Sylow subgroups whenever the order of H is divisible by at least two different prime numbers.* It remains only to consider the case when H is the hamiltonian group of 2^m and $p > 2$.

Since H contains only three abelian subgroups of order 2^{m-1}, t would have to transform each of these into itself if $p > 3$. As the group generated by t and such a subgroup would be abelian, t would again be commutative with every operator of H and we have proved that if the order of G is not of the form $2^m.3$ G is always the direct product of an abelian group of odd order and a hamiltonian group of order 2^m. We shall therefore assume that the order of G is $2^m.3$ and consider the possible value of m.

The order of the subgroup generated by three operators of order 4 in H cannot exceed 16. Hence if we assume that t is not commutative with every operator of order 4 in H it transforms a subgroup of H whose order cannot exceed 16 into itself. Since the group generated by t and this subgroup is neither abelian nor hamiltonian the order of G cannot exceed 48 unless it is the direct product of its Sylow subgroups. We may assume that t does not transform each of the three abelian subgroups of order 2^{m-1} into itself, otherwise it would be commutative with each operator of H, and hence the group which the three conjugate operators of order 4 generate is the hamiltonian group of order 16 or the quaternion group. As the former contains exactly four quaternion

subgroups t and this quaternion subgroup generate the group of order 24 which has no subgroup of order 12.

The main results of this section may be stated as follows: If a group contains at least one hamiltonian subgroup and if all its subgroups are either abelian or hamiltonian it is the direct product of the hamiltonian group of order 2^m and an abelian group of odd order, unless it is the group of order 24 which does not contain a subgroup of order 12. Hence *there are only two non-hamiltonian groups which contain at least one hamiltonian subgroup and whose other subgroups are either abelian or hamiltonian.* The orders of these well-known groups are 16 and 24 respectively. While there is an infinite number of non-abelian groups in which every subgroup is abelian, there are only two other non-hamiltonian groups in which every non-abelian subgroup is hamiltonian.

ON MODULAR GROUPS ISOMORPHIC WITH A GIVEN LINEAR GROUP*

BY

H. F. BLICHFELDT

THEOREM. *Given a group G of linear homogeneous substitutions in n variables, transitive (irreducible) and of finite order. Then there exists an infinitude of prime numbers p for each of which we can construct a simply isomorphic transitive group G' of linear homogeneous substitutions in n variables, the elements of whose matrices are integers taken modulo p.*

Let the operators of the abstract group G'' simply isomorphic with G be S''_i, $i = 1, 2, \cdots, N$. Write down N matrices in n variables with undetermined coefficients

$$S'_i = \| a^i_{jk} \|,$$

and form the N^2 products $S'_i S'_j$. Writing $S'_i S'_j = S'_k$ whenever $S''_i S''_j = S''_k$, we obtain $n^2 N^2$ equations in the elements a^i_{jk}. This system of equations shall be denoted by A. Any system of elements a^i_{jk} satisfying A will furnish a linear group G_1 isomorphic with G''. That this group may be transitive in n variables we must, furthermore, have no equation of the form †

$$\sum_{j,k} b_{jk} a^i_{jk} = 0 \qquad (i = 1, 2, \cdots, N), \tag{1}$$

the coefficients b_{jk} being independent of i. In other words, zero cannot be the value of every determinant of $(n^2)^2$ elements of the matrix of n^2 columns and N rows, the ith row of which is formed of the n^2 elements a^i_{jk}. We shall denote by B' the system of equations obtained by equating to zero all the determinants mentioned. Furthermore, in order that G_1 may not contain two transformations that are identical, we must exclude all possible sets of solutions of A for which two rows of the matrix of $n^2 N$ elements just mentioned are identical. This condition expressed in analytical form is as follows: the expression

* Presented to the Society (Chicago), April 14, 1906. Received for publication February 13, 1906.

† BURNSIDE, Proceedings of the London Mathematical Society, vol. 3 (1905), p. 433; FROBENIUS, Sitzungsberichte der K. Preussischen Ak. der Wissenschaften, 1897, p. 1004.

$$P \equiv \prod_{i,i'} \left\{ \sum_{j,k} \lambda_{jk} (a^i_{jk} - a^{i'}_{jk}) \right\} \quad \begin{pmatrix} j, k = 1, 2, \cdots, n^2; \\ i, i' = 1, 2, \cdots, N; \; i \neq i' \end{pmatrix}$$

must not vanish for a set of n^2 arbitrary parameters λ_{jk}. We shall modify the system B' by multiplying each of its equations by P, and we shall denote the resulting set of equations by B.

Now, because the transitive group G exists, the system A can be solved, and solutions exist which will not satisfy all the equations of B. To solve A we may, by a well known process, form a normal equation of the system, an algebraic equation whose coefficients are integers and which has no double roots. Let this equation be

$$F = a_e x^e + a_{e-1} x^{e-1} + \cdots + a_0 = 0. \tag{2}$$

Denoting by x any one of the roots of this equation, we can write every corresponding value of a^i_{jk} as an integral function of x, the coefficients of which are definitely given rational numbers (the same for any root x of (2) considered). Substituting in the system B we have a series of equations in x with rational coefficients, known functions of the parameters λ_{jk}, which equations could not all be satisfied for every root x of (2). Hence $F = 0$ has at least one root not found in one (say $C = 0$) of the equations B. Let us suppose $F \equiv F_1 F_2$, where $F_1 = 0$ has no root in common with $C = 0$.* Then we can construct an identity of the form

$$\alpha F_1 + \beta C \equiv K_1 \not\equiv 0,$$

where α, β and βC are integral functions of x whose coefficients, as well as K_1, are integral functions of the parameters λ_{jk} with integral coefficients. To every root x of $F_1 = 0$ will correspond a transitive group G_1 simply isomorphic with G''.

The question whether or not there exists a transitive linear group in n variables simply isomorphic with G'' with coefficients modulo p can now be solved. We start as above with the N matrices

$$S'_i = \| a^i_{jk} \|$$

and write down all the congruences (mod p) following from the equations $S'_i S'_j = S'_k$. The system A above will merely be replaced by congruences, and instead of $F = F_1 F_2 = 0$ we will have $F = F_1 F_2 \equiv 0$ (mod p). We remark that the coefficients of F, F_1 and F_2 are all known integers, although p is, as yet, not known. The elements a^i_{jk} are, as above, expressed as integral functions of a root x of $F_1 \equiv 0$ (mod p), the coefficients of which functions are known fractions. Let the least common multiple of all the denominators entering in these functions be denoted by M. We shall replace the parameters λ_{jk} by such a system of integers that K_1 does not vanish. The resulting value of K_1 (an integer) will be denoted by K.

* We seek the highest common factor of F and C, etc. The coefficients of F_1 and F_2 will be supposed to be integers.

Suppose that $F_1 = b_m x^m + \cdots + b_0$. We may assume that $b_0 \neq 0$, as we may replace x by $x + h$. Let us substitute for x in $F_1 \equiv 0 \pmod{p}$ the quantity $MKb_0 y$. We obtain

$$b_0 \{ MK(c_m y^m + \cdots + c_1 y) + 1 \} \equiv 0 \pmod{p},$$

the coefficients of the left-hand member being known integers evidently not all zero.

If we substitute any integer y' for y such that

$$MK(c_m y'^m + \cdots + c_1 y') + 1 = L \neq 1 \text{ or } 0,$$

and choose for p any prime factor > 1 of L, we have a modulus p fulfilling the conditions of the problem. For, p is prime to MK, and $F_1 \equiv 0 \pmod{p}$ has a solution $x = MKb_0 y'$. Accordingly, the system of congruences A is satisfied, but not the system B (by virtue of the identity $\alpha F_1 + \beta C \equiv K$). Because A is satisfied, we have a modular group H isomorphic with G''. If this group is intransitive modulo p, it may be transformed into a group of type

$$\left.\begin{array}{c|c} H_1 & 0 \\ \hline 0 & H_2 \end{array}\right.,$$

from which it follows that the elements of H satisfy at least one system of congruences corresponding to (1), from which again would follow the system B, and therefore also $C \equiv 0 \pmod{p}$. Again, if H were not simply isomorphic with G, the factor P would vanish $(\bmod\, p)$, and therefore also every equation of B. But this is not the case, according to our procedure.

BERLIN,
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DENUMERANTS OF DOUBLE DIFFERENTIANTS*

BY

WILLIAM EDWARD STORY

I. *Introduction.*

1.† In 1856 ‡ CAYLEY gave (without rigorous proof) a formula for the number of linearly independent yx-differentiants (as I call them) of a given type belonging to a single binary quantic,—a formula that he derived from the *assumption* that the necessary and sufficient conditions, as they naturally appear, are linearly independent; this formula was first proved in 1877 § by SYLVESTER, who extended it to a system of any number of binary quantics. The formula as originally given by CAYLEY expresses the number in question as the excess of the number of terms in the general form of the type of the differentiants over the number of terms in the general form of a certain other type, precisely as in formula (37) of this paper (when x and y are the only variables); but the number of terms in a general *binary* form is readily expressible as the sum of the numbers of partitions of a certain kind of certain numbers, and it was the formula in terms of such partition-numbers that SYLVESTER proved and extended. About 1885 it occurred to me, while giving a course of lectures on invariants at the Johns Hopkins University, that CAYLEY's original formula and SYLVESTER's proof of it,—indeed, the latter's extension of it, when expressed in the original terms,—are valid for yx-differentiants (*simple* differentiants, as I call them below), even if x and y are only two out of *any number* of variables. In 1892 ‖ I discovered operators that produce all the differentiants of a given type and certain combinations of characters, namely, all forms that are at once yx-, zx-, sx-, $\cdots$, vx-differentiants and all forms that are at once xy-, xz-, xs-, $\cdots$, xv-differentiants. Suspecting that the method by which these operators were obtained could be utilized for the discovery and proof of a formula for the number of linearly independent forms of any given type that possess any given combination of differentiant characters, I have devoted much time during the

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† I shall refer to these numbered *sections* as § 1, etc. For Table of Contents see p. 70.

‡ *A Second Memoir upon Quantics*, Philosophical Transactions, London, vol. 146, p. 107.

§ Philosophical Magazine, ser. 5, vol. 5, March, 1878.

‖ Mathematische Annalen, vol. 41 (1893), pp. 485–489; Proceedings of the London Mathematical Society, vol. 23 (1892), pp. 265-272.

last fourteen years to attempts to obtain such formulæ. I have succeeded in the cases of *double* differentiants of all kinds, as the present paper will show; the extension of the formulæ to triple and higher differentiants is almost self-evident, but the proof of the extended formulæ presents a difficulty that I have not yet overcome.

When I began these attempts I was not aware of what DERUYTS had done in a similar line;* but his work differs from mine in essential respects, both as to methods and results. He has, to be sure, given a formula for the denumerant of the semi-invariants (chain-differentiants of order 0 — that is, not involving the variables explicitly) which is *formally* identical with the formula (51) of this paper for three variables, but my formula is proved for xyz-differentiants of *any* type, even if this type implies any number of variables other than x, y and z, and the explicit presence of all these variables in the differentiant, while his formula supposes that the variables do not enter explicitly and that x, y, and z are the only variables in the system. Moreover, I have obtained formulæ for other combinations of differentiant characters not considered by DERUYTS. But the most important differences in our work will be found in the methods used. DERUYTS has employed the methods of "sources" and of "symbolic representation," while my main object has been to obtain formulæ by direct consideration of the "actual" forms, without recourse to those (from my point of view) indirect methods. At all events, it must be for the advantage of mathematical science that various methods should be exploited.

2. We assume a system of quantics (homogeneous polynomials), which we shall call simply *the quantics*, finite in number and of finite orders m, m', m'', $\cdots$, respectively, in a finite number n of variables $x, y, z, \cdots$. Any such quantic of order m is assumed in the form

$$(1) \qquad \sum \frac{m!}{g!\ h!\ i! \cdots} a_{g,h,i\ldots} x^g y^h z^i \cdots,$$

where the summation extends to all integral values (positive and 0) of $g, h, i, \cdots$, *whose sum is* m. For convenience, we speak of $a_{g,h,i,\ldots}$ freed from the corresponding polynomial coefficient as a *coefficient* of the quantic. If it were necessary to represent the coefficients of the different quantics, we should denote those of the first quantic by a, those of the second by b, those of the third by c, etc., each with its proper suffixes;† the coefficients of the quantics will be used as so many sets of additional variables (though not so called) having no determined numerical values. To each coefficient is assigned a *weight* in each var-

* *Essai d'une théorie générale des formes algébriques*, Mémoires de la société royale des sciences de Liège, ser. 2, vol. 17 (1891).

† Of course the sum of the suffixes of any coefficient is equal to the order of the quantic to which that coefficient belongs.

iable, namely, the *weight of any coefficient in either variable* is the sum of all its suffixes excepting that which denotes the power of the variable in question in the term of the quantic to which the coefficient belongs; thus, in (1) the weight of $a_{g,h,i,\dots}$ in z is $m-i$. The *weight of any variable* in itself is 1 and its weight in any other variable is 0. The *weight of any product* of variables and coefficients in any variable is the sum of the weights of its factors in that variable; thus the weight of any power of a variable or coefficient is the weight of that variable or coefficient multiplied by the exponent of the power.

The *order* of any product of powers of variables and coefficients is the sum of the exponents of the powers of the variables and its *degree* in the coefficients of either quantic is the sum of the exponents of the powers of such coefficients in it. If the terms of a polynomial in the variables and coefficients of the quantics are all of one order and of one degree in the coefficients of *each* quantic of the system (even if of different degrees in the coefficients of the different quantics), the polynomial is *homogeneous* of that order and those degrees. If the terms of a polynomial are all of one weight in each variable (even if of different weights in the different variables), the polynomial is *isobaric* of those weights in the several variables. *Any polynomial is a sum of homogeneous isobaric polynomials.* The order, degrees in the coefficients of the several quantics, and weights in the several variables, of a homogeneous isobaric polynomial, or *form*, together constitute the *type* of that form. One consequence (and advantage) of the definition of *weight* here given (which has not been adopted by all writers) is that each of the quantics is isobaric in each of the variables of a weight equal to its order. The numbers that characterize the type of any form are connected by a simple relation; namely, if the quantics are of orders $m, m', m'', \cdots$, respectively, in n variables, $x, y, z, \cdots$, and any form of degrees $j, j', j'', \cdots$, in the coefficients of these quantics, respectively, and of order ϑ in the variables, is of weights $w_x, w_y, w_z, \cdots$ in these variables, respectively, then

$$w_x + w_y + w_z + \cdots = (n-1)\sum mj + \vartheta, \tag{2}$$

where $\sum mj = mj + m'j' + m''j'' + \cdots$; so that the weight in either variable is completely determined when the other characteristics of its type are given.

If a linear transformation is imposed upon the variables, that is, if the n variables $x, y, z, \cdots$ are expressed as homogeneous linear functions of n new variables $x', y', z', \cdots$, and these expressions are substituted for $x, y, z, \cdots$ in any one of the quantics, this becomes a quantic of the same order in $x', y', z', \cdots$, whose coefficients are homogeneous linear functions of the original coefficients of the quantic and homogeneous functions of the parameters of transformation of a degree equal to the order of the quantic: the coefficients of the new quantic in $x', y', z', \cdots$ are the *transformed coefficients* of the quantic in question. It is assumed that the determinant of the parameters of transformation is not 0, so

that the new variables $x', y', z', \cdots$ can be expressed as homogeneous linear functions of the old variables $x, y, z, \cdots$. Any polynomial in the variables and the coefficients of the quantics is transformed by replacing the variables by the expressions of the corresponding new variables in terms of the old ones and the coefficients of the quantics by the expressions of the corresponding transformed coefficients in terms of the original coefficients. A homogeneous polynomial is thus transformed into a homogeneous polynomial of the same order and the same degrees; but the weights are not generally preserved by an arbitrary linear transformation,—indeed an isobaric form does not remain isobaric in general.

The linear transformation by which the variable x is increased by a multiple of the variable y, that is, the transformation $x = x' + \lambda y'$, $y = y'$, $z = z'$, $\cdots$, shall be called a *yx-shear;* and similarly for other pairs of variables. A shear involves a single parameter (λ, in the example just given), which we shall generally suppose to have an arbitrary value. A polynomial in the variables and the coefficients of the quantics that is not altered by an *arbitrary* yx-shear (that is, by a yx-shear whose parameter is arbitrary) shall be called a *yx-differentiant*, and similarly for other pairs of variables. A polynomial subject to only one such condition is a *simple* differentiant; if subject to more than one such condition, it is a *multiple* differentiant, namely, double, triple, etc., according to the number of *independent* conditions of this kind to which it is subject (we shall see that certain combinations of simple differentiant conditions imply others). The aggregate of differentiant conditions to which a polynomial is subject constitute its *character*. The object of this paper is the determination of the number of linearly independent *double* differentiants of a given type, more specifically, the number of xy- and xz-differentiants, of xz- and yz-differentiants, of xy- and yz-differentiants (which are, as we shall see, also xz-differentiants), and of xy- and zs-differentiants. There are as many kinds of simple shears and, therefore, of simple differentiants and simple characters as there are pairs of variables, if we have regard to the sequence of variables in the pairs (so that xy and yx are regarded as different pairs), that is $n(n-1)$ kinds.

A very important class of multiple differentiants is that defined by a set of simple characters such that the second letter of each defining pair of variables is the same as the first letter of the next pair, when the pairs are taken in the proper sequence; such differentiants may be called *chain-differentiants* (SYLVESTER called them "seminvariants"). Any chain-differentiant character is completely determined by the letters in it and their sequence in the chain, so that it may be unambiguously designated by this sequence; thus an xy-, yz-, zs-, $\cdots$, tu-differentiant may be called simply an $xyzs \cdots tu$-differentiant. Any simple differentiant character may be regarded as a chain-differentiant character of two letters. Of course, the number of letters in any given differentiant character

has no particular relation to the whole number of variables in the quantics and differentiants in question.

The name "differentiant" is derived from the fact that a yx-differentiant not explicitly involving the variables is a function of the *differences* of the roots of the equations in $x:y$ formed by equating the quantics to 0, if x and y are the only variables ($n = 2$).

3. The result of imposing the yx-shear $x = x' + \lambda y'$, $y = y'$, $z = z'$, $\cdots$ on any polynomial ϕ is

$$\phi + \lambda \cdot \widehat{yx} \cdot \phi + \frac{\lambda^2}{2!} \cdot \widehat{yx}^2 \cdot \phi + \frac{\lambda^3}{3!} \cdot \widehat{yx}^3 \cdot \phi + \cdots, \tag{3}$$

the operator $\widehat{yx}$ being defined by *

$$\widehat{yx} \cdot \phi \equiv \sum\sum h \cdot a_{g+1,\, h-1,\, i,\, \ldots} \cdot \frac{\partial \phi}{\partial a_{g,\, h,\, i,\, \ldots}} - y \cdot \frac{\partial \phi}{\partial x}, \tag{4}$$

where the double summation extends to the coefficients $a_{g,\, h,\, i,\, \ldots}$, $b_{g,\, h,\, i,\, \ldots}$, $c_{g,\, h,\, i,\, \ldots}$, $\cdots$ of all the quantics and to all values of the suffixes of the coefficients of each quantic whose sum is the order of that quantic. From (3) it is evident that the necessary and sufficient condition that ϕ shall be a yx-differentiant is the identity

$$\widehat{yx} \cdot \phi \equiv 0\,; \tag{5}$$

that is, the condition is that the multipliers of the different products of powers of the variables and of the coefficients of the quantics that occur in $\widehat{yx} \cdot \phi$ (these multipliers are homogeneous linear functions of the multipliers of such products in ϕ) shall all be 0, a condition that gives as many homogeneous linear equations between the multipliers of the terms of ϕ as there are terms of $\widehat{yx} \cdot \phi$, but it does not yet appear that these equations are linearly independent. From (4) it appears that $\widehat{yx}$ leaves unchanged the order and degrees of any *form*, as well as its weights in all the variables excepting x and y, diminishes the weight in x by 1, and increases the weight in y by 1, so that (5) must be satisfied by each of the homogeneous isobaric parts of ϕ of different types if it is satisfied by ϕ as a whole; that is, every yx-differentiant is a sum of homogeneous isobaric yx-differentiants, — and the same is true of differentiants of any simple or multiple character. It appears from (5) that a yx-differentiant is a polynomial that is annihilated by the *shear-operator* $\widehat{yx}$.

Being concerned with linearly independent differentiants alone, we shall henceforth assume that every polynomial with which we have to do is homogeneous and isobaric, that is, is a *form*. Moreover, the forms to be considered at any one time will all be obtained by applying operators analogous to $\widehat{yx}$ to forms of one and the same given type; the orders and degrees of all forms thus obtained

* The sign $\equiv$ is used throughout this paper to denote *identity*.

are the same, as are also the weights in the several variables excepting those variables that occur in the shear-operators used in obtaining the forms, and the sum of the weights in these excepted variables is the same for all the forms. In giving the type of a form it will, then, suffice to give its weights in those variables in which the forms to be considered are not all of the same weight; thus, if the forms considered are all of a given order, of given degrees, and of given weights in all the variables excepting x, y, and z, — so that the weights in these three variables are the only characteristics of the type that vary from form to form, — the type of any particular form may be represented by (w_x, w_y, w_z), where w_x, w_y, w_z are its weights in x, y, z, respectively; and similarly for any number of variables. It is to be observed that the expression of the type of a form by its weights in certain variables does not imply anything with regard to the whole number of variables, but only that the weights in all the variables that do not enter into this expression of the type are the same for all the forms considered. To avoid confusion, we suppose the variables to be arranged in a certain sequence $x, y, z, \cdots$ and we write the weights in any expression of a type in the corresponding sequence.

4. By a *complete* system of forms of given type or types and given differentiant characters, subject to other conditions or not, we mean a system of such *linearly independent* forms satisfying the given conditions that *any* form of the given type or types satisfying the conditions can be expresed as a linear function of the forms of the system. The forms of any complete system of *one* type can, evidently, be replaced by the same number of linearly independent linear functions of them, chosen arbitrarily, without affecting the completeness of the system or the character of its forms (because the shear-operators are distributive over their operands). By the *rank* of any form relatively to any given shear-operator we mean the exponent of the highest power of that operator that does not annihilate the form; thus, ϕ is of rank r quâ* $\widehat{xy}$ if $\widehat{xy}^r \cdot \phi \not\equiv 0$ but $\widehat{xy}^{r+1} \cdot \phi \equiv 0$. A complete system of forms of a given type and given characters shall be called *reduced* quâ $\widehat{xy}$ if its forms are so taken as to include the greatest possible number of forms of rank not greater than r quâ $\widehat{xy}$ for *each* value of r from 0 to the highest rank of any form † of the system, inclusive. A complete system of forms of a given type and given characters reduced quâ $\widehat{xy}$ can be constructed thus: out of the (N) forms ψ of *any* complete system of the given type and characters construct the greatest possible number (N_0) of linearly independent linear functions ϕ_0 of rank 0 quâ $\widehat{xy}$ and select as many $(N_0' = N - N_0)$ other forms ψ_0 of the system that are linearly independent of

* *Quâ* signifies relatively to, with respect to, according to.

† We employ this abbreviated expression instead of the more cumbrous "rank of the forms that are of highest rank."

each other and the ϕ_0's as may be necessary to constitute a complete system with the ϕ_0's; out of these (N'_0) other forms ψ_0 construct the greatest possible number (N_1) of linearly independent linear functions ϕ_1 of rank 1 quâ $\widehat{xy}$ and select as many $(N'_1 = N'_0 - N_1)$ other forms ψ_1 from among the ψ_0's that are linearly independent of each other and the ϕ_1's as may be necessary to constitute a complete system with the ϕ_0's and ϕ_1's; out of the (N'_1) forms ψ_1 construct the greatest possible number (N_2) of linearly independent linear functions ϕ_2 of rank 2 quâ $\widehat{xy}$ and select as many $(N'_2 = N'_1 - N_2)$ other forms ψ_2 from among the ψ_1's that are linearly independent of each other and the ϕ_2's as may be necessary to constitute a complete system with the ϕ_0's, ϕ_1's and ϕ_2's; and so proceed until the forms ψ_{k-1} last selected are all expressible as linear functions of the forms ϕ_k last constructed. The forms $\phi_0, \phi_1, \phi_2, \cdots, \phi_k$ thus constructed constituted a complete system of forms of the given type and given characters reduced quâ $\widehat{xy}$. Of course, the forms ψ_{k-1} are forms ϕ_k.

The peculiarity of a system *reduced* quâ $\widehat{xy}$ is that any linear function of the forms of such a system is of exactly the rank quâ $\widehat{xy}$ of the forms of highest rank in it. For no linear function of forms of a system reduced quâ $\widehat{xy}$ that are all of the rank r can be annihilated by $\widehat{xy}^r$ (otherwise, the system would not contain the greatest possible number of forms of rank less than r, that is, would not be reduced), but every linear function whose terms are all of the rank r is annihilated by $\widehat{xy}^{r+1}$; so that the result of applying $\widehat{xy}^r$ to a linear function of forms of the reduced system of which some are of rank r and others of rank less than r is the same as the result of applying $\widehat{xy}^r$ to the aggregate of those terms of the linear function whose rank is r alone, and this result is not identically 0.

If ϕ is a form of rank r quâ $\widehat{xy}$, then $\widehat{xy}^r \cdot \phi \not\equiv 0$ but $\widehat{xy}^{r+1} \cdot \phi \equiv 0$, that is, $\widehat{xy}^r \cdot \phi$ is an xy-differentiant (of rank 0 quâ $\widehat{xy}$). The form derived from any form ϕ by operating upon it with the power of $\widehat{xy}$ whose exponent is the rank quâ $\widehat{xy}$ of the form ϕ in question shall be called the *derivative* quâ $\widehat{xy}$ of that form ϕ; the derivative quâ $\widehat{xy}$ of any form is, then, an xy-differentiant, and the derivatives quâ $\widehat{xy}$ of the forms of a system of *one* type *reduced* quâ $\widehat{xy}$ are *linearly independent* xy-differentiants (namely, the derivatives of forms of the same type that are of different ranks are themselves of different types, so that a linear relation between the forms of a reduced system implies a linear relation between the derivatives of forms that are of one rank, and this again implies a linear function of these forms that is of lower rank than they, which is impossible, as above shown).

5. In the present investigation we are concerned chiefly with what we may call *denumerants*, that is, expressions for the numbers of linearly independent forms of given types that satisfy certain conditions. A denumerant shall be

represented by N followed by the type with the conditions attached to the N as suffixes and exponents. Conditions shall be expressed thus:

that the forms in question have certain differentiant characters shall be indicated by the pairs and sequences of variables that define these characters, attached to N as suffixes (thus, the suffix xy shall indicate that the forms are xy-differentiants);

that the forms are of given ranks relatively to certain shear-operators in a system *reduced* relatively to each of these operators shall be indicated by the highest powers of the respective operators that do not annihilate the forms, attached to N as exponents (thus, the exponent $\widehat{xy}^r$ indicates that the forms are of rank r quâ $\widehat{xy}$ in a system *reduced* quâ $\widehat{xy}$);

other conditions will generally be attached to N as suffixes;

several suffixes or exponents attached to the same N shall be separated by commas; and

the absence of suffix or exponent shall indicate that no condition such as would generally be written in that place is imposed.

Thus,

$N(w_x, w_y, \cdots)$ is the number of linearly independent forms of the type $(w_x, w_y, \cdots)$, which is also the number of terms in the *general* form of this type;

$N_{xy}(w_x, w_y, \cdots)$ is the number of linearly independent xy-differentiants of the type $(w_x, w_y, \cdots)$;

$N_{xy,\,xz}(w_x, w_y, w_z, \cdots)$ is the number of linearly independent xy- and xz-differentiants of the type $(w_x, w_y, w_z, \cdots)$;

$N^{\widehat{xy}^r}(w_x, w_y, \cdots)$ is the number of forms of a complete system of type $(w_x, w_y, \cdots)$ *reduced* quâ $\widehat{xy}$ that are of rank r quâ $\widehat{xy}$;

$N^{\widehat{yz}^r}_{xy,\,xz}(w_x, w_y, w_z, \cdots)$ is the number of xy- and xz-differentiants of rank r quâ $\widehat{yz}$ in a complete system of type $(w_x, w_y, w_z, \cdots)$ *reduced* quâ $\widehat{yz}$;

$N_{xyz}(w_x, w_y, w_z, \cdots)$ is the number of linearly independent xyz-differentiants of the type $(w_x, w_y, w_z, \cdots)$;

$N_K(w_x, w_y, \cdots)$ is the number of linearly independent forms of the type $(w_x, w_y, \cdots)$ that satisfy the conditions K (whatever they may be);

$N_{K,\,xy}(w_x, w_y, \cdots)$ is the number of linearly independent xy-differentiants of the type $(w_x, w_y, \cdots)$ that satify the further conditions K; etc. Evidently, the exponent $\widehat{xy}^0$ is equivalent to the suffix xy, so that

$N^{\widehat{yz}^0}_{xy,\,xz}(w_x, w_y, w_z) = N_{xyz}(w_x, w_y, w_z)$.

The conditions to which the forms under consideration are subject are generally that the forms shall have certain differentiant characters and that certain numerical characteristics (rank, etc.) of the forms shall have given values or values lying between given limits; it will be convenient to speak of the former as *differentiant* conditions and of the latter as *numerical* conditions; there can even be no objection to speaking of the differentiant characters assumed and the

values or limitations of value of the numerical characteristics as themselves *conditions*, because they determine the conditions.

If ϕ is a form of type $(w_x, w_y, \cdots)$, $\widehat{xy}\cdot\phi$ is a form of type $(w_x+1, w_y-1, \cdots)$, — provided it is not identically 0; accordingly, using the symbol $\overline{xy}$ to represent the operator that changes the *type* of any form ϕ into the *type* of $\widehat{xy}\cdot\phi$, we have (as definition of the symbol $\overline{xy}$).

$$\overline{xy}\cdot(w_x, w_y, \cdots) = (w_x+1, w_y-1, \cdots),$$

so that

$$\overline{xy}^k\cdot(w_x, w_y, \cdots) = (w_x+k, w_y-k, \cdots).$$

The types obtained from any given type by applying powers of $\overline{xy}$ whose exponents are positive integers or 0 shall be called types *subordinate* to the given type quâ $\widehat{xy}$; thus, the types $(w_x+k, w_y-k, \cdots)$ for all positive integral values of k and 0 are types *subordinate* to $(w_x, w_y, \cdots)$. Of course the same notation and nomenclature applies to the *type-operators* $\overline{xy}$, etc., that correspond to *all* shear-operators $\widehat{xy}$, etc.

Any denumerant may be regarded as a *function of the type*, and we shall apply the type-operators $\overline{xy}$, etc., to any denumerant as to a function of the type, observing that they affect only the *type*, but neither the *suffixes* nor the *exponents* of N; thus,

$$\overline{xy}^k\cdot N_{xz}(w_x, w_y, w_z) = N_{xz}(w_x+k, w_y-k, w_z),$$

$$(1-\overline{xy})(1-\overline{xz})\cdot N(w_x, w_y, w_z) = N(w_x, w_y, w_z) - N(w_x+1, w_y-1, w_z)$$
$$- N(w_x+1, w_y, w_z-1) + N(w_x+2, w_y-1, w_z-1), \text{ etc.}$$

It is to be observed that the operators $\overline{xy}$, etc., like the operators $\widehat{xy}$, etc., are distributive over their operands, but, unlike the latter, the former operators are *commutative*, so that, in their effects on any function of a type,

$$\overline{xy}\cdot\overline{yx} = 1, \qquad \overline{xy}\cdot\overline{yz} = \overline{yz}\cdot\overline{xy} = \overline{xz}, \text{ etc.}$$

As $\widehat{xy}^k\cdot\phi \equiv 0$ if k is greater than the weight of ϕ in y, we must take $\overline{xy}^k\cdot N(w_x, w_y, \cdots) = 0$ if $w_y < k$, whatever suffixes and exponents N may have; that is, the result of applying any product of powers of type-operators to a denumerant must be taken to be 0 if it makes the weight in any variable negative.* But it may be that some less value of k than w_y+1 will make $\overline{xy}^k\cdot N(w_x, w_y, \cdots) = 0$, because, with the notation of § 2, the weight in y of any form can be 0 only if the weights in each of the other variables is as least as great as $\sum mj$ for the form. The result of applying a product of powers of type-operators to a denumerant may be 0 also because there are no forms of the

* Any form of weight 0 in either variable has, evidently, every differentiant character that is defined by a pair in which the variable in question occupies the *second* place.

resultant type that satisfy the conditions imposed by the suffixes and exponents of N in the case in question; thus,

$$\overline{yz}^k \cdot N_{xy,\,xz}(w_x, w_y, w_z) = N_{xy,\,xz}(w_x, w_y + k, w_z - k) = 0$$

if $w_x - w_y < k$, because there are no xy-differentiants whose weight in y is greater than that in x, as we shall see in § 8.

6. We shall make use of the notations

$$\alpha^{(i)} = \alpha(\alpha - 1)(\alpha - 2)\cdots(\alpha - i + 1), \qquad i^{(i)} = i! \tag{6}$$

for *any* value of α and any *positive integer* i; from this follows

$$\alpha^{(i)} = \alpha(\alpha - 1)^{(i-1)} = \frac{\alpha^{(i+1)}}{\alpha - i}, \tag{7}$$

by means of which the symbol $\alpha^{(i)}$ can be extended to negative integral values of i and 0; thus, still for a positive integer i,

$$\alpha^{(-i)} = \frac{1}{(\alpha+1)(\alpha+2)(\alpha+3)\cdots(\alpha+i)} = \frac{1}{(\alpha+i)^{(i)}}, \qquad \alpha^{(0)} = 1. \tag{8}$$

We shall write also, for any *integer* i (positive, negative or 0),

$$\frac{\alpha^{(i)}}{i!} = \binom{\alpha}{i}, \tag{9}$$

which is the coefficient of x^i in the development of $(1 + x)^\alpha$ according to positive or negative powers of x, according as i is positive or negative. In particular, if α is a *positive integer* or 0,

$$\binom{\alpha}{i} = 0 \text{ unless } 0 \leqq i \leqq \alpha; \tag{10}$$

but if α is a *negative integer*,

$$\binom{\alpha}{i} = 0 \text{ for } \alpha < i \leqq -1; \tag{11}$$

while

$$i! = \infty \text{ if } i \leqq -1. \tag{12}$$

It is evident from (7) and (8) that

$$(-\alpha)^{(i)} = (-1)^i(\alpha + i - 1)^{(i)}, \qquad \alpha^{(-i)} = (-1)^i \frac{1}{(-\alpha - 1)^{(i)}}. \tag{13}$$

If α and i are both *integers*,

$$\alpha^{(i)} = 0 \text{ if } 0 \leqq \alpha < i, \text{ and only then.} \tag{14}$$

We shall have occasion to use the formula of summation

$$(15)\qquad S_{p,q}^{m,n} = \sum_{i=0}^{n} (-1)^i \cdot \binom{n}{i} \cdot \frac{(p-i)^{(m+n-1)}}{(q+i)^{(m)}} = \frac{\binom{p-n}{m-1} \cdot (p+q-m+1)^{(n)}}{\binom{q+n}{m+n} \cdot (m+n)},$$

where m, n, p, q are given integers, of which n is not negative; from which follows

$$(16)\qquad S_{p,q}^{m,n} \neq 0 \text{ if } 1 \leqq m \leqq q \text{ and } m+n-1 \leqq p.$$

7. The results of applying the shear-operators defined by (4) for different pairs of letters successively are connected by these simple relations, where ϕ is any form of type $(w_x, w_y, w_z, w_s, \cdots)$:

$$(17)\qquad \begin{cases} \widehat{xy} \cdot \widehat{yx} \cdot \phi = \widehat{yx} \cdot \widehat{xy} \cdot \phi + (w_x - w_y) \cdot \phi, & \widehat{xy} \cdot \widehat{xz} \cdot \phi = \widehat{xz} \cdot \widehat{xy} \cdot \phi, \\ \widehat{xy} \cdot \widehat{zx} \cdot \phi = \widehat{zx} \cdot \widehat{xy} \cdot \phi + \widehat{zy} \cdot \phi, & \widehat{xy} \cdot \widehat{zy} \cdot \phi = \widehat{zy} \cdot \widehat{xy} \cdot \phi, \\ \widehat{xy} \cdot \widehat{yz} \cdot \phi = \widehat{yz} \cdot \widehat{xy} \cdot \phi - \widehat{xz} \cdot \phi, & \widehat{xy} \cdot \widehat{zs} \cdot \phi = \widehat{zs} \cdot \widehat{xy} \cdot \phi, \end{cases}$$

where x, y, z, s are any four different variables. The last three identities of (17) show that any two shear-symbols and their powers are commutative unless they have a common letter that occupies *different* places in them. Repetitions of the first three formulæ of (17) give

$$(18)\qquad \begin{aligned} \widehat{xy}^{\alpha} \cdot \widehat{yx}^{\beta} \cdot \phi &\equiv \sum_{i=0}^{\alpha,\beta} \frac{\alpha^{(i)}\beta^{(i)}}{i!} (w_x - w_y + \alpha - \beta)^{(i)} \cdot \widehat{yx}^{\beta-i} \widehat{xy}^{\alpha-i} \cdot \phi, \\ \widehat{xy}^{\alpha} \cdot \widehat{zx}^{\beta} \cdot \phi &\equiv \sum_{i=0}^{\alpha,\beta} \frac{\alpha^{(i)}\beta^{(i)}}{i!} \cdot \widehat{zx}^{\beta-i} \widehat{zy}^{i} \widehat{xy}^{\alpha-i} \cdot \phi, \\ \widehat{xy}^{\alpha} \cdot \widehat{yz}^{\beta} \cdot \phi &\equiv \sum_{i=0}^{\alpha,\beta} (-1)^i \frac{\alpha^{(i)}\beta^{(i)}}{i!} \cdot \widehat{yz}^{\beta-i} \widehat{xz}^{i} \widehat{xy}^{\alpha-i} \cdot \phi, \end{aligned}$$

where each sum extends to all integral values of i from 0 to the smaller of the two numbers α and β.

8. Any form ϕ of type (w_x, w_y) is annihilated by a certain lowest power of $\widehat{xy}$ and by a certain lowest power of $\widehat{yx}$; let α and β be the exponents of the highest powers of $\widehat{xy}$ and $\widehat{yx}$, respectively, that do not annihilate ϕ (the ranks of ϕ quâ $\widehat{xy}$ and $\widehat{yx}$, respectively), so that

$$\widehat{xy}^{\alpha} \cdot \phi \not\equiv 0, \qquad \widehat{xy}^{\alpha+1} \cdot \phi \equiv 0, \qquad \widehat{yx}^{\beta} \cdot \phi \not\equiv 0, \qquad \widehat{yx}^{\beta+1} \cdot \phi \equiv 0;$$

then, by the first formula of (18) as applied to $\widehat{xy}^{\alpha+\beta+1} \cdot \widehat{yx}^{\beta+1} \cdot \phi$ and $\widehat{yx}^{\alpha+\beta+1} \cdot \widehat{xy}^{\alpha+1} \cdot \phi$,

$$0 \equiv (\alpha+\beta+1)^{(\beta+1)} \cdot (w_x - w_y + \alpha)^{(\beta+1)} \cdot \widehat{xy}^{\alpha} \cdot \phi$$

and

$$0 \equiv (\alpha+\beta+1)^{(\alpha+1)} \cdot (w_y - w_x + \beta)^{(\alpha+1)} \cdot \widehat{yx}^{\beta} \cdot \phi\,;$$

therefore,

$$(w_x - w_y + \alpha)^{(\beta+1)} = 0 \qquad \text{and} \qquad (w_y - w_x + \beta)^{(\alpha+1)} = 0\,,$$

that is, by (14),

$$w_x - w_y + \alpha \leqq \beta \qquad \text{and} \qquad w_y - w_x + \beta \leqq \alpha\,,$$

so that

$$\beta = w_x - w_y + \alpha \qquad \text{or} \qquad \alpha = w_y - w_x + \beta\,. \tag{19}$$

If ϕ is an xy-differentiant, $\alpha = 0$ and, therefore, $\beta = w_x - w_y$, so that

$$w_y \leqq w_x \tag{20}$$

and

A. Every xy-differentiant is of at least as great a weight in x as in y.

If ϕ is an xy-differentiant and $w_x = w_y$, then $\beta = 0$, so that

B. Every xy-differentiant whose weights in x and y are equal is also a yx-differentiant.

From the third identity of (17) follows that $\widehat{xz} \cdot \phi \equiv 0$ if both $\widehat{xy} \cdot \phi \equiv 0$ and $\widehat{yz} \cdot \phi \equiv 0$; that is,

C. Every xy- and yz-differentiant is also an xz-differentiant (xyz-differentiant). From this theorem it appears that every chain-differentiant has every simple differentiant character that is defined by two letters of the corresponding chain, taken in the sequence in which they occur.

It is evident that

D. If two given shear-operators are commutative, the result of applying any power of the first to a form that is annihilated by the second is also a form annihilated by that second operator;

that is, the repeated application of one of two *commutative* shear-operators does not affect the simple differentiant character defined by the other. Similarly, from the second and third formulae of (18), together with theorem D, it appears that,

E. If ϕ is an xy- and xz-differentiant, so is also $\widehat{yz}^k \cdot \phi$: and, *if ϕ is an xz- and yz-differentiant, so is also $\widehat{xy}^k \cdot \phi$;*

provided, in each case, the result is not identically 0.

A polynomial that has *all* the simple differentiant characters that are defined by pairs of variables taken from a certain set (including both orders of each pair) may be called a *complete* differentiant relatively to that set of variables. In the paper in vol. 41 of the Annalen to which I have referred, I have proved that every *isobaric* complete differentiant in *all* the variables is a *covariant* (or invariant); but the proof shows also that an isobaric complete differentiant in *any set* of variables is simply multiplied by a power of the determinant

of transformation when the variables of *that set alone* are subjected to any linear transformation.

From (19) follows also that

F. If ϕ is a form of weights w_x and w_y in x and y, respectively, and of rank r quâ $\widehat{xy}$, then $w_y - w_x \leqq r$ (of consequence only if $w_x < w_y$).

Theorems B and C show that an isobaric chain-differentiant is a complete differentiant in the variables defining the chain provided it is of one and the same weight in those variables (which, by theorem A, requires only that it shall be of the same weight in the first and last variables of the chain).

9. In the papers cited in § 1* I have shown that the operators

$$(21)\quad \begin{aligned} &\{v;\ u, \cdots, z, y, x\} \\ &\equiv \sum(-1)^{\sigma} \cdot \frac{(w_x - w_v + \nu)!\ \widehat{vx}^{\alpha} \cdot \widehat{vy}^{\beta} \cdot \widehat{vz}^{\gamma} \cdots \widehat{vu}^{\epsilon} \cdot \widehat{uv}^{\epsilon} \cdots \widehat{zv}^{\gamma} \cdot \widehat{yv}^{\beta} \cdot \widehat{xv}^{\alpha}}{\alpha!\ \beta!\ \gamma!\ \cdots \epsilon!\ (\sigma + w_x - w_v + \nu)!} \end{aligned}$$

and

$$(22)\quad \begin{aligned} &\{x, y, z, \cdots, u;\ v\} \\ &\equiv \sum(-1)^{\sigma} \cdot \frac{(w_v - w_x + \nu)!\ \widehat{xv}^{\alpha} \cdot \widehat{yv}^{\beta} \cdot \widehat{zv}^{\gamma} \cdots \widehat{uv}^{\epsilon} \cdot \widehat{vu}^{\epsilon} \cdots \widehat{vz}^{\gamma} \cdot \widehat{vy}^{\beta} \cdot \widehat{vx}^{\alpha}}{\alpha!\ \beta!\ \gamma!\ \cdots \epsilon!\ (\sigma + w_v - w_x + \nu)!} \end{aligned}$$

where ν is the number of variables $x, y, z, \cdots, u$ (without v), the summations extend to all integral but not negative values of $\alpha, \beta, \gamma, \cdots, \epsilon$, and $\sigma = \alpha + \beta + \gamma + \cdots + \epsilon$ for *each* term, have, when applied to a form ϕ of weights $w_x, w_y, w_z, \cdots, w_u, w_v$ in $x, y, z, \cdots, u, v$, respectively (whatever the number of other variables and the weights of ϕ in them may be), these properties:

If ϕ is an xy-, xz-, $\cdots$, xu-differentiant, $\{v;\ u, \cdots, z, y, x\} \cdot \phi$ is an xy-, xz-, $\cdots$, xu-, xv-differentiant of the same type as ϕ (if it is not identically 0), *is the most general differentiant of that type and these characters if ϕ is the most general differentiant of its type and the characters stated, and is ϕ itself if that form is an xy-, xz-, $\cdots$, xu-, xv-differentiant.*

If ϕ is a yx-, zx-, $\cdots$, ux-differentiant, $\{x, y, z, \cdots, u;\ v\} \cdot \phi$ is a yx-, zx-, $\cdots$, ux-, vx-differentiant of the same type as ϕ (if it is not identically 0), *is the most general differentiant of that type and these characters if ϕ is the most general differentiant of its type and the characters stated, and is ϕ itself if that is a yx-, zx-, $\cdots$, ux-, vx-differentiant.*

The summation in (21) is naturally limited to values of $\alpha, \beta, \gamma, \cdots, \epsilon$ such that $\sigma \leqq w_v$, and that in (22) to values of $\alpha \leqq w_x$, $\beta \leqq w_y$, $\gamma \leqq w_z$, $\cdots$, $\epsilon \leqq w_u$.

In particular, if ϕ is any form of type (w_x, w_y),†

* E. g., Mathematische Annalen, vol. 41 (1893), pp. 485–489.

† If ϕ is a form involving only the *coefficients* of the quantics (that is, a form of order 0) for which $w_x = w_y$, this formula is equivalent to that for $[\phi]$ given by HILBERT, Mathematische Annalen, vol. 36 (1890), p. 523.

$$(23) \qquad \{y; x\} \cdot \phi \equiv \sum_{\alpha=0}^{w_y} (-1)^\alpha \cdot \frac{(w_x - w_y + 1)!}{\alpha!\,(w_x - w_y + \alpha + 1)!} \cdot \widehat{yx}^\alpha \cdot \widehat{xy}^\alpha \cdot \phi$$

is an xy-differentiant of the same type as ϕ (if it is not identically 0), *is the most general xy-differentiant of that type if* ϕ *is the most general form of the type, and is* ϕ *itself if that is an xy-differentiant.*

In applying (21), (22) and (23) to any particular form ϕ, it is to be observed that the values of $w_x, w_y, w_z, \cdots, w_u, w_v$ to be used are the weights of *that form.*

If ϕ is any form of type $(w_x, w_y, w_z, \cdots, w_u, w_v)$,

$$(24) \quad [\phi]' \equiv \{v; u, \cdots, z, y, x\} \cdot \{u; \cdots, z, y, x\} \cdots \{z; y, x\} \cdot \{y; x\} \cdot \phi$$

is an xy-, xz-, $\cdots$, xu, xv-differentiant of that type, and

$$(25) \quad [\phi]_1 \equiv \{x, y, z, \cdots, u; v\} \cdot \{x, y, z, \cdots; u\} \cdots \{x, y; z\} \cdot \{x; y\} \cdot \phi$$

is a yx-, zx-, $\cdots$, ux, vx-differentiant of that type.

II. *Determination of denumerants in general.*

Let xy be any one of the differentiant characters included in the conditions for the denumerant, let the aggregate of all the other conditions be represented by K, and let (a, b) be any given type (to specify the weights in x and y alone). If the shear-operator $\widehat{xy}$, or differentiant character xy, is so related (or unrelated) to the conditions K that the form $\widehat{xy}^k \cdot \phi$ satisfies the conditions K for every form ϕ that satisfies them and every value of k for which $\widehat{xy}^k \cdot \phi$ is not identically 0, we shall say that the operator $\widehat{xy}$, or the differentiant character xy, *does not interfere* with the conditions K.

10. If the differentiant character xy does not interfere with the conditions K, the forms $\widehat{xy}^r \cdot \phi$ derived from the forms ϕ of a complete system of type (a, b) satisfying the conditions K and *reduced* quâ $\widehat{xy}$ by operating on each such form ϕ with the power of $\widehat{xy}$ whose exponent is the rank of that form ϕ quâ $\widehat{xy}$ are *linearly independent* xy-differentiants of the types $(a + r, b - r)$ subordinate to the type (a, b) quâ $\widehat{xy}$, by § 4. Therefore, the number of linearly independent xy-differentiants of any type $(a + r, b - r)$ subordinate to (a, b) quâ $\widehat{xy}$ is at least as great as the number of forms of rank r in a complete system of type (a, b) satisfying the conditions K *reduced* quâ $\widehat{xy}$; that is, for $0 \leqq r \leqq r'$,

$$(26) \qquad N_K^{\widehat{xy}^r}(a, b) \leqq N_{K, xy}(a + r, b - r),$$

from which follows

$$(27) \qquad N_K(a, b) = \sum_{r=0}^{r'} N_K^{\widehat{xy}^r}(a, b) \leqq \sum_{r=0}^{r'} N_{K, xy}(a + r, b - r),$$

where r' is the greatest value of r for any possible type $(a+r, b-r)$ subordinate to (a, b) quâ $\widehat{xy}$.

11. If, for *each* value of r from 0 to r', the greatest value of r for any possible type subordinate to (a, b) quâ $\widehat{xy}$, there exists a *determinate** distributive operator ω_r such that the forms $\omega_r \psi_r$ obtained from a complete system of xy-differentiants ψ_r of *all* types $(a+r, b-r)$ subordinate to (a, b) quâ $\widehat{xy}$ that satisfy the conditions K are *linearly independent*† forms of type (a, b) that satisfy the conditions K, then the number of linearly independent forms of type (a, b) that satisfy the conditions K is at least as great as the number of linearly independent xy-differentiants of all types $(a+r, b-r)$ subordinate to (a, b) quâ $\widehat{xy}$ that satisfy the conditions K: that is,

$$(28) \qquad \sum_{r=0}^{r'} N_{K, xy}(a+r, b-r) \leqq \sum_{r=0}^{r'} N_K^{\widehat{xy}^r}(a, b) = N_K(a, b).$$

12. If the conditions of *both* § 10 and § 11 are satisfied, we have, by (27) and (28),

$$(29) \qquad \sum_{r=0}^{r'} N_{K, xy}(a+r, b-r) = \sum_{r=0}^{r'} N_K^{\widehat{xy}^r}(a, b) = N_K(a, b),$$

and, because (29) cannot be satisfied unless the sign of equality is to be taken in (26) for *every* value of r from 0 to r', inclusive,

$$(30) \qquad N_{K, xy}(a+r, b-r) = N_K^{\widehat{xy}^r}(a, b) \quad \text{for } 0 \leqq r \leqq r'.$$

In this case, for any *particular* value of r, the forms $\widehat{xy}^r \cdot \phi$ of § 10, being, as we have there seen, linearly independent xy-differentiants of type $(a+r, b-r)$ that satisfy the conditions K, and being, by (30), just equal in number to the number of such linearly independent xy-differentiants, in themselves constitute a complete system of linearly independent xy-differentiants of type $(a+r, b-r)$ that satisfy the conditions K: for any *particular* value of r, the forms ψ_r of § 11 are, then, linearly independent linear functions of the forms $\widehat{xy}^r \cdot \phi$ of § 10, and vice versâ, and the former forms may be replaced by the latter, whenever it may be convenient; that is, when we are considering simply a complete system of linearly independent xy-differentiants of a type $(a+r, b-r)$ subordinate to (a, b) quâ $\widehat{xy}$ satisfying the conditions K, we may take these xy-differentiants to be the forms $\widehat{xy}^r \cdot \phi$ derived from the forms ϕ of rank r in a complete system of type (a, b), to which $(a+r, b-r)$ is subordinate quâ $\widehat{xy}$,

* By a *determinate* operator ω_r I mean one that depends on the type (a, b), the conditions K, the differentiant character xy, and the value of r, alone, and not at all on the particular form ψ_r to which it is to be applied.

† And, therefore, *non-vanishing* forms: for a vanishing form may be regarded as a linear function of any forms, having all its coefficients 0.

satisfying the conditions K and *reduced* quâ $\widehat{xy}$. Again, in this same case, the forms $\omega_r \psi_r$ of § 11 for *all* values of r, being, as we have there seen, linearly independent forms of type (a, b) that satisfy the conditions K, and being, by (29), just equal in number to the number of such linearly independent forms, in themselves constitute a complete system of linearly independent forms of type (a, b) that satisfy the conditions K; the forms ϕ of § 10 are, then, linearly independent linear functions of the forms $\omega_r \psi_r$ of § 11 for *all* values of r, and vice versâ: whenever we are considering simply a complete system of linearly independent forms of type (a, b) that satisfy the conditions K, we may, then, take them to be the forms $\omega_r \psi_r$ obtained from complete systems of linearly independent xy-differentiants ψ_r of *all* types $(a + r, b - r)$ subordinate to (a, b) quâ $\widehat{xy}$ that satisfy the conditions K: but it is to be observed that we have *not* shown that the forms ψ_r, although they may be replaced by the same number of *arbitrary* linearly independent linear functions of them for each value of r, can be so taken that the forms $\omega_r \psi_r$ for *all* values of r shall constitute a system *reduced* quâ $\widehat{xy}$, — however, it would be so if the forms $\omega_r \psi_r$ were of rank r for *each* value of r, as a little consideration will show.

13. If the differentiant character xy does not interfere with the conditions K, so that (26) and (27) hold, and if, for each value of r from 0 to r' [the greatest value of r for which $(a + r, b - r)$ is a possible type subordinate to the *given* type (a, b) quâ $\widehat{xy}$], there exists a determinate distributive operator ω_r such that the form $\omega_r \psi_r$ is a form of type (a, b) that satisfies the conditions K for every xy-differentiant ψ_r of type $(a + r, b - r)$ that satisfies the same conditions, and also such that

$$\widehat{xy}^r \cdot \omega_r \psi_r \equiv S_r \cdot \psi_r, \tag{31}$$

where S_r is a *non-vanishing* constant, then the forms $\omega_r \psi_r$ obtained from complete systems of xy-differentiants of *all* types $(a + r, b - r)$ subordinate to (a, b) quâ $\widehat{xy}$ that satisfy the conditions K are *linearly independent.* For, by (31),

$$\widehat{xy}^k \cdot \omega_r \psi_r \equiv S_r \cdot \widehat{xy}^{k-r} \cdot \psi_r \equiv 0 \qquad \text{if} \qquad r < k,$$

so that any linear relation between forms $\omega_r \psi_r$ for which the greatest value of r is k would be turned by the operator $\widehat{xy}^k$ into a linear relation between the forms ψ_k with non-vanishing coefficients, whereas no such relation exists; therefore, such operators ω_r satisfy the conditions of § 11, so that their existence proves (28) and, because (26) and (27) hold, also (29) and (30); then, by § 12, the linearly independent xy-differentiants of any type $(a + r, b - r)$ subordinate to the given type (a, b) quâ $\widehat{xy}$ that satisfy the conditions K may be taken to be the forms $\widehat{xy}^r \cdot \phi$ derived from the forms ϕ of rank r in a complete system of type (a, b) satisfying the conditions K and *reduced* quâ $\widehat{xy}$.

In this case, for any *given* value of k from 0 to r', inclusive, the forms $\widehat{xy}^k \cdot \phi$ of type $(a+k, b-k)$ obtained from the forms ϕ *of rank as great as* k are *linearly independent*, for any linear relation between such of them as are obtained from forms ϕ of rank h and less quâ $\widehat{xy}$ $(k \leqq h \leqq r')$ would be turned by the operator $\widehat{xy}^{h-k}$ into a linear relation between the linearly independent forms $\widehat{xy}^h \cdot \phi$ derived from forms ϕ of rank h quâ $\widehat{xy}$. Again, the types $(a+r, b-r)$ subordinate to (a, b) quâ $\widehat{xy}$ for *all* values of r *as great as* k are also the types subordinate to $(a+k, b-k)$ quâ $\widehat{xy}$, and the xy-differentiants $\widehat{xy}^r \cdot \phi \equiv \widehat{xy}^{r-k} \cdot \widehat{xy}^k \cdot \phi$ derived from the forms ϕ that are of rank r quâ $\widehat{xy}$ are the xy-differentiants derived from the forms $\widehat{xy}^k \cdot \phi$ that are of rank $r-k$ quâ $\widehat{xy}$, so that the latter forms constitute a system *reduced* quâ $\widehat{xy}$. Finally, the operators $\widehat{xy}^k \cdot \omega_r$ for *all* values of r *as great as* k turn the linearly independent xy-differentiants $\widehat{xy}^r \cdot \phi$ into *linearly independent* forms of type $(a+k, b-k)$ and rank $r-k$ quâ $\widehat{xy}$ that satisfy the conditions K and for which, by (31),

$$\widehat{xy}^{r-k} \cdot \widehat{xy}^k \cdot \omega_r \cdot \widehat{xy}^r \cdot \phi \equiv \widehat{xy}^r \cdot \omega_r \cdot \widehat{xy}^r \cdot \phi \equiv S_r \cdot \widehat{xy}^r \cdot \phi,$$

which is what (31) becomes when the system of forms ϕ is replaced by the system of forms $\widehat{xy}^k \cdot \phi$, so that formulæ (29) and (30) hold also if the type (a, b) is replaced by the type $(a+k, b-k)$ and, of course, the limit r' of r by the limit $r'-k$ of $r-k$. The resultant formulæ are, evidently, equivalent to

$$(32) \qquad \sum_{r=k}^{r'} N_{K,\,xy}(a+r, b-r) = N_K(a+k, b-k)$$

and

$$(33) \qquad N_{K,\,xy}(a+r, b-r) = N_K^{\widehat{xy}^{r-k}}(a+k, b-k) \quad \text{for} \quad k \leqq r \leqq r'.$$

On subtracting (32) from (29), member from member, we get, for $1 \leqq k \leqq r'$,

$$(34) \qquad \sum_{r=0}^{k-1} N_{K,\,xy}(a+r, b-r) = N_K(a, b) - N_K(a+k, b-k) = (1-\overline{xy}^k) \cdot N_K(a, b),$$

and, in particular, for $k = 1$,

$$(35) \qquad N_{K,\,xy}(a, b) = N_K(a, b) - N_K(a+1, b-1) = (1-\overline{xy}) \cdot N_K(a, b),$$

by which (if the conditions for its validity are satisfied) the number of linearly independent forms of *given* type that satisfy the conditions K *and have the additional differentiant character* xy is determined when the numbers of linearly independent forms of *certain* types that satisfy the conditions K are known. In other words (35) is a formula for *the addition*, in a denumerant, *of a differentiant character* to other conditions with which it does not interfere.

14. Whatever differentiant and numerical conditions may be given, it seems easy to select one of the differentiant characters that shall not interfere with the

aggregate of the other conditions and, for each of the possible values of r, to find an operator ω_r that shall have all the properties stated in § 13, *excepting the last*, that is, that the constant S_r shall be *different from* 0 for each value of r. At least, that is the only difficulty that has presented itself to me in my attempts to determine the denumerants for triple and higher differentiants.

Evidently, the reasoning of §§ 11–13 requires that there shall be forms of type (a, b) that satisfy the conditions K *if there are forms of any type subordinate to* (a, b) *quâ* $\widehat{xy}$ *that satify them* and we shall find that, at least in the cases we consider, the operators ω_r that naturally suggest themselves do not satisfy (31) with a *non-vanishing* constant S_r for *all* the values of r in question unless $b \leqq a$, which, by theorem A of § 8, is a necessary, though not sufficient, condition that there shall be xy-differentiants of the type (a, b). In other words, we shall find it impossible to use (35) unless the type (a, b) is such that the condition of theorem A of § 8 is satisfied by the weights in *all* the pairs of variables that define the differentiant characters for the denumerant to be determined. But this is really no limitation to the use of (35), because the left member of this formula implies these relations between the weights, or else its value is 0.

15. The numerical conditions included in K may be of very different kinds, of which it would be impossible to state all that it might be useful to take into account, but we ought to mention one class that we shall have occasion to consider. Numerical conditions satisfied by the *derivatives* of a system of forms relatively to a given shear-operator may be regarded as conditions satisfied by the forms of the system. In particular, that the *rank* quâ $\widehat{yz}$ of the *derivative* quâ $\widehat{xy}$ of any form ϕ of a complete system of type (a, b) satisfying certain other conditions shall *lie within given limits* is admissible as a numerical condition to restrict the system further, and will be satisfied by the *derivatives* of the restricted system; the xy-differentiants ψ_r of types $(a + r, b - r)$ must then be taken to satisfy the same condition, namely, that their *ranks* quâ $\widehat{yz}$ *shall lie within the given limits.* It may be that the conditions K can be separated into two sets, of which the second set contains only conditions that are satisfied by *all* forms ϕ of type (a, b) that satisfy the conditions of the first set set *and by their derivatives* quâ $\widehat{xy}$*:* the conditions of the second set need be *specified* only for the xy-differentiants ψ_r of types $(a + r, b - r)$.

For example, if the differentiant character xy does not interfere with the conditions K (representing only the conditions of the *first* set); if the rank quâ $\widehat{yz}$ of the form $\widehat{xy}^k \cdot \phi$ lies within limits l_k to l'_k dependent only on the given type (a, b) and the value of k, inclusive of both limits, for *every* form of type (a, b) that satisfies the conditions K and for *every* value of k for which $\widehat{xy}^k \cdot \phi$ is not identically 0 (the one condition of the *second* set); and if, for *each* value of r

from 0 to r' [the greatest value of r for which $(a+r, b-r)$ is a type subordinate to (a, b) quâ $\widehat{xy}$], inclusive, there exists a determinate distributive operator ω_r such that the forms $\omega_r \psi_r$ obtained from the xy-differentiants ψ_r *whose ranks* quâ $\widehat{yz}$ *lie within the limits* l_r *to* l'_r, inclusive, in complete systems of *all* types $(a+r, b-r)$ subordinate to (a, b) quâ $\widehat{xy}$ satisfying the conditions K and *reduced* quâ $\widehat{yz}$ are forms of type (a, b) that satisfy the conditions K and the identity (31), where S_r is a non-vanishing constant; then, by (29),

$$(36) \qquad \sum_{r=0}^{r'} \sum_{\rho=l_k}^{l'_k} N_{K,xy}^{\widehat{yz}\rho}(a+r, b-r) = N_K(a, b).$$

III. *Simple differentiants.*

16. In this section we consider only forms of a given order in the variables, of given degrees in the coefficients of the several quantics, and of given weights in the several variables *excepting two*, say x and y. For any such form of type (w_x, w_y) we have, by (2), $w_x + w_y = \sigma$, where σ is a constant, that is, is the same number for *all* the forms considered. It will be convenient to represent w_x temporarily by a simpler symbol a (or a'): then, $w_y = \sigma - a$ (or $\sigma - a'$).

For xy-differentiants of a *given* type $(a, \sigma - a)$ there are *no* conditions excepting the differentiant character xy and the types subordinate to $(a, \sigma - a)$ quâ $\widehat{xy}$ are the types $(a+r, \sigma - a - r)$ for which $0 \leqq r \leqq \sigma - a$ (it may even be that r has a smaller upper limit than $\sigma - a$). If $\psi_{a'-a}$ is any xy-differentiant of one of these subordinate types, say $(a', \sigma - a')$ for $r = a' - a$, where $a \leqq a' \leqq \sigma$ and $\frac{1}{2}\sigma \leqq a'$, by (20), then $\widehat{yx}^{a'-a} \cdot \psi_{a'-a}$ is a form of type $(a, \sigma - a)$ for which, by (18), because $\widehat{xy} \cdot \psi_{a'-a} \equiv 0$,

$$\widehat{xy}^{a'-a} \cdot \widehat{yx}^{a'-a} \cdot \psi_{a'-a} \equiv (a' - a)!\,(2a' - \sigma)^{(a'-a)} \cdot \psi_{a'-a},$$

which is not identically 0 if $\sigma - a' \leqq a \leqq a' \leqq \sigma$: therefore, by (35), if $\frac{1}{2}\sigma \leqq a \leqq \sigma$,

$$N_{xy}(a, \sigma - a) = (1 - \overline{xy}) \cdot N(a, \sigma - a)$$

or, if we replace a by w_x and $\sigma - a$ by w_y,

$$(37) \qquad N_{xy}(w_x, w_y) = (1 - \overline{xy}) \cdot N(w_x, w_y) \qquad \text{for} \qquad w_y \leqq w_x.$$

This is CAYLEY's formula as extended to a system of any number of quantics in any number of variables.

Formula (37) shows that the number of linearly independent linear equations between the coefficients of the general form ϕ of type (w_x, w_y) implied by the identity $\widehat{xy} \cdot \phi \equiv 0$ is just the number of terms in the general form of type $\overline{xy} \cdot (w_x, w_y)$, that is, that the linear equations implied by this identity are *linearly independent*, as Cayley *assumed*. This proves also that, if ϕ is the *general*

form of type (w_x, w_y), then $\widehat{xy}\cdot\phi$ is the *general* form of type (w_x+1, w_y-1) and, therefore, $\widehat{xy}^k\cdot\phi$ is the most *general* form of type (w_x+k, w_y-k) for *every* value of k for which this type exists.

Incidentally, formula (37) shows that, of the general forms of two types that differ only in the weights in two variables, that has the more terms in which the weights in the two variables in question are the more nearly equal (unless they have the same number of terms, in which case the general forms of all intermediate types, — types subordinate to one and to which the other is subordinate, — have that same number of terms).

Again, formula (37) gives the number of linearly independent *covariants* or *invariants* of any system of *binary* quantics if x and y are the only variables and $w_x = w_y$, by theorems A and B of § 8, because every xy-differentiant for which these conditions are satisfied is a *complete* differentiant in x and y and, therefore, a *covariant* or *invariant*.

IV. *Double differentiants.*

In §§ 17–22 we shall consider only forms of a *given* order in the variables, of *given* degrees in the coefficients of the several quantics, and of *given* weights in the several variables *excepting three*, say x, y, and z. The type of such a form shall be regularly denoted by (w_x, w_y, w_z), representing only the weights in x, y, and z, in this sequence, so that, by (2), $w_x + w_y + w_z = \sigma$, where σ has the same value for *all* forms considered; but, in the deduction of formulae, it will be more convenient to represent two of these three weights by simple letters, and for this purpose we shall put either $w_x = a$, $w_y = b$ and $w_z = \sigma - a - b$, —or $w_x = \sigma - b - c$, $w_y = b$, and $w_z = c$, —accenting a, b, or c when we have occasion to represent more than one value of either.

17. If χ is an xz- and yz-differentiant of type $(\sigma - b' - c, b', c)$, — where, by theorem A of § 8, $c \leqq b' \leqq \sigma - 2c$, — so that $\widehat{xz}\cdot\chi \equiv 0$ and $\widehat{yz}\cdot\chi \equiv 0$, then, by (17) and (18),

$$\widehat{xz}\cdot\widehat{xy}^k\cdot\chi \equiv \widehat{xy}^k\cdot\widehat{xz}\cdot\chi \equiv 0 \quad\text{and}\quad \widehat{yz}\cdot\widehat{xy}^k\cdot\chi \equiv \widehat{xy}^k\cdot\widehat{yz}\cdot\chi + k\cdot\widehat{xy}^{k-1}\cdot\widehat{xz}\cdot\chi \equiv 0,$$

so that the differentiant character xy *does not interfere* with the differentiant characters xz and yz. The *derivatives* of xz- and yz-differentiants of the *given* type $(\sigma - b' - c, b', c)$ quâ $\widehat{xy}$ are, then, xyz-differentiants of types $(\sigma - b - c, b, c)$ subordinate to $(\sigma - b' - c, b', c)$ quâ $\widehat{xy}$ for which $0 \leqq c \leqq b \leqq b' \leqq \sigma - b - c$ and $b \leqq \frac{1}{2}(\sigma - c)$, by theorems A and F of § 8. If $\psi_{b'-b}$ is any xyz-differentiant of type $(\sigma - b - c, b, c)$, — for which, in the notation of §§ 10–13, $r = b' - b$ and $0 \leqq r \leqq b' - c$, — then, by (17) and (18),

$$\widehat{xz}\cdot\widehat{yx}^{b'-b}\cdot\psi_{b'-b} \equiv \widehat{yx}^{b'-b}\cdot\widehat{xz}\cdot\psi_{b'-b} + (b'-b)\cdot\widehat{yx}^{b'-b-1}\cdot\widehat{yz}\cdot\psi_{b'-b} \equiv 0$$

and

$$\widehat{yz}\cdot\widehat{yx}^{b'-b}\cdot\psi_{b'-b}\equiv\widehat{yx}^{b'-b}\cdot\widehat{yz}\cdot\psi_{b'-b}\equiv 0,$$

so that $\widehat{yx}^{b'-b}\cdot\psi_{b'-b}$ is an xz- and yz-differentiant of type $(\sigma-b'-c,\,b',\,c)$, if not identically 0, for which, by (18),

$$\widehat{xy}^{b'-b}\cdot\widehat{yx}^{b'-b}\cdot\psi_{b'-b}\equiv(b'-b)!\;(\sigma-2b-c)^{(b'-b)}\cdot\psi_{b'-b},$$

where the multiplier of $\psi_{b'-b}$ is a *non-vanishing* constant if $b\leqq b'\leqq\sigma-b-c$. Therefore, if $0\leqq c\leqq b'\leqq\frac{1}{2}(\sigma-c)$,

$$N_{xz,\,yz}(\sigma-b'-c,\,b',\,c)=\sum_{b=c}^{b'}N_{xyz}(\sigma-b-c,\,b,\,c),$$

by (28), and

$$N_{xyz}(\sigma-b'-c,\,b',\,c)=(1-\overline{xy})\cdot N_{xz,\,yz}(\sigma-b'-c,\,b',\,c),$$

by (35), while, if $0\leqq c\leqq b\leqq b'\leqq\frac{1}{2}(\sigma-c)$,

$$N_{xz,\,yz}^{\widehat{xy}^{b'-b}}(\sigma-b'-c,\,b',\,c)=N_{xyz}(\sigma-b-c,\,b,\,c),$$

by (30); or, when $\sigma-b'-c$ is replaced by w_x, b' by w_y, c by w_z, and b by $b'-r$, if $w_z\leqq w_y\leqq w_x$,

$$N_{xz,\,yz}(w_x,\,w_y,\,w_z)=\sum_{r=0}^{w_y-w_z}N_{xyz}(w_x+r,\,w_y-r,\,w_z)\tag{38}$$

and

$$N_{xyz}(w_x,\,w_y,\,w_z)=(1-\overline{xy})\cdot N_{xz,\,yz}(w_x,\,w_y,\,w_z),\tag{39}$$

while, if $w_z\leqq w_y\leqq w_x$ and $0\leqq r\leqq w_y-w_z$,

$$N_{xz,\,yz}^{\widehat{xy}^r}(w_x,\,w_y,\,w_z)=N_{xyz}(w_x+r,\,w_y-r,\,w_z).\tag{40}$$

Also, by § 12, the forms $\widehat{yx}^{b'-b}\cdot\psi_{b'-b}$ obtained from *complete* systems of xyz-differentiants of *all* types $(\sigma-b-c,\;b,\;c)$ subordinate to the *given* type $(\sigma-b'-c,\,b',\,c)$ quâ xy constitute a *complete* system of xz- and yz-differentiants of the latter type if $0\leqq c\leqq b'\leqq\frac{1}{2}(\sigma-c)$.

18. If χ is an xy- and xz-differentiant of type $(a,\,b',\,\sigma-a-b')$,—where, by theorem A of § 8, $\sigma-2a\leqq b'\leqq a$,—so that $\widehat{xy}\cdot\chi\equiv 0$ and $\widehat{xz}\cdot\chi\equiv 0$, then, by (17) and (18),

$$\widehat{xy}\cdot\widehat{yz}^k\cdot\chi\equiv\widehat{yz}^k\cdot\widehat{xy}\cdot\chi-k\cdot\widehat{yz}^{k-1}\cdot\widehat{xz}\cdot\chi\equiv 0\qquad\text{and}\qquad\widehat{xz}\cdot\widehat{yz}^k\cdot\chi\equiv\widehat{yz}^k\cdot\widehat{xz}\cdot\chi\equiv 0,$$

so that the differentiant character yz *does not interfere* with the differentiant characters xy and xz. The *derivatives* of xy- and xz-differentiants of the *given* type $(a,\;b',\;\sigma-a-b')$ quâ $\widehat{yz}$ are, then, xyz-differentiants of types $(a,\;b,\;\sigma-a-b)$ subordinate to $(a,\;b',\;\sigma-a-b')$ quâ $\widehat{yz}$ for which

$\sigma - a - b \leqq b' \leqq b \leqq a \leqq \sigma - b$ and $\frac{1}{2}(\sigma - a) \leqq b$, by theorems A and F of § 8. If $\psi_{b-b'}$ is any xyz-differentiant of type $(a, b, \sigma - a - b)$,—for which, in the notation of §§ 10–13, $r = b - b'$ and $0 \leqq r \leqq a - b'$,—then, by (17) and (18),

$$\widehat{xy} \cdot \widehat{zy}^{b-b'} \cdot \psi_{b-b'} \equiv \widehat{zy}^{b-b'} \cdot \widehat{xy} \cdot \psi_{b-b'} \equiv 0$$

and

$$\widehat{xz} \cdot \widehat{zy}^{b-b'} \cdot \psi_{b-b'} \equiv \widehat{zy}^{b-b'} \cdot \widehat{xz} \cdot \psi_{b-b'} - (b - b') \cdot \widehat{zy}^{b-b'-1} \cdot \widehat{xy} \cdot \psi_{b-b'} \equiv 0,$$

so that $\widehat{zy}^{b-b'} \cdot \psi_{b-b'}$ is an xy- and xz-differentiant of type $(a, b', \sigma - a - b')$, if not identically 0, for which, by (18),

$$\widehat{yz}^{b-b'} \cdot \widehat{zy}^{b-b'} \cdot \psi_{b-b'} \equiv (b - b')!\ (a + 2b - \sigma)^{(b-b')} \cdot \psi_{b-b'},$$

where the multiplier of $\psi_{b-b'}$ is a *non-vanishing* constant if $\sigma - a - b \leqq b' \leqq b$. Therefore, if $\frac{1}{2}(\sigma - a) \leqq b' \leqq a \leqq \sigma - b'$,

$$N_{xy,\,xz}(a, b', \sigma - a - b') = \sum_{b=b'}^{a} N_{xyz}(a, b, \sigma - a - b),$$

by (28), and

$$N_{xyz}(a, b', \sigma - a - b') = (1 - \overline{yz}) \cdot N_{xy,\,xz}(a, b', \sigma - a - b'),$$

by (35), while, if $\frac{1}{2}(\sigma - a) \leqq b' \leqq b \leqq a \leqq \sigma - b$,

$$N_{xy,\,xz}^{\widehat{yz}^{b-b'}}(a, b', \sigma - a - b') = N_{xyz}(a, b, \sigma - a - b),$$

by (30), or when a is replaced by w_x, b' by w_y, $\sigma - a - b'$ by w_z, and b by $b' + r$, if $w_z \leqq w_y \leqq w_x$,

$$N_{xy,\,xz}(w_x, w_y, w_z) = \sum_{r=0}^{w_x - w_y} N_{xyz}(w_x, w_y + r, w_z - r) \tag{41}$$

and

$$N_{xyz}(w_x, w_y, w_z) = (1 - \overline{yz}) \cdot N_{xy,\,xz}(w_x, w_y, w_z), \tag{42}$$

while, if $w_z \leqq w_y \leqq w_x$ and $0 \leqq r \leqq w_x - w_y$,

$$N_{xy,\,xz}^{\widehat{yz}^{r}}(w_x, w_y, w_z) = N_{xyz}(w_x, w_y + r, w_z - r). \tag{43}$$

Also, by § 12, the forms $\widehat{zy}^{b-b'} \cdot \psi_{b-b'}$ obtained from *complete* systems of xyz-differentiants of *all* types $(a, b, \sigma - a - b)$ subordinate to the given type $(a, b', \sigma - a - b')$ quâ $\widehat{yz}$ constitute a *complete* system of xy- and xz-differentiants of the latter type if $\frac{1}{2}(\sigma - a) \leqq b' \leqq a \leqq \sigma - b'$.

19. If ϕ is an xy-differentiant of type $(a', b', \sigma - a' - b')$, where, by theorem A of § 8, $b' \leqq a' \leqq \sigma - b'$, so that $\widehat{xy} \cdot \phi \equiv 0$, then, by (17),

$$\widehat{xy} \cdot \widehat{xz}^{k} \cdot \phi \equiv \widehat{xz}^{k} \cdot \widehat{xy} \cdot \phi \equiv 0,$$

that is, the differentiant character xz *does not interfere* with the differentiant

character xy. The derivatives of xy-differentiants of *given type* $(a', b', \sigma - a' - b')$ quâ $\widehat{xz}$ are, then, xy- and xz-differentiants of types $(a, b', \sigma - a - b')$ for which $\sigma - a - a' \leqq b' \leqq a' \leqq a \leqq \sigma - b'$, by theorems A and F of § 8. If $\chi_{a-a'}$ is any xy- and xz-differentiant of type $(a, b', \sigma - a - b')$, —for which, in the notation of §§ 10 — 13, $r = a - a'$ and $0 \leqq r \leqq \sigma - a' - b'$, —then, by § 9, $\{y : x\} \cdot \widehat{zx}^{a-a'} \cdot \chi_{a-a'}$ is an xy-differentiant of type $(a', b', \sigma - a' - b')$, if not identically 0.

Now, by § 18, if $\frac{1}{2}(\sigma - a) \leqq b' \leqq a \leqq \sigma - b'$, we may take a complete system of xy- and xz-differentiants $\chi_{a-a'}$ of type $(a, b', \sigma - a - b')$ in such manner that any one of them of rank $b - b'$ quâ $\widehat{yz}$ can be represented as $\chi_{a-a'} \equiv \widehat{zy}^{b-b'} \cdot \psi_{b-b'}$, where $\psi_{b-b'}$ is an xyz-differentiant of some type $(a, b, \sigma - a - b)$ subordinate to $(a, b', \sigma - a - b')$ quâ $\widehat{yz}$ for which $b' \leqq b \leqq a$ and $b \leqq \sigma - a$. Then,

$$\widehat{xy} \cdot \chi_{a-a'} \equiv 0, \qquad \widehat{xz} \cdot \chi_{a-a'} \equiv 0,$$

$$\widehat{xy} \cdot \psi_{b-b'} \equiv 0, \qquad \widehat{xz} \cdot \psi_{b-b'} \equiv 0, \qquad \widehat{yz} \cdot \psi_{b-b'} \equiv 0,$$

and, therefore,

$$\widehat{xz} \cdot \widehat{zy}^{b-b'+\alpha} \cdot \psi_{b-b'} \equiv 0 \quad \text{and} \quad \widehat{xz}^{\beta} \cdot \widehat{zx}^{a-a'-\alpha} \cdot \widehat{zy}^{b-b'+\alpha} \cdot \psi_{b-b'} \equiv 0 \text{ if } a - a' - \alpha < \beta,$$

so that, by (17) and (18),

$$\begin{aligned}
\widehat{xy}^{\alpha} \cdot \widehat{zx}^{a-a'} \cdot \chi_{a-a'} &\equiv (a - a')^{(\alpha)} \cdot \widehat{zx}^{a-a'-\alpha} \cdot \widehat{zy}^{\alpha} \cdot \chi_{a-a'} \\
&\equiv (a - a')^{(\alpha)} \cdot \widehat{zx}^{a-a'-\alpha} \cdot \widehat{zy}^{b-b'+\alpha} \cdot \psi_{b-b'},
\end{aligned}$$

$$\begin{aligned}
\widehat{xz}^{a-a'} \cdot \widehat{yx}^{\alpha} \cdot \widehat{xy}^{\alpha} \cdot \widehat{zx}^{a-a'} \cdot \chi_{a-a'} &\equiv (a - a')^{(\alpha)} \cdot \widehat{yz}^{\alpha} \cdot \widehat{xz}^{a-a'-\alpha} \cdot \widehat{xy}^{\alpha} \cdot \widehat{zx}^{a-a'} \cdot \chi_{a-a'} \\
&\equiv (a - a')!\,(a - a')^{(\alpha)} \cdot (2a + b' - \sigma - \alpha)^{(a-a'-\alpha)} \cdot \widehat{yz}^{\alpha} \cdot \widehat{zy}^{b-b'+\alpha} \cdot \psi_{b-b'} \\
&\equiv (a - a')!\,(a - a')^{(\alpha)} \cdot (2a + b' - \sigma - \alpha)^{(a-a'-\alpha)} \cdot (b - b' + \alpha)^{(\alpha)} \\
&\qquad \times (a + b + b' - \sigma)^{(\alpha)} \cdot \widehat{zy}^{b-b'} \cdot \psi_{b-b'} \\
&\equiv \frac{(a - a')!\,(a + b + b' - \sigma)!}{(a + a' + b' - \sigma)!\,(b - b')!} \cdot (a - a')^{(\alpha)} \cdot \frac{(2a + b' - \sigma - \alpha)!\,(b - b' + \alpha)!}{(a + b + b' - \sigma - \alpha)!} \cdot \chi_{a-a'},
\end{aligned}$$

and, by (23),

$$\widehat{xz}^{a-a'} \cdot \{y;\, x\} \cdot \widehat{zx}^{a-a'} \cdot \chi_{a-a'} \equiv S_{a-a'} \cdot \chi_{a-a'}, \tag{44}$$

where, by (15)

$$\begin{aligned}
S_{a-a'} &= \frac{(a - a')!\,(a' - b' + 1)!\,(a + b + b' - \sigma)!}{(b - b')!\,(a + a' + b' - \sigma)!} \cdot \sum_{\alpha=0}^{a-a'} (-1)^{\alpha} \\
&\qquad \times \binom{a - a'}{\alpha} \cdot \frac{(2a + b' - \sigma - \alpha)^{(a-b)}}{(a' - b' + \alpha + 1)^{(a'-b+1)}} \\
&= \frac{(a - a')!\,(a' - b' + 1)!\,(a + b + b' - \sigma)!}{(b - b')!\,(a + a' + b' - \sigma)!}\, S^{a'-b+1,\, a-a'}_{2a+b'-\sigma,\, a'-b'+1},
\end{aligned}$$

which, by (14) and (16), is a *non-vanishing* constant if $\sigma - a - b \leqq b' \leqq b \leqq a' \leqq a$; so that, at least when these conditions are satisfied, the *derivative* of the form $\{y;\, x\} \cdot \widehat{zx}^{a-a'} \cdot \chi_{a-a'}$ quâ $\widehat{xz}$ is a non-vanishing constant multiple of $\chi_{a-a'}$. Observe that the condition $\sigma - a - b \leqq b'$ is satisfied if $\frac{1}{2}(\sigma - a') \leqq b' \leqq b$.

But, if ϕ is an xy-differentiant of type $(a', b', \sigma - a' - b')$ and of rank $a - a'$ quâ $\widehat{xz}$, its derivative quâ $\widehat{xz}$, namely $\widehat{xz}^{a-a'} \cdot \phi$, is an xy- and xz-differentiant of type $(a, b', \sigma - a - b')$ and, say, of rank $b - b'$ quâ $\widehat{yz}$, so that $\widehat{yz}^{b-b'} \cdot \widehat{xz}^{a-a'} \cdot \phi$ is a non-vanishing xyz-differentiant of type $(a, b, \sigma - a - b)$, where, by theorems A and F of § 8, $\sigma - a - a' \leqq b' \leqq a' \leqq a \leqq \sigma - b$, $b' \leqq b \leqq a$, and $\sigma - a - b' \leqq b$; then, by (18), because $\widehat{xy} \cdot \phi \equiv 0$,

$$\widehat{xy}^{a-a'} \cdot \widehat{yz}^{a+b-a'-b'} \cdot \phi \equiv (-1)^{a-a'} \cdot (a + b - a' - b')^{(a-a')} \cdot \widehat{yz}^{b-b'} \cdot \widehat{xz}^{a-a'} \cdot \phi \not\equiv 0,$$

by (14), while

$$\widehat{xy}^{a-a'+1} \cdot \widehat{yz}^{a+b-a'-b'} \cdot \phi \equiv 0;$$

therefore, $\widehat{yz}^{a+b-a'-b'} \cdot \phi$ is a non-vanishing form of type $(a', a+b-a', \sigma-a-b)$ and rank $a - a'$ quâ $\widehat{xy}$, so that, by theorem F of § 8, $a + b - 2a' \leqq a - a'$, that is, $b \leqq a'$; the types of ϕ and its *successive* derivatives quâ $\widehat{xz}$ and quâ $\widehat{yz}$ are, then, so related that $\sigma - a - b \leqq b' \leqq b \leqq a' \leqq a \leqq \sigma - b$.

If, then, $\frac{1}{2}(\sigma - a) \leqq b' \leqq a' \leqq a \leqq \sigma - b'$, the xy- and xz-differentiant $\chi_{a-a'}$ of type $(a, b', \sigma - a - b')$ subordinate to $(a', b', \sigma - a' - b')$ quâ $\widehat{xz}$ and of rank $b - b'$ quâ $\widehat{yz}$ is turned by the operator $\{y;\, x\} \cdot \widehat{zx}^{a-a'}$ into an xy-differentiant of type $(a', b', \sigma - a' - b')$ satisfying the condition (44) with a non-vanishing value of the constant $S_{a-a'}$ if $b' \leqq b \leqq a'$; the condition (44) is equivalent to (31) if, in the latter, r is replaced by $a - a'$, $\widehat{xy}$ by $\widehat{xz}$, ψ_r by $\chi_{a-a'}$, and ω_r by $\{y;\, x\} \cdot \widehat{zx}^{a-a'}$; and the conditions $b' \leqq b \leqq a'$ are satisfied by *every* xy-differentiant of type $(a', b', \sigma - a' - b')$ whose derivative quâ $\widehat{xz}$ of type $(a, b', \sigma - a - b')$ is of rank $b - b'$ quâ $\widehat{yz}$. Therefore, by (36), if $\frac{1}{2}(\sigma - a') \leqq b' \leqq a' \leqq \sigma - b'$,

$$\sum_{b=b'}^{a'} \sum_{a=a'}^{\sigma-b} N_{xy,\, xz}^{\widehat{yz}^{b-b'}}(a, b', \sigma - a - b') = N_{xy}(a', b', \sigma - a' - b')$$

or, by (43),

$$\sum_{b=b'}^{a'} \sum_{a=a'}^{\sigma-b} N_{xyz}(a, b, \sigma - a - b) = N_{xy}(a', b', \sigma - a' - b'), \tag{45}$$

where, however, the upper limit of b is $\sigma - a'$ if $\frac{1}{2}\sigma \leqq a'$, because otherwise there could be no summation quâ a. Evidently, the summations quâ r and ρ in (36) may be performed in either order if the limits are properly taken, and here it is more convenient to sum first with respect to r $(a - a')$ and then with respect to ρ $(b - b')$.

If $\frac{1}{2}(\sigma - a') \leqq b' \leqq a' \leqq \sigma - b' - 1$, (45) holds also if a' is replaced by $a' + 1$, and when we subtract the result of the replacement from (45), member from member, we find

$$(46)\quad \left\{ \begin{aligned} &\sum_{b=b'}^{a'} N_{xyz}(a', b, \sigma - a' - b) - \sum_{a=a'+1}^{\sigma-a'-1} N_{xyz}(a, a' + 1, \sigma - a - a' - 1) \\ &\qquad\qquad = (1 - \overline{xz}) \cdot N_{xy}(a', b', \sigma - a' - b') \end{aligned} \right.$$

if $a' \leqq \frac{1}{2}\sigma$, but

$$(47)\quad \sum_{b=b'}^{\sigma-a'} N_{xyz}(a', b, \sigma - a' - b) = (1 - \overline{xz}) \cdot N_{xy}(a', b', \sigma - a' - b')$$

if $\frac{1}{2}\sigma \leqq a'$; the special case $a' = \frac{1}{2}(\sigma - 1)$ requires a' to be taken as the upper limit of b before the replacement of a' by $a' + 1$ and $\sigma - a' - 1 = a'$ afterward, and the result of the subtraction is (46); in the special case $a' = \frac{1}{2}\sigma$, (46) and (47) are identical; the second sum of (46) vanishes if $a' = \frac{1}{2}\sigma$ or $\frac{1}{2}(\sigma - 1)$.

If $\frac{1}{2}(\sigma - a') \leqq b' \leqq a' = \sigma - b'$, that is, if $\frac{1}{2}\sigma \leqq a'$ and $b' = \sigma - a'$, (45) gives directly

$$(48)\qquad N_{xyz}(a', \sigma - a', 0) = N_{xy}(a', \sigma - a', 0),$$

which is consistent with (47) [and with (46) if $b' = a' = \frac{1}{2}\sigma$],—because $\overline{xz} \cdot N_{xy}(a', \sigma - a', 0) = 0$, by § 5,—and true, because every xy-differentiant of weight 0 in z is an xyz-differentiant, by the foot-note to § 5. So that (46) holds if $\frac{1}{2}(\sigma - a') \leqq b' \leqq a' \leqq \sigma - b'$ and $a' \leqq \frac{1}{2}\sigma$, while (47) holds if $\frac{1}{2}(\sigma - a') \leqq b' \leqq a' \leqq \sigma - b'$ and $\frac{1}{2}\sigma \leqq a'$.

If $\frac{1}{2}(\sigma - a') \leqq b' < a' \leqq \sigma - b' - 1$, (46) holds for $a' \leqq \frac{1}{2}\sigma$ and (47) for $\frac{1}{2}\sigma \leqq a'$ when b' is replaced by $b' + 1$, the difference of the formulæ before and after replacement, in either case, being

$$(49)\quad N_{xyz}(a', b', \sigma - a' - b') = (1 - \overline{yz})(1 - \overline{xz}) \cdot N_{xy}(a', b', \sigma - a' - b'),$$

which, because $\overline{yz}$, $\overline{xz}$ and $\overline{yz} \cdot \overline{xz}$ annihilate $N_{xy}(a', \sigma - a', 0)$, includes also (48),—that being the case of (49) for which $a' = \sigma - b'$.

If $\frac{1}{2}(\sigma - a') \leqq b' = a' \leqq \sigma - b'$, that is, if $\frac{1}{3}\sigma \leqq a' \leqq \frac{1}{2}\sigma$ and $b' = a'$, (45) gives

$$\sum_{a=a'}^{\sigma-a'} N_{xyz}(a, a', \sigma - a - a') = N_{xy}(a', a', \sigma - 2a')$$

or, with the replacement of $N_{xyz}(a, a', \sigma - a - a')$ by its expression from (49) for all values of *a greater than a'* [for which, evidently, (49) holds],

$$N_{xyz}(a', a', \sigma - 2a') = N_{xy}(a', a', \sigma - 2a') - (1 - \overline{yz}) \cdot \sum_{a=a'+1}^{\sigma-a'} (1 - \overline{xz}) \times N_{xy}(a, a', \sigma - a - a')$$

$$=N_{xy}(a', a', \sigma-2a')-N_{xy}(a'+1, a', \sigma-2a'-1)+\overline{yz}\cdot N_{xy}(a'+1, a', \sigma-2a'-1)$$
$$=(1-\overline{xz}+\overline{xz}\cdot\overline{yz})\cdot N_{xy}(a', a', \sigma-2a')$$
$$=(1-\overline{yz})(1-\overline{xz})\cdot N_{xy}(a', a', \sigma-2a'),-$$

because $\overline{yz}\cdot N_{xy}(a', a', \sigma-2a')=N_{xy}(a', a'+1, \sigma-2a'-1)=0$, by theorem A of § 8, — and this result is what (49) becomes for $b'=a'$.* Therefore, (49) holds for $\frac{1}{2}(\sigma-a')\leqq b'\leqq a'\leqq\sigma-b'$, that is, for *all* types of which there can be xyz-differentiants in accordance with theorem A of § 8.

On replacing $N_{xy}(a', b', \sigma-a'-b')$ in (49) by its expression from (37), we have, if $\frac{1}{2}(\sigma-a')\leqq b'\leqq a'\leqq\sigma-b'$,

$$N_{xyz}(a', b', \sigma-a'-b')=(1-\overline{yz})(1-\overline{xz})(1-\overline{xy})\cdot N(a', b', \sigma-a'-b');$$

after the restoration of w_x, w_y, w_z for $a', b', \sigma-a'-b'$, respectively, we have, if $w_z\leqq w_y\leqq w_x$,

$$(50)\quad N_{xyz}(w_x, w_y, w_z)=(1-\overline{yz})(1-\overline{xz})(1-\overline{xy})\cdot N(w_x, w_y, w_z).$$

By § 5, $\overline{xy}$, $\overline{xz}$, and $\overline{yz}$ are commutative and $\overline{xy}\cdot\overline{yz}=\overline{xz}$; therefore, (50) may be written

$$(51)\quad \begin{cases} N_{xyz}\cdot(w_x, w_y, w_z)=(1-\overline{xy}-\overline{yz}+\overline{xz})(1-\overline{xz})\cdot N(w_x, w_y, w_z) \\ \qquad\qquad =[1-\overline{xy}-\overline{yz}+(\overline{xy}+\overline{yz}-\overline{xz})\cdot\overline{xz}]\cdot N(w_x, w_y, w_z). \end{cases}$$

In particular, if $w_y=w_x$,

$$\overline{yz}\cdot N(w_x, w_x, w_z)=N(w_x, w_x+1, w_z-1)=N(w_x+1, w_x, w_z-1)$$
$$=\overline{xz}\cdot N(w_x, w_x, w_z),$$

so that

$$N_{xyz}(w_x, w_x, w_z)=(1-\overline{xz}+\overline{xz}\cdot\overline{yz})(1-\overline{xy})\cdot N(w_x, w_x, w_z),$$

as we have already seen in the special case of (49) for which $b'=a'$.

20. In combination with (43), (41), and (37), formula (49) gives

$$(52)\quad \begin{cases} N^{\widehat{yz^r}}_{xy,\,xz}(w_x, w_y, w_z)=(1-\overline{yz})(1-\overline{xz})\cdot\overline{yz}^r\cdot N_{xy}(w_x, w_y, w_z) \\ \qquad\qquad =(1-\overline{yz})(1-\overline{xz})(1-\overline{xy})\cdot\widehat{yz^r}\cdot N(w_x, w_y, w_z) \end{cases}$$

if $w_z\leqq w_y\leqq w_x$ and $0\leqq r\leqq w_x-w_y$, and

$$N_{xy,\,xz}(w_x, w_y, w_z)=(1-\overline{xz})\cdot\sum_{r=0}^{w_x-w_y}(1-\overline{yz})\cdot\overline{yz}^r\cdot N_{xy}(w_x, w_y, w_z)$$
$$=(1-\overline{xz})\cdot[N_{xy}(w_x, w_y, w_z)-\overline{yz}\cdot N_{xy}(w_x, w_x, w_y+w_z-w_x)]$$

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* If $a'=\frac{1}{2}\sigma$, the sum in the first expression for $N_{xyz}(a', a', \sigma-2a')$ vanishes and we have $N_{xyz}(\frac{1}{2}\sigma, \frac{1}{2}\sigma, 0)=N_{xy}(\frac{1}{2}\sigma, \frac{1}{2}\sigma, 0)$, which is a particular case of (49), — because $\overline{xz}$ and $\overline{yz}$ annihilate $N_{xy}(\frac{1}{2}\sigma, \frac{1}{2}\sigma, 0)$, by § 5, — and this formula is true, by (48).

$$= (1 - \overline{xz}) \cdot N_{xy}(w_x, w_y, w_z) + \overline{xz} \cdot \overline{yz} \cdot N_{xy}(w_x, w_x, w_y + w_z - w_x),$$

that is, by (37),

$$(53)\begin{cases} N_{xy,\,xz}(w_x, w_y, w_z) \\ \quad = (1 - \overline{xz}) \cdot N_{xy}(w_x, w_y, w_z) + N_{xy}(w_x + 1, w_x + 1, w_y + w_z - w_x - 2) \\ \quad = (1 - \overline{xy})(1 - \overline{xz}) \cdot N(w_x, w_y, w_z) \\ \qquad\qquad + (1 - \overline{xy}) \cdot N(w_x + 1, w_x + 1, w_y + w_z - w_x - 2) \end{cases}$$

if $w_z \leqq w_y \leqq w_x$. If $w_y + w_z < w_x + 2$, the last term of each of the expressions for $N_{xy,\,xz}(w_x, w_y, w_z)$ given by (53) vanishes.

Because the sequence of variables is unessential, (52) gives, if $w_y \leqq w_z \leqq w_x$ and $0 \leqq r \leqq w_x - w_z$,

$$N_{xy,\,xz}^{\widehat{yz}^r}(w_x, w_z, w_y) = (1 - \overline{yz})(1 - \overline{xz})(1 - \overline{xy}) \cdot N(w_x, w_z + r, w_y - r)$$

$$= N_{xz,\,xy}^{\widehat{zy}^r}(w_x, w_y, w_z) = N_{xy,\,xz}^{\widehat{yz}^{w_z - w_y + r}}(w_x, w_y, w_z),$$

by (19), from which follows, when r is replaced by $r + w_y - w_z$,

$$N_{xy,\,xz}^{\widehat{yz}^r}(w_x, w_y, w_z) = (1 - \overline{yz})(1 - \overline{xz})(1 - \overline{xy}) \cdot N(w_x, w_y + r, w_z - r)$$

if $w_y \leqq w_z \leqq w_x$ and $w_z - w_y \leqq r \leqq w_x - w_y$, but this formula is identical with (52); therefore, (52) holds if $w_y \leqq w_x$, $w_z \leqq w_x$, and $w_z - w_y \leqq r \leqq w_x - w_y$, without regard to the relative values of w_y and w_z (of course, r is not negative); that is, (52) holds for *all* types of which there can be xy- and xz-differentiants in accordance with theorem A of § 8 and for *all* possible ranks of such xy- and xz-differentiants quâ $\widehat{yz}$, by theorem F of § 8 and by § 18.

Similarly, (53) gives, if $w_y \leqq w_z \leqq w_x$,

$$\begin{aligned} &N_{xy,\,xz}(w_x, w_z, w_y) \\ &\quad = (1 - \overline{xy})(1 - \overline{xz}) \cdot N(w_x, w_y, w_z) + (1 - \overline{xy}) \cdot N(w_x + 1, w_x + 1, w_y + w_z - w_x - 2) \\ &\quad = N_{xz,\,xy}(w_x, w_y, w_z) \\ &\quad = (1 - \overline{xz})(1 - \overline{xy}) \cdot N(w_x, w_y, w_z) + (1 - \overline{xy}) \cdot N(w_x + 1, w_x + 1, w_y + w_z - w_x - 2), \end{aligned}$$

which is identical with (53), so that (53) holds if $w_y \leqq w_x$ and $w_z \leqq w_x$, without regard to the relative values of w_y and w_z; that is, (53) holds for *all* types of which there can be xy- and xz-differentiants in accordance with theorem A of § 8.

21. If ϕ is a yz-differentiant of type $(\sigma - b' - c', b', c')$, — where, by theorem A of § 8, $c' \leqq b' \leqq \sigma - c'$, — so that $\widehat{yz} \cdot \phi \equiv 0$; then, by (17),

$$\widehat{yz} \cdot \widehat{xz}^k \cdot \phi \equiv \widehat{xz}^k \cdot \widehat{yz} \cdot \phi \equiv 0,$$

that is, the differentiant character xz *does not interfere* with the differentiant character yz; the derivatives of yz-differentiants of *given* type $(\sigma - b' - c', b', c')$ quâ $\widehat{xz}$ are, then, xz- and yz-differentiants of types $(\sigma - b' - c, b', c)$ for which $0 \leqq c \leqq c' \leqq b' \leqq \sigma - c - c'$, by theorems A and F of § 8; if $\chi_{c'-c}$ is any xz- and yz-differentiant of type $(\sigma - b' - c, b', c)$, — for which, in the notation of §§ 10–13, $r = c' - c$ and $0 \leqq r \leqq c'$, — then, by § 9, $\{z;\ y\} \cdot \widehat{zx}^{c'-c} \cdot \chi_{c'-c}$ is a yz-differentiant of type $(\sigma - b' - c', b', c')$, if not identically 0.

Now, by § 17, if $0 \leqq c \leqq b' \leqq \frac{1}{2}(\sigma - c)$, we may take a complete system of xz- and yz-differentiants $\chi_{c'-c}$ of type $(\sigma - b' - c, b', c)$ in such manner that any one of them of rank $b' - b$ quâ $\widehat{xy}$ can be represented as $\chi_{c'-c} \equiv \widehat{yx}^{b'-b} \cdot \psi_{b'-b}$, where $\psi_{b'-b}$ is an xyz-differentiant of some type $(\sigma - b - c, b, c)$ subordinate to $(\sigma - b' - c, b', c)$ quâ $\widehat{xy}$ for which $c \leqq b \leqq b'$; then,

$$\widehat{xz} \cdot \chi_{c'-c} \equiv 0, \qquad \widehat{yz} \cdot \chi_{c'-c} \equiv 0,$$

$$\widehat{xy} \cdot \psi_{b'-b} \equiv 0, \qquad \widehat{xz} \cdot \psi_{b'-b} \equiv 0, \qquad \widehat{yz} \cdot \psi_{b'-b} \equiv 0,$$

and, therefore,

$$\widehat{xz} \cdot \widehat{yx}^{b'-b+\alpha} \cdot \psi_{b'-b} \equiv 0 \quad \text{and} \quad \widehat{xz}^{\beta} \cdot \widehat{zx}^{c'-c-\alpha} \cdot \widehat{yx}^{b'-b+\alpha} \cdot \psi_{b'-b} \equiv 0 \text{ if } c'-c-\alpha < \beta,$$

so that, by (17) and (18),

$$\begin{aligned}
\widehat{yz}^{\alpha} \cdot \widehat{zx}^{c'-c} \cdot \chi_{c'-c} &\equiv (-1)^{\alpha} \cdot (c'-c)^{(\alpha)} \cdot \widehat{zx}^{c'-c-\alpha} \cdot \widehat{yx}^{\alpha} \cdot \chi_{c'-c} \\
&\equiv (-1)^{\alpha} \cdot (c'-c)^{(\alpha)} \cdot \widehat{zx}^{c'-c-\alpha} \cdot \widehat{yx}^{b'-b+\alpha} \cdot \psi_{b'-b}, \\
\widehat{xz}^{c'-c} \cdot \widehat{zy}^{\alpha} \cdot \widehat{yz}^{\alpha} \cdot \widehat{zx}^{c'-c} \cdot \chi_{c'-c} &\equiv (-1)^{\alpha} \cdot (c'-c)^{(\alpha)} \cdot \widehat{xy}^{\alpha} \cdot \widehat{xz}^{c'-c-\alpha} \cdot \widehat{yz}^{\alpha} \cdot \widehat{zx}^{c'-c} \cdot \chi_{c'-c} \\
&\equiv (c'-c)!\,(c'-c)^{(\alpha)} \cdot (\sigma - b' - 2c - \alpha)^{(c'-c-\alpha)} \cdot \widehat{xy}^{\alpha} \cdot \widehat{yx}^{b'-b+\alpha} \cdot \psi_{b'-b} \\
&\equiv (c'-c)!\,(c'-c)^{(\alpha)} \cdot (\sigma - b' - 2c - \alpha)^{(c'-c-\alpha)} \cdot (b'-b+\alpha)^{(\alpha)} \\
&\qquad \times (\sigma - b - b' - c)^{(\alpha)} \cdot yx^{b'-b} \cdot \psi_{b'-b} \\
&\equiv \frac{(c'-c)!\,(\sigma - b - b' - c)!}{(\sigma - b' - c - c')!\,(b'-b)!} \cdot (c'- \quad)^{(\alpha)} \cdot \frac{(\sigma - b' - 2c - \alpha)!\,(b'-b+\alpha)!}{(\sigma - b - b' - c - \alpha)!} \cdot \chi_{c'-c}
\end{aligned}$$

and, by (23),

$$\widehat{xz}^{c'-c} \cdot \{z;\ y\} \cdot \widehat{zx}^{c'-c} \cdot \chi_{c'-c} \equiv S_{c'-c} \cdot \chi_{c'-c}, \tag{54}$$

where, by (15),

$$\begin{aligned}
S_{c'-c} &= \frac{(c'-c)!\,(b'-c'+1)!\,(\sigma - b - b' - c)!}{(b'-b)!\,(\sigma - b' - c - c')!} \cdot \sum_{\alpha=0}^{c'-c} (-1)^{\alpha} \cdot \binom{c'-c}{\alpha} \\
&\qquad \times \frac{(\sigma - b' - 2c - \alpha)^{(b-c)}}{(b'-c'+\alpha+1)^{(b-c'+1)}} \\
&= \frac{(c'-c)!\,(b'-c'+1)!\,(\sigma - b - b' - c)!}{(b'-b)!\,(\sigma - b' - c - c')!} \cdot S^{b-c'+1,\ c'-c}_{\sigma - b' - 2c,\ b'-c'+1},
\end{aligned}$$

which, by (14) and (16), is a *non-vanishing* constant if $c \leqq c' \leqq b \leqq b' \leqq \sigma - b - c$; so that, at least when these conditions are satisfied, the derivative of the form $\{z;\, y\} \cdot zx^{c'-c} \cdot \chi_{c'-c}$ quâ $\widehat{xz}$ is a non-vanishing constant multiple of $\chi_{c'-c}$. Observe that the condition $b' \leqq \sigma - b - c$ is satisfied if $b \leqq b' \leqq \frac{1}{2}(\sigma - c)$.

But if ϕ is a yz-differentiant of type $(\sigma - b' - c',\, b',\, c')$ and of rank $c' - c$ quâ $\widehat{xz}$, its derivative quâ $\widehat{xz}$, namely $\widehat{xz}^{c'-c} \cdot \phi$, is an xz- and yz-differentiant of type $(\sigma - b' - c,\, b',\, c)$ and, say, of rank $b' - b$ quâ $\widehat{xy}$, so that $\widehat{xy}^{b'-b} \cdot \widehat{xz}^{c'-c} \cdot \phi$ is a *non-vanishing* xyz-differentiant of type $(\sigma - b - c,\, b,\, c)$ for which, by theorems A and F of § 8, $0 \leqq c \leqq c' \leqq b' \leqq \sigma - c - c'$, $c \leqq b \leqq b'$, and $b \leqq \sigma - b' - c$; then, by (18), because $\widehat{yz} \cdot \phi \equiv 0$,

$$\widehat{yz}^{c'-c} \cdot \widehat{xy}^{b'+c'-b-c} \cdot \phi \equiv (b' + c' - b - c)^{(c'-c)} \cdot \widehat{xy}^{b'-b} \cdot \widehat{xz}^{c'-c} \cdot \phi \not\equiv 0,$$

by (14), while

$$\widehat{yz}^{c'-c+1} \cdot \widehat{xy}^{b'+c'-b-c} \cdot \phi \equiv 0;$$

therefore, $\widehat{xy}^{b'+c'-b-c} \cdot \phi$ is a non-vanishing form of type $(\sigma - b - c,\, b + c - c',\, c')$ and rank $c' - c$ quâ $\widehat{yz}$, so that, by theorem F of § 8, $2c' - b - c \leqq c' - c$, that is, $c' \leqq b$: the types of ϕ and its *successive* derivatives quâ $\widehat{xz}$ and quâ $\widehat{xy}$ are, then, so related that $0 \leqq c \leqq c' \leqq b \leqq b' \leqq \sigma - b - c$.

If, then, $0 \leqq c \leqq c' \leqq b' \leqq \frac{1}{2}(\sigma - c')$, the xz- and yz- differentiant $\chi_{c'-c}$ of type $(\sigma - b' - c,\, b',\, c)$ subordinate to $(\sigma - b' - c',\, b',\, c')$ quâ $\widehat{xz}$ and of rank $b' - b$ quâ $\widehat{xy}$ is turned by the operator $\{z;\, y\} \cdot \widehat{zx}^{c'-c}$ into a yz-differentiant of type $(\sigma - b' - c',\, b',\, c')$ satisfying the condition (54) with a non-vanishing value of the constant $S_{c'-c}$ if $c' \leqq b \leqq b'$: the condition (54) is equivalent to (31) if, in the latter, r is replaced by $c' - c$, $\widehat{xy}$ by $\widehat{xz}$, ψ_r by $\chi_{c'-c}$, and ω_r by $\{z;\, y\} \cdot \widehat{zx}^{c'-c}$; and the conditions $c' \leqq b \leqq b'$ are satisfied by *every* yz-differentiant of type $(\sigma - b' - c',\, b',\, c')$ whose derivative quâ $\widehat{xz}$ of type $(\sigma - b' - c,\, b',\, c)$ is of rank $b' - b$ quâ $\widehat{xy}$. Therefore, by (36), if $0 \leqq c' \leqq b' \leqq \frac{1}{2}(\sigma - c')$,

$$\sum_{c=0}^{c'} \sum_{b=c'}^{b'} N_{xz,\, yz}^{\widehat{xy}^{b'-b}}(\sigma - b' - c,\, b',\, c) = N_{yz}(\sigma - b' - c',\, b',\, c')$$

or, by (40),

$$\sum_{c=0}^{c'} \sum_{b=c'}^{b'} N_{xyz}(\sigma - b - c,\, b,\, c) = N_{yz}(\sigma - b' - c',\, b',\, c'). \tag{55}$$

If $1 \leqq c' \leqq b' \leqq \frac{1}{2}(\sigma - c')$, (55) holds also when c' is replaced by $c' - 1$, and when we substract the result of the replacement from (55), member from member, we find

$$\left\{ \begin{aligned} &\sum_{b=c'}^{b'} N_{xyz}(\sigma - b - c',\, b,\, c') - \sum_{c=0}^{c'-1} N_{xyz}(\sigma - c - c' + 1,\, c' - 1,\, c) \\ &\qquad\qquad = (1 - \overline{xz}) \cdot N_{yz}(\sigma - b' - c',\, b',\, c'). \end{aligned} \right. \tag{56}$$

If $c' = 0$, (56) is identical with (55), because $\overline{xz} \cdot N_{yz}(\sigma - b', b', 0) = 0$ and the second sum in (56) vanishes by the general principles of summation; therefore, (56) holds for $0 \leqq c' \leqq b' \leqq \frac{1}{2}(\sigma - c')$.

If $0 \leqq c' < b' \leqq \frac{1}{2}(\sigma - c')$, (56) holds when b' is replaced by $b' - 1$, the difference of the formulæ before and after replacement being

$$(57)\quad N_{xyz}(\sigma - b' - c', b', c') = (1 - \overline{xy})(1 - \overline{xz}) \cdot N_{yz}(\sigma - b' - c', b', c').$$

If $0 \leqq c' = b' \leqq \frac{1}{2}(\sigma - c')$, that is, if $0 \leqq c' \leqq \frac{1}{3}\sigma$ and $b' = c'$, (55) gives

$$\sum_{c=0}^{c'} N_{xyz}(\sigma - c - c', c', c) = N_{yz}(\sigma - 2c', c', c');$$

or, with the replacement of $N_{xyz}(\sigma - c - c', c', c)$ by its expression from (57) for all values of c *less than* c' [for which, evidently, (57) holds],

$$\begin{aligned} N_{xyz}(\sigma - 2c', c', c') &= N_{yz}(\sigma - 2c', c', c') - (1 - \overline{xy}) \cdot \sum_{c=0}^{c'-1} (1 - \overline{xz}) \\ &\times N_{yz}(\sigma - c - c', c', c) = N_{yz}(\sigma - 2c', c', c') - N_{yz}(\sigma - 2c' + 1, c', c' - 1) \\ &+ \overline{xy} \cdot N_{yz}(\sigma - 2c' + 1, c', c' - 1) = (1 - \overline{xz} + \overline{xy} \cdot \overline{xz}) \cdot N_{yz}(\sigma - 2c', c', c') \\ &= (1 - \overline{xy})(1 - \overline{xz}) \cdot N_{yz}(\sigma - 2c', c', c'), - \end{aligned}$$

because $\overline{xy} \cdot N_{yz}(\sigma - 2c', c', c') = N_{yz}(\sigma - 2c' + 1, c' - 1, c') = 0$, by theorem A of § 8, — and this result is what (57) becomes for $b' = c'$.* Therefore, (57) holds for $0 \leqq c' \leqq b' \leqq \frac{1}{2}(\sigma - c')$, that is, for *all* types of which there can be xyz-differentiants in accordance with theorem A of § 8.

On replacing $N_{yz}(\sigma - b' - c', b', c')$ in (57) by its expression from (37), we have, if $0 \leqq c' \leqq b' \leqq \frac{1}{2}(\sigma - c')$,

$$N_{xyz}(\sigma - b' - c', b', c') = (1 - \overline{xy})(1 - \overline{xz})(1 - \overline{yz}) \cdot N(\sigma - b' - c', b', c');$$

after the restoration of w_x, w_y, w_z for $\sigma - b' - c', b', c'$, respectively, we have, if $w_z \leqq w_y \leqq w_x$,

$$(58)\quad N_{xyz}(w_x, w_y, w_z) = (1 - \overline{xy})(1 - \overline{xz})(1 - \overline{yz}) \cdot N(w_x, w_y, w_z),$$

which is identical with (50) and (51). In particular, if $w_y = w_z$,

$$\begin{aligned} \overline{xy} \cdot N(w_x, w_z, w_z) &= N(w_x + 1, w_z - 1, w_z) \\ &= N(w_x + 1, w_z, w_z - 1) = \overline{xz} \cdot N(w_x, w_z, w_z), \end{aligned}$$

so that

$$N_{xyz}(w_x, w_z, w_z) = (1 - \overline{xz} + \overline{xy} \cdot \overline{xz})(1 - \overline{yz}) \cdot N(w_x, w_z, w_z),$$

as we have already seen in the special case of (57) for which $b' = c'$.

* If $c' = 0$, the sum in the first expression for $N_{xyz}(\sigma - 2c', c', c')$ vanishes, but the result is valid, by (55) for $b' = c' = 0$.

22. In combination with (40), (38), and (37), formula (57) gives

$$(59)\quad \begin{cases} N^{\widehat{xy}^r}_{xz,\,yz}(w_x, w_y, w_z) = (1-\overline{xy})(1-\overline{xz})\cdot\overline{xy}^r\cdot N_{yz}(w_x, w_y, w_z) \\ \qquad\qquad = (1-\overline{xy})(1-\overline{xz})(1-\overline{yz})\cdot\overline{xy}^r\cdot N(w_x, w_y, w_z) \end{cases}$$

if $w_z \leqq w_y \leqq w_x$ and $0 \leqq r \leqq w_y - w_z$, and

$$\begin{aligned} N_{xz,\,yz}(w_x, w_y, w_z) &= (1-\overline{xz})\cdot\sum_{r=0}^{w_y-w_z}(1-\overline{xy})\cdot\overline{xy}^r\cdot N_{yz}(w_x, w_y, w_z) \\ &= (1-\overline{xz})\cdot[N_{yz}(w_x, w_y, w_z) - \overline{xy}\cdot N_{yz}(w_x+w_y-w_z, w_z, w_z)] \\ &= (1-\overline{xz})\cdot N_{yz}(w_x, w_y, w_z) + \overline{xy}\cdot\overline{xz}\cdot N_{yz}(w_x+w_y-w_z, w_z, w_z), \end{aligned}$$

that is, by (37),

$$(60)\quad \begin{cases} N_{xz,\,yz}(w_x, w_y, w_z) \\ = (1-\overline{xz})\cdot N_{yz}(w_x, w_y, w_z) + N_{yz}(w_x+w_y-w_z+2, w_z-1, w_z-1) \\ = (1-\overline{xz})(1-\overline{yz})\cdot N(w_x, w_y, w_z) + (1-\overline{yz})\cdot N(w_x+w_y-w_z+2, w_z-1, w_z-1) \end{cases}$$

if $w_z \leqq w_y \leqq w_x$. If $w_z = 0$, the last term in each of the expressions for $N_{xz,\,yz}(w_x, w_y, w_z)$ vanishes and we have

$$N_{xz,\,yz}(w_x, w_y, 0) = N_{yz}(w_x, w_y, 0) = N(w_x, w_y, 0),$$

as it should be, because every form of weight 0 in z is an xz- and yz-differentiant, by the footnote to § 5.

Because the sequence of variables is unessential, (59) gives, if $w_z \leqq w_x \leqq w_y$ and $0 \leqq r \leqq w_x - w_z$,

$$\begin{aligned} N^{\widehat{xy}^r}_{xz,\,yz}(w_y, w_x, w_z) &= (1-\overline{xy})(1-\overline{xz})(1-yz)\cdot N(w_y+r, w_x-r, w_z) \\ &= N^{\widehat{y.x}^r}_{yz,\,xz}(w_x, w_y, w_z) = N^{\widehat{xy}^{w_y-w_x+r}}_{xz,\,yz}(w_x, w_y, w_z), \end{aligned}$$

by (19), from which follows, when r is replaced by $r + w_x - w_y$,

$$N^{\widehat{xy}^r}_{xz,\,yz}(w_x, w_y, w_z) = (1-\overline{xy})(1-\overline{xz})(1-\overline{yz})\cdot N(w_x+r, w_y-r, w_z)$$

if $w_z \leqq w_x \leqq w_y$ and $w_y - w_x \leqq r \leqq w_y - w_z$; but this formula is identical with (59); therefore, (59) holds if $w_z \leqq w_x$, $w_z \leqq w_y$ and $w_y - w_x \leqq r \leqq w_y - w_z$, without regard to the relative values of w_x and w_y (of course r is not negative); that is, (59) holds for *all* types of which there can be xz- and yz-differentiants in accordance with theorem A of § 8 and for *all* possible ranks of such xz- and yz-differentiants quâ $\widehat{xy}$, by theorem F of § 8 and by § 17.

Similarly, (60) gives, if $w_z \leqq w_x \leqq w_y$,

$$\begin{aligned} N_{xz,\,yz}(w_y, w_x, w_z) &= (1-\overline{xz})(1-\overline{yz})\cdot N(w_y, w_x, w_z) \\ &\quad + (1-\overline{yz})\times N(w_y+w_x-w_z+2, w_z-1, w_z-1) = N_{yz,\,xz}(w_x, w_y, w_z) \\ &= (1-\overline{yz})(1-\overline{xz})\times N(w_x, w_y, w_z) \\ &\qquad + (1-\overline{yz})\cdot N(w_y+w_x-w_z+2, w_z-1, w_z-1), \end{aligned}$$

which is identical with (60); therefore (60) holds if $w_z \leqq w_x$ and $w_z \leqq w_y$, without regard to the relative values of w_x and w_y, that is, for *all* types of which there can be xz- and yz-differentiants in accordance with theorem A of § 8. Formulae (50) or (58), (53) and (60) express the denumerants of *all* kinds of double differentiants whose defining pairs involve only *three* of the variables.

23. The expression for the denumerant of double differentiants of a given type whose two defining pairs involve *four* different variables is found in the same manner in which that for simple differentiants was found in § 16. We consider only forms of a *given* order in the variables, of *given* degrees in the coefficients of the several quantics, and of *given* weights in each of the variables *excepting four*, say x, y, z, and s; in representing the type of such a form, we shall write the weights in these four variables alone, following the sequence of the variables as just given, thus: (w_x, w_y, w_z, w_s), so that, by (2), $w_x+w_y+w_z+w_s$ has the same value for *all* forms of our system. In determining the expression for the denumerant $N_{xy,\,zs}(w_x, w_y, w_z, w_s)$, we shall confine our attention to the zs-differentiants that are of *given* weights w_z and w_s in z and s, respectively, — where, by theorem A of § 8, $w_s \leqq w_z$, —and, in writing the type of such a zs-differentiant, shall express the weights in x and y alone, in this sequence, as (w_x, w_y). If four weights are written in the type, it is to be understood that they are the weights in x, y, z, and s, respectively: but if only two weights are written, they are the weights in x and y, respectively, and the weights in z and s are constant for all forms whose types are so written. In the latter representation of the type, we shall replace w_x by a (or a') and w_y by $\sigma - a$ (or $\sigma - a'$), where σ has the same value for *all* the forms considered.

If ϕ is a zs-differentiant of given type $(a, \sigma - a)$, $\widehat{zs}\cdot\phi \equiv 0$ and, by (17), $\widehat{zs}\cdot\widehat{xy}^k\cdot\phi \equiv \widehat{xy}^k\cdot\widehat{zs}\cdot\phi \equiv 0$, so that the differentiant character xy *does not interfere* with the differentiant character zs. The types subordinate to $(a, \sigma - a)$ quâ $\widehat{xy}$ are types $(a', \sigma - a')$ for which $a \leqq a' \leqq \sigma$ (in the notation of § 10, the zs character is the only condition K and $r = a' - a$). If $\psi_{a'-a}$ is an xy- and zs-differentiant of type $(a', \sigma - a')$ subordinate to $(a, \sigma - a)$ quâ $\widehat{xy}$, where $a \leqq a' \leqq \sigma$ and $\frac{1}{2}\sigma \leqq a'$, by theorem A of § 8, then $yx^{a'-a}\cdot\psi_{a'-a}$ is a form of type $(a, \sigma - a)$ for which $\widehat{zs}\cdot\widehat{yx}^{a'-a}\cdot\psi_{a'-a} \equiv \widehat{yx}^{a'-a}\cdot\widehat{zs}\cdot\psi_{a'-a} \equiv 0$, by (17), and

$$\widehat{xy}^{a'-a}\cdot\widehat{yx}^{a'-a}\cdot\psi_{a'-a} \equiv (a'-a)!\,(2a'-\sigma)^{(a'-a)}\cdot\psi_{a'-a},$$

by (18), that is, $\widehat{yx}^{a'-a}\cdot\psi_{a'-a}$ is a zs-differentiant that satisfies condition (31) $[r=a'-a,\ \omega_r \equiv \widehat{yx}^{a'-a}$, and $S_r=(a'-a)!\,(2a'-\sigma)^{(a'-a)}]$ if $0 \leqq a \leqq a' \leqq \sigma$ and $\sigma - a \leqq a'$. Therefore, by (35), if $\frac{1}{2}\sigma \leqq a \leqq \sigma$,

$$N_{xy,\,zs}(a, \sigma - a) = (1 - \overline{xy})\cdot N_{zs}(a, \sigma - a);$$

that is, on restoring w_x for a and w_y for $\sigma - a$, writing the weights in z and s,

and expressing the denumerant of zs-differentiants by (37), we have, if $w_y \leqq w_x$ and $w_s \leqq w_z$,

$$(61) \qquad N_{xy,\,zs}(w_x, w_y, w_z, w_s) = (1 - xy)(1 - \overline{zs}) \cdot N(w_x, w_y, w_z, w_s).$$

Formula (61) shows that, if ϕ is the *most general* form of any type of which there can be xy- and zs-differentiants in accordance with theorem A of § 8, the linear equations between the multipliers of the terms of ϕ that are implied in the identities $\widehat{xy} \cdot \phi \equiv 0$ and $\widehat{zs} \cdot \phi \equiv 0$ are linearly independent in this sense that, when ϕ shall have been determined as the *most general* form of its type $(\cdots)$ that satisfies the identity $\widehat{zs} \cdot \phi \equiv 0$, so that the number of independent arbitrary multipliers of its terms is $N_{zs}(\cdots)$, the number of linearly independent linear equations between these multipliers that are implied in the identity $\widehat{xy} \cdot \phi \equiv 0$ is $\overline{xy} \cdot N_{zs}(\cdots)$.

It is evident that the method of this section can be employed to determine the denumerant of forms of a proper type, in accordance with theorem A of § 8, that have any multiple differentiant character that can be *compounded* of two or more simple or multiple characters of which *no two have a common variable* in their defining pairs when we know the general formulæ for the denumerants of forms having the *several* component characters. For example, if H and K represent two simple or multiple differentiant characters such that no variable in the pairs that define H occurs in the pairs that define K, if we know that $N_H(\cdots) = P \cdot N(\cdots)$ for *every* type $(\cdots)$ of which there can be H-differentiants and that $N_K(\cdots) = Q \cdot N(\cdots)$ for *every* type $(\cdots)$ of which there can be K-differentiants, in accordance with theorem A of § 8, where P and Q are certain type-operators, then, if $(\cdots)$ is any type of which there can be H- and K-differentiants, we shall have $N_{H,K}(\cdots) = PQ \cdot N(\cdots)$.

24. If $w_x = w_y = w_z = w$, we have, by (50) and (51),

$$(62) \left\{ \begin{aligned} N_{xyz}(w, w, w) &= N(w, w, w) - 2N(w+1, w, w-1) \\ &\quad + N(w+2, w-1, w-1) + N(w+1, w+1, w-2) - N(w+2, w, w-2). \end{aligned} \right.$$

This is the culminating formula for denumerants of double differentiants. It gives, if x, y, and z are the only variables, the number of linearly independent *invariants* or *covariants* (according as the order is or is not 0) of any possible type (w, w, w) that belong to any system of *ternary* quantics; namely, as we have stated in § 8, every homogeneous and isobaric xyz-differentiant of equal weights in x and z and (therefore) y is a *complete* differentiant in x, y and z and, therefore, an *invariant* or *covariant* if x, y, and z are the only variables; and, conversely, every invariant or covariant in x, y, and z is a homogeneous isobaric xyz-differentiant of equal weights in x, y, and z. What Cayley's formula does for the invariants and covariants of a *binary* quantic and Sylves-

ter's extension of it for the invariants and covariants of any system of binary quantics, this formula (62) does for the invariants and covariants of any system of *ternary* quantics.

25. Just as (30) follows from (26) and (29) so from theorem D and the equation preceding (45), together with (52) and (53), follows, if $w_z \leqq w_y \leqq w_x$ and $0 \leqq r \leqq w_z$,

$$N_{xy}^{\widehat{xz}^r}(w_x, w_y, w_z) = \sum_{k=0}^{w_z - w_y} N_{xy,\,xz}^{\widehat{yz}^k}(w_x + r, w_y, w_z - r)$$

$$= (1 - \overline{xz}) \cdot \sum_{k=0}^{w_z - w_y} (1 - \overline{yz}) \cdot \overline{yz}^k \cdot N_{xy}(w_x + r, w_y, w_z - r)$$

$$= (1 - \overline{xz}) \cdot [N_{xy}(w_x + r, w_y, w_z - r) - N_{xy}(w_x + r, w_z + 1, w_y + w_z - w_x - r - 1)]$$

$$(63) \quad = N_{xy,\,xz}(w_x + r, w_y, w_z - r) - N_{xy,\,xz}(w_x + r, w_x + 1, w_y + w_z - w_x - r - 1).$$

But the condition $\frac{1}{2}(\sigma - a') \leqq b'$ for the formula preceding (45) might have been replaced by $\sigma - a - b \leqq b'$, as was previously stated, so that the first expression given above for $N_{xy}^{\widehat{xz}^r}(w_x, w_y, w_z)$ holds also if $w_y \leqq w_z \leqq w_x$ and $0 \leqq r \leqq w_z$; excepting that,

if $w_y \leqq w_z \leqq w_x$ and $0 \leqq r \leqq w_z - w_y$, the lower limit of k is $w_z - w_y - r$ and the first term of the developed expression is $(1 - \overline{xz}) \cdot N_{xy}(w_x + r, w_z - r, w_y)$, which, by (37), is the same as $(1 - \overline{xz}) \cdot N_{xy}(w_x + r, w_y, w_z - r)$, as before, and

if $w_y \leqq w_x$, $w_z \leqq w_x$, and $w_y + w_z - w_x \leqq r \leqq w_z$, the upper limit of k is $w_z - r$ and the second term of the developed expression vanishes, together with the second term of (63). Therefore, (63) holds if $w_y \leqq w_x$, $w_z \leqq w_x$, and $0 \leqq r \leqq w_z$, that is, for *every* type of which there can be xy- and xz-differentiants in accordance with theorem A and for *every* possible rank quâ $\widehat{xz}$ of such differentiants.

Formula (63) shows that the number of linearly independent xy- and xz-differentiants of any type $(w_x + r, w_y, w_z - r)$ subordinate to (w_x, w_y, w_z) quâ $\widehat{xz}$ that must be annexed to those derivable quâ $\widehat{xz}$ from xy-differentiants of type (w_x, w_y, w_z) in order to produce a *complete* system is

$$N_{xy,\,xz}(w_x + r, w_x + 1, w_y + w_z - w_x - r - 1) \text{ if } w_y \leqq w_x \text{ and } w_z \leqq w_x.$$

If $w_y + w_z - w_x \leqq r \leqq w_z$, *every* xy- and xz-differentiant of type $(w_x + r, w_y, w_z - r)$ is derivable quâ $\widehat{xz}$ from the xy-differentiants of type (w_x, w_y, w_z), provided $w_y \leqq w_x$ and $w_z \leqq w_x$. In particular, we find (on putting $r = 1$ and replacing w_x by $w_x - 1$) that the number of linearly independent xy- and xz-differentiants that must be annexed to those derivable quâ $\widehat{xz}$ from the xy-differentiants of type $(w_x - 1, w_y, w_z + 1)$, where $w_y \leqq w_x - 1$ and $w_z \leqq w_x - 1$, in order to produce a *complete* system of xy- and xz-differentiants of type (w_x, w_y, w_z) is

$N_{xy,\,xz}(w_x, w_x, w_y + w_z - w_x)$, which is evidently $N_{xyz}(w_x, w_x, w_y + w_z - w_x)$, because, by theorems B and C of § 8, every xy- and xz-differentiant of equal weights in x and y is also a yx- and yz-differentiant and, therefore, an xyz-differentiant; but no such differentiants have to be annexed if $w_y + w_z < w_x$.

It is evident that the result of applying $\widehat{xz}$ to any xy-differentiant of type $(w_x - 1, w_y, w_z + 1)$, where $w_y \leqq w_x - 1$, is an xy-differentiant of type (w_x, w_y, w_z), if not identically 0, and the number of xy-differentiants in a complete system of type $(w_x - 1, w_y, w_z + 1)$ *reduced* quâ $\widehat{xz}$ that are annihilated by $\widehat{xz}$ is $N_{xy,\,xz}(w_x - 1, w_y, w_z + 1)$, so that the number of linearly independent xy-differentiants of type (w_x, w_y, w_z) that can be obtained by applying the operator $\widehat{xz}$ to xy-differentiants of type $(w_x - 1, w_y, w_z + 1)$ is

$$(64)\qquad \begin{cases} N_{xy}(w_x - 1, w_y, w_z + 1) - N_{xy,\,xz}(w_x - 1, w_y, w_z + 1) \\ \qquad = N_{xy}(w_x, w_y, w_z) - N_{xy}(w_x, w_x, w_y + w_z - w_x), \end{cases}$$

by (53), if $w_y \leqq w_x - 1$ and $w_z \leqq w_x - 2$; that is, the number of linearly independent xy-differentiants of type (w_x, w_y, w_z) that must be annexed to those obtained by applying the operator $\widehat{xz}$ to xy-differentiants of type $(w_x - 1, w_y, w_z + 1)$ in order to produce a *complete* system of xy-differentiants of type (w_x, w_y, w_z) is $N_{xy}(w_x, w_x, w_y + w_z - w_x)$; but this number is 0 if $w_y + w_z < w_x$, and then *no* such differentiants have to be annexed in order to complete the system. If $w_x - 1 \leqq w_z$, there are *no* xz-differentiants of type $(w_x - 1, w_y, w_z + 1)$, and the forms obtained by applying $\widehat{xz}$ to a complete system of xy-differentiants of that type are *linearly independent* xy-differentiants of type (w_x, w_y, w_z), which, however, do *not* generally constitute a complete system.

Similarly, from the equation preceding (55), together with (59) and (60), follows, if $w_z \leqq w_x$, $w_z \leqq w_y$, and $0 \leqq r \leqq w_z$,

$$N_{yz}^{\widehat{xz}^r}(w_x, w_y, w_z) = \sum_{k=0}^{w_y - w_z} N_{xz,\,yz}^{\widehat{xy}^k}(w_x + r, w_y, w_z - r)$$

$$= (1 - \overline{xz}) \cdot \sum_{k=0}^{w_y - w_z} (1 - \overline{xy}) \cdot \overline{xy}^k \cdot N_{yz}(w_x + r, w_y, w_z - r)$$

$$= (1 - \overline{xz}) \cdot [N_{yz}(w_x + r, w_y, w_z - r) - N_{yz}(w_x + w_y - w_z + r + 1, w_z - 1, w_z - r)]$$

$$(65)\quad = N_{xz,\,yz}(w_x + r, w_y, w_z - r) - N_{xz,\,yz}(w_x + w_y - w_z + r + 1, w_z - 1, w_z - r);$$

so that the number of linearly independent xz- and yz-differentiants of type $(w_x + r, w_y, w_z - r)$ that must be annexed to those derivable quâ $\widehat{xz}$ from yz-differentiants of type (w_x, w_y, w_z) in order to produce a *complete* system is $N_{xz,\,yz}(w_x + w_y - w_z + r + 1, w_z - 1, w_z - r)$. In particular, we find (on putting $r = 1$ and replacing w_z by $w_z + 1$) that the number of linearly inde-

pendent xz- and yz-differentiants of type (w_x, w_y, w_z) that must be annexed to those derivable quâ $\widehat{xz}$ from yz-differentiants of type $(w_x - 1, w_y, w_z + 1)$, where $w_z + 1 \leqq w_x$ and $w_z + 1 \leqq w_y$, in order to produce a *complete* system of xz- and yz-differentiants of type (w_x, w_y, w_z) is $N_{xz,yz}(w_x + w_y - w_z, w_z, w_z)$, which is also $N_{xyz}(w_x + w_y - w_z, w_z, w_z)$.

From (60) follows, if $w_z + 2 \leqq w_x$ and $w_z + 1 \leqq w_y$,

$$(66) \quad \begin{cases} N_{yz}(w_x - 1, w_y, w_z + 1) - N_{xz,yz}(w_x - 1, w_y, w_z + 1) \\ \qquad\qquad = N_{yz}(w_x, w_y, w_z) - N_{yz}(w_x + w_y - w_z, w_z, w_z), \end{cases}$$

which shows that the number of linearly independent yz-differentiants of type (w_x, w_y, w_z) that must be annexed to those obtained by applying the operator $\widehat{xz}$ to yz-differentiants of type $(w_x - 1, w_y, w_z + 1)$ in order to produce a *complete* system of yz-differentiants of type (w_x, w_y, w_z) is $N_{yz}(w_x + w_y - w_z, w_z, w_z)$. If $w_x \leqq w_z + 1$, there are *no* xz-differentiants of type $(w_x - 1, w_y, w_z + 1)$ and the forms obtained by applying $\widehat{xz}$ to a complete system of yz-differentiants of that type are *linearly independent* yz-differentiants of type (w_x, w_y, w_z), which, however, do *not* generally constitute a complete system.

26. Because we have always excluded denumerants of such types that their values as given by our formulæ might come out negative, these formulæ may serve to determine the *relative* values of some denumerants involved in the expressions of others, just as (37) determined the relative numbers of terms in the general forms of certain types.

It follows from (45) that, if $w_z \leqq w_y \leqq w_x$,

$$N_{xy}(w_x, w_y + 1, w_z - 1) \leqq N_{xy}(w_x, w_y, w_z);$$

that is, of two types of which there can be xyz-differentiants in accordance with theorem A of § 8 and of which the one is subordinate to the other quâ $\overline{yz}$, that one has the more linearly independent xy-differentiants for which the weights in y and z are the more nearly equal, unless they have the same number.

It follows from (55) that, if $w_z \leqq w_y \leqq w_x$,

$$N_{yz}(w_x + 1, w_y - 1, w_z) \leqq N_{yz}(w_x, w_y, w_z);$$

or, because the sequence of variables is immaterial, if $w_y \leqq w_x \leqq w_z$,

$$N_{xy}(w_x - 1, w_y, w_z + 1) \leqq N_{xy}(w_x, w_y, w_z);$$

that is, of two types of which there can be zxy-differentiants in accordance with theorem A and of which the one is subordinate to the other quâ $\widehat{zx}$ (or $\widehat{xz}$), that one has the more linearly independent xy-differentiants for which the weights in x and z are the more nearly equal, unless they have the same number.

It follows from (61) and (37) that, if $w_y \leqq w_x$ and $w_s \leqq w_z$,

$$N_{xy,zs}(w_x, w_y, w_z, w_s) = (1 - \overline{zs}) \cdot N_{xy}(w_x, w_y, w_z, w_s)$$

and, therefore,

$$N_{xy}(w_x, w_y, w_z + 1, w_s - 1) \leqq N_{xy}(w_x, w_y, w_z, w_s).$$

Similarly, if $w_y \leqq w_x$ and $w_z \leqq w_s$,

$$N_{xy}(w_x, w_y, w_z - 1, w_s + 1) \leqq N_{xy}(w_x, w_y, w_z, w_s);$$

that is, of two types of which there can be xy-differentiants in accordance with theorem A and of which the one is subordinate to the other relatively to some pair of *other* variables than x and y, that one has the more linearly independent xy-differentiants for which the weights in the two other variables are the more nearly equal, unless they have the same number.

It follows from (39) that, if $w_z \leqq w_y \leqq w_x$,

$$N_{xz,\,yz}(w_x + 1, w_y - 1, w_z) \leqq N_{xz,\,yz}(w_x, w_y, w_z)$$

and, if $w_z \leqq w_x \leqq w_y$,

$$N_{xz,\,yz}(w_x - 1, w_y + 1, w_z) \leqq N_{xz,\,yz}(w_x, w_y, w_z);$$

that is, of two types of which there can be xz- and yz-differentiants and of which the one is subordinate to the other quâ xy, that one has the more linearly independent xz- and yz-differentiants for which the weights in x and y are the more nearly equal, unless they have the same number.

It follows from (42) that if $w_z \leqq w_y \leqq w_x$,

$$N_{xy,\,xz}(w_x, w_y + 1, w_z - 1) \leqq N_{xy,\,xz}(w_x, w_y, w_z)$$

and, if $w_y \leqq w_z \leqq w_x$,

$$N_{xy,\,xz}(w_x, w_y - 1, w_z + 1) \leqq N_{xy,\,xz}(w_x, w_y, w_z);$$

that is, of two types of which there can be xy- and xz-differentiants and of which the one is subordinate to the other quâ $\widehat{yz}$, that one has the more linearly independent xy- and xz-differentiants for which the weights in y and z are the more nearly equal, unless they have the same number.

For the actual numerical calculation of the denumerants considered in this paper nothing is wanting but a formula for the *number of terms* in the *general form* of any given type, that is, a formula for the denumerant $N(w_x, w_y, w_z, \cdots)$ for *any* type $(w_x, w_y, w_z, \cdots)$.

In conclusion, I wish to call attention to the fact that the real basis of this whole investigation is SYLVESTER's method, described in his *Proof of the hitherto undemonstrated Fundamental Theorem of Invariants* in the Philosophical Magazine for March, 1878, by which we pass from (29) through (26) to (30).

To facilitate reference I append here a brief table of contents.

Contents.

Introduction:

Historical notice . . . . . § 1

Types and characters of differentiants 2

Analytical conditions for differentiants, — the shear-operator $\widehat{yx}$ . . 3

Complete and reduced systems, — ranks, derivatives 4

Notation for denumerants, — the type-operator $\overline{xy}$ 5

Numerical symbols and the sum $S_{p,q}^{m,n}$ 6

Commutation of shear-operators 7

Certain properties of differentiants, theorems A-F 8

Operators that produce differentiants, $\{y;\ x\}$ etc. 9

Determination of denumerants in general 10–15

Simple differentiants and their denumerants 16

Double differentiants and their denumerants:

Derivation of xyz-differentiants from xz- and yz-differentiants . . . 17

Derivation of xyz-differentiants from xy- and xz-differentiants . . . 18

Derivation of xyz-differentiants from xy-differentiants, — final form of denumerants for xyz-differentiants 19

Denumerants for xy- and xz-differentiants 20

Derivation of xyz-differentiants from yz-differentiants 21

Denumerants for xz- and yz-differentiants 22

Denumerants for xy- and zs-differentiants 23

Denumerants for ternary co- and invariants 24

Completion of derived systems 25

Relative numbers of terms in differentiants of different types . . . 26

Worcester, Mass.,
October 20, 1906.

THE GROUP OF CLASSES OF CONGRUENT MATRICES WITH APPLICATION TO THE GROUP OF ISOMORPHISMS OF ANY ABELIAN GROUP*

BY

ARTHUR RANUM

Introduction.

An ordinary linear congruence group, or JORDAN † group, may be considered as a group of matrices whose elements (coefficients) are all residues of the same modulus m, or in other words as a group of classes of congruent matrices, mod m. Among the writers on these groups, besides JORDAN, are CAUCHY, MATHIEU, MOORE, DICKSON (for the case where m is prime) and GIERSTER (for the case where m is composite).

The subject of the first part of this paper is a generalization of the JORDAN group, and is obtained by using different moduli for the different elements of the matrices, so that each element is a residue of its own particular modulus. In this way a more general "group of classes of congruent matrices" is obtained, which includes the JORDAN group as a special case. For the sake of brevity it will sometimes be designated by the shorter title "linear congruence group," although its operators are matrices and not linear substitutions.

The second part is devoted to the application of these more general linear congruence groups to the group of isomorphisms of any abelian group. For the purpose of this application it will be shown that there is no loss of generality in restricting the moduli to being powers of the same prime p. Therefore that case alone will be considered in the first part.

Since every group of finite order can be represented as a group of matrices, the problem of finding the minimum degree of its representation is of considerable interest. In a large number of cases the degree can be made lower by means of these general groups than is possible by means of JORDAN groups alone. In other words, many of these groups cannot be represented as subgroups of JORDAN groups of the same degree; they are new groups of that degree.

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† JORDAN, *Traité des Substitutions*, 1870, pp. 91–110.

PART I.

The group of classes of congruent matrices.

Classes of congruent matrices.

1. Let (l_{ij}) represent an n-ary matrix, or square array, of n^2 integers l_{ij} $(i, j = 1, 2, \cdots, n)$, l_{ij} being the element in the ith row and jth column. We introduce once for all a fixed matrix

$$(p^{a_{ij}})$$

whose elements $p^{a_{ij}}$ are powers of the same prime p with exponents $a_{ij} \geqq 0$, and call two matrices (l_{ij}), (m_{ij}) congruent with respect to the matrix $(p^{a_{ij}})$ as a modulus, in notation

$$(l_{ij}) \equiv (m_{ij}) \mod (p^{a_{ij}}),$$

in case

$$l_{ij} \equiv m_{ij} \mod p^{a_{ij}} \qquad (i, j = 1, 2, \cdots, n).$$

The elements $p^{a_{ij}}$ of the modular matrix $(p^{a_{ij}})$ are referred to as the moduli.

All matrices congruent to (l_{ij}) will be said to belong to the *class* * $((l_{ij}))$. In this way all matrices are distributed among a finite number of mutually exclusive classes.

Composition of classes.

2. Taking the usual law of composition or multiplication of matrices, viz.,

$$(1) \qquad (l_{ij})(l'_{ij}) = (l''_{ij}),\ l''_{ik} = \sum_{j=1}^{n} l_{ij} l'_{jk} \qquad (i, k = 1, \cdots, n),$$

we proceed to prove

THEOREM 1. *If (l_{ij}) and (l'_{ij}) are given matrices and if the matrices (m_{ij}) and (m'_{ij}) range over the classes $((l_{ij}))$ and $((l'_{ij}))$ respectively, the product $(m_{ij})(m'_{ij}) = (m''_{ij})$ will always belong to the class $((l''_{ij}))$, if, and only if, the conditions*

$$(2) \qquad p^{a_{ij}+a_{jk}} \equiv 0, \quad l_{ij} p^{a_{jk}} \equiv 0, \quad p^{a_{ij}} l'_{jk} \equiv 0, \quad \mathit{mod}\ p^{a_{ik}} \quad (i, j, k = 1, \cdots, n),$$

are satisfied.

When the conditions (2) are satisfied, there is defined a *unique law of composition of the classes* $((l_{ij}))$ and $((l'_{ij}))$, in virtue of which their product, in this order, is $((l''_{ij}))$.

* The explicit use of classes of matrices was suggested to me by Professor DICKSON.

3. Proof. The general values of the elements of (m_{ij}) and (m'_{ij}) are

$$m_{ij} = l_{ij} + h_{ij} p^{a_{ij}}, \qquad m'_{ij} = l'_{ij} + h'_{ij} p^{a_{ij}},$$

where h_{ij} and h'_{ij} are any integers. Therefore (m''_{ij}) will always belong to the class $((l''_{ij}))$, if and only if it is possible, whatever be the values of h_{ij} and h'_{ij}, to find values of the integers h''_{ij} $(i, j = 1, \cdots, n)$ to satisfy the equations

$$l''_{ik} + h''_{ik} p^{a_{ik}} = \sum_j (l_{ij} + h_{ij} p^{a_{ij}})(l'_{jk} + h'_{jk} p^{a_{jk}}) \quad (i, k = 1, \cdots, n).$$

All summations, unless otherwise specified, have the range $1, \cdots, n$.

By expansion and the subtraction of (1), these become

$$\sum_j h'_{jk} l_{ij} p^{a_{jk}} + \sum_j h_{ij} l'_{jk} p^{a_{ij}} + \sum_j h_{ij} h'_{jk} p^{a_{ij}+a_{jk}} = h''_{ik} p^{a_{ik}} \quad (i, k = 1, \cdots, n),$$

which are to be identities in the indeterminates h_{ij} and h'_{jk} and are therefore equivalent to the series of congruences (2).

4. Corollary. *The modular matrix* $(p^{a_{ij}})$ *admits the possibility of the composition of certain classes, if, and only if,*

$$a_{ij} + a_{jk} \geqq a_{ik} \qquad (i, j, k = 1, \cdots, n). \tag{3}$$

Sets of classes.

5. Conditions (2) show that unless all the moduli are alike, the composition of classes cannot apply to every * pair of classes. But it will be seen that if the moduli satisfy (3), a certain limited *set* of classes can be found, to which composition applies.

Suppose that $((l^{(s)}_{ij}))$ $(s = 1, 2, \cdots, N)$ are the classes of any such set S, not necessarily the largest set and not necessarily forming a group. Let $p^{\beta_{ij}}$ $(\beta_{ij} \geqq 0)$ be the highest power of p dividing $l^{(1)}_{ij}, \cdots, l^{(N)}_{ij}$, and put

$$l^{(s)}_{ij} = p^{\beta_{ij}} \lambda^{(s)}_{ij} \qquad (i, j = 1, \cdots, n;\ s = 1, \cdots, N), \tag{4}$$

where the factors $\lambda^{(s)}_{ij}$ $(s = 1, \cdots, N)$ cannot all be divisible by p. Since $l^{(s)}_{ij}$ is a residue of $p^{a_{ij}}$, $\lambda^{(s)}_{ij}$ is obviously a residue of $p^{a_{ij}-\beta_{ij}}$. Call $p^{\beta_{ij}}$ a *p-factor*, β_{ij} a *p-exponent*, $\lambda^{(s)}_{ij}$ a *λ-factor*, and $p^{a_{ij}-\beta_{ij}}$ a *λ-modulus*, of the set S.

6. THEOREM 2. *If the moduli* $p^{a_{ij}}$ *satisfy* (3), *the classes of a set* S *are subject to the law of composition, if, and only if, its p-factors* $p^{\beta_{ij}}$ *satisfy the conditions*

$$\beta_{ij} \geqq \begin{cases} a_{ik} - a_{jk}, \\ a_{kj} - a_{ki} \end{cases} \qquad (i, j, k = 1, \cdots, n). \tag{5}$$

* E. g., if $a_{ik} > a_{jk}$, l_{ij} must be a multiple of $p^{a_{ik}-a_{jk}}$.

The conditions (5) for i, j fixed will be designated as $(5)_{ij}$.

Proof. By means of (4) we see that conditions (2), applied to every two equal or distinct classes in S, take the form of the congruences

$$\left.\begin{aligned} p^{\alpha_{ij}+\alpha_{jk}} &\equiv 0 \\ \lambda_{ij}^{(s)} p^{\beta_{ij}+\alpha_{jk}} &\equiv 0 \\ \lambda_{jk}^{(s)} p^{\alpha_{ij}+\beta_{jk}} &\equiv 0 \end{aligned}\right\}, \bmod p^{\alpha_{ik}} \qquad \begin{pmatrix} i, j, k=1, \cdots, n \\ s=1, \cdots, N \end{pmatrix},$$

which are equivalent to the inequalities

$$\begin{aligned} \alpha_{ij} + \alpha_{jk} &\geqq \alpha_{ik}, \\ \beta_{ij} + \alpha_{jk} &\geqq \alpha_{ik}, \\ \alpha_{ij} + \beta_{jk} &\geqq \alpha_{ik}, \end{aligned} \qquad (i, j, k=1, \cdots, n).$$

If in the third inequality k is replaced by j, j by i, and i by k, it becomes $\alpha_{ki} + \beta_{ij} \geqq \alpha_{kj}$, which, with the second inequality, gives the required conditions.

Sets closed under composition.

7. Let S_β be the set obtained by giving to $\lambda_{ij}^{(s)}$ in (4) all integral values, mod $p^{\alpha_{ij}-\beta_{ij}}$.

THEOREM 3. *The set S_β of classes, whose p-exponents β_{ij} satisfy* (5), *is closed under composition, if, and only if, they also satisfy the conditions*

$$\beta_{ij} + \beta_{jk} \geqq \beta_{ik} \qquad (i, j, k=1, \cdots, n). \tag{6}$$

Proof. If S_β is closed under composition, then the product

$$((l_{ij}^{(s)}))((l_{ij}^{(t)})) = ((l_{ij}^{(u)}))$$

is in S_β, when its factors are. If in the equations

$$l_{ik}^{(u)} = \sum_j l_{ij}^{(s)} l_{jk}^{(t)} \qquad \begin{pmatrix} i, k=1, \cdots, n \\ s, t=1, \cdots, N \end{pmatrix}$$

the p-factors are introduced by means of (4), they become

$$p^{\beta_{ik}} \lambda_{ik}^{(u)} = \sum_j p^{\beta_{ij}+\beta_{jk}} \lambda_{ik}^{(s)} \lambda_{ik}^{(t)} \quad (i, k=1, \cdots, n;\ s, t=1, \cdots, N),$$

which must be satisfied for integral values of $\lambda_{ik}^{(u)}$. This can happen only if (6) is satisfied. Conversely, if (6) is satisfied, S_β is closed.

The largest set S_α closed under composition.

8. Denote by α_{ij} the *smallest* value of β_{ij} which satisfies (5), i. e. for $i, j = 1, \cdots, n$,

$(7)_{ij}$ α_{ij} = the *greatest* of the $2n+1$ quantities $0, a_{ik}-a_{jk}, a_{kj}-a_{ki}$ $(k=1, \cdots, n)$. Then the set S_α of all classes $((l_{ij}^{(s)}))$, in which

$$(8) \qquad 0 \leqq l_{ij}^{(s)} < p^{\alpha_{ij}}, \qquad l_{ij}^{(s)} \equiv 0, \quad \text{mod } p^{\alpha_{ij}} \qquad (i, j=1, \cdots, n),$$

is the totality of all the classes to which composition applies, for given moduli $p^{\alpha_{ij}}$ satisfying (3).

By comparing (7) and (3), we see that

$$(9) \qquad \alpha_{ij} \leqq a_{ij} \qquad (i, j=1, \cdots, n).$$

THEOREM 4. *The set S_α is closed under composition.*

Proof: From $(7)_{ij}$ and $(7)_{ik}$ we have for $r=1, \cdots, n$,

$$\alpha_{ij} \geqq a_{ir} - a_{jr}, \qquad \alpha_{jk} \geqq a_{jr} - a_{kr},$$

so that

$$\alpha_{ij} + \alpha_{jk} \geqq a_{ir} - a_{kr}.$$

Similarly

$$\alpha_{ij} + \alpha_{jk} \geqq a_{rk} - a_{ri},$$

and obviously

$$\alpha_{ij} + \alpha_{jk} \geqq 0.$$

Hence, by $(7)_{ik}$,

$$(10) \qquad \alpha_{ij} + \alpha_{jk} \geqq \alpha_{ik},$$

that is, the p-exponents of S_α satisfy (6).

The chief group.

9. The classes of S_α clearly do not form a group.* We shall prove however

THEOREM 5. *If the moduli satisfy* (3) *and the p-exponents are defined by* (7), *the totality G_α of classes of matrices $((l_{ij}^{(s)}))$ $(s=1, \cdots, N_\alpha)$, of elements $l_{ij}^{(s)}$ satisfying* (8) *with determinants $\Delta = |l_{ij}|$ prime to p,*

$$(11) \qquad (\Delta, p) = 1,$$

form a group† G_α of order N_α.

As G_α is the largest group satisfying (11), we shall call it the *chief group of classes of congruent matrices modulo* $(p^{\alpha_{ij}})$, *or the chief linear congruence group modulo* $(p^{\alpha_{ij}})$.

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* E. g., although $((l_{ij}))$, where $l_{ij}=0$ $(i, j=1, \cdots, n)$, is a class of S_α, it cannot belong to a group of order >1.

† We might also select classes, for which Δ is *divisible by* p, to form a group. E. g., with moduli all 3, the classes $L=\left(\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)\right)$ and $L^2=\left(\left(\begin{smallmatrix}2&2\\2&2\end{smallmatrix}\right)\right)=I$ form a group of order 2.

10. Proof.* (*a*) G_a is closed under composition. For, S_a is closed, and if the determinants of two matrices are prime to p, the determinant of their product is prime to p.

(*b*) The identity class I for all groups satisfying (11) is clearly $((\delta_{ij}))$, where

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

This class occurs in G_a, because by (7) we see that $\alpha_{ii} = 0$ $(i = 1, \cdots, n)$, i. e., every p factor in the axis is 1.

(*c*) Finally if $((l_{ij}))$ is any class of G_a, its inverse $((l'_{ij}))$ exists and is in G_a. For, let m_{ij} be the cofactor of l_{ij} in the expansion of $\Delta = |l_{ij}|$. Then, in view of (8), setting $l_{ij} = p^{\alpha_{ij}}\lambda_{ij}$, we write the identities

$$\sum_j l_{ij} m_{kj} = \Delta\delta_{ik} \qquad (i, k = 1, \cdots, n)$$

in the form

$$\sum_j \lambda_{ij} \cdot p^{\alpha_{ij}} m_{kj} = \Delta\delta_{ik} \qquad (i, k = 1, \cdots, n). \tag{12}$$

Now, in view of (11), there exists precisely one class $((l'_{jk}))$ satisfying the conditions

$$0 \leqq l'_{jk} < p^{\alpha_{jk}}, \qquad \Delta l'_{jk} \equiv m_{kj}, \qquad \bmod p^{\alpha_{jk}} \qquad (j, k = 1, \cdots, n). \tag{13}$$

From (13), in view of (7), we have

$$\Delta p^{\alpha_{ij}} l'_{jk} \equiv p^{\alpha_{ij}} m_{kj}, \qquad \bmod p^{\alpha_{ik}} \qquad (i, j, k = 1, \cdots, n),$$

where the modulus is independent of j. Therefore by substitution in (12) we have

$$\Delta \sum_j p^{\alpha_{ij}} \lambda_{ij} l'_{jk} \equiv \Delta\delta_{ik}, \qquad \bmod p^{\alpha_{ik}} \qquad (i, k = 1, \cdots, n).$$

Dividing this by Δ and restoring l_{ij}, we have

$$\sum_j l_{ij} l'_{jk} \equiv \delta_{ik}, \qquad \bmod p^{\alpha_{ik}} \qquad (i, k = 1, \cdots, n),$$

which shows that $((l'_{ij}))$ is the inverse of $((l_{ij}))$ and that $\Delta' = |l'_{ij}|$ is prime to p.

Further $((l'_{ij}))$ satisfies (8). For let $(i_1, i_2, i_3, \cdots, i_r)(j_1, \cdots, j_s), \cdots, (\cdots)$ be any substitution on the integers $1, 2, \cdots, n$ expressed in terms of its inde-

* The main features of the argument used in leading up to this theorem and in proving it are derived from a set of group-postulates given in lectures by MOORE in 1897. Compare also MOORE, these Transactions, vol. 3 (1902), p. 485; vol. 6 (1905), p. 179; and DICKSON, *Ibid.*, vol. 6 (1905), p. 198.

pendent cycles, and such that i_1 goes into i_2. Then by the definition of determinants

$$m_{i_1, i_2} = \sum \pm l_{i_2, i_3} \cdots l_{i_r, i_1} \prod (l_{j_1, j_2} \cdots l_{j_s, j_1}) \quad (i_1, i_2 = 1, \cdots, n),$$

where the product sign extends over all the cycles after the first, and the summation covers all the substitutions which replace i_1 by i_2. Introducing p-factors, we have

$$m_{i_1, i_2} = \sum \pm p^{\alpha_{i_2, i_3} + \alpha_{i_3, i_4} + \cdots + \alpha_{i_r, i_1}} \lambda_{i_2, i_3} \lambda_{i_3, i_4} \cdots \lambda_{i_r, i_1} \prod (l_{j_1, j_2} \cdots l_{j_2, j_1}) \quad (i_1, i_2 = 1, \cdots, n).$$

But by the repeated application of (10) we see that

$$\alpha_{i_2, i_3} + \alpha_{i_3, i_4} + \cdots + \alpha_{i_r, i_1} \geqq \alpha_{i_2, i_1},$$

and therefore that

$$m_{i_1, i_2} \equiv 0, \qquad \bmod p^{\alpha_{i_2, i_1}} \qquad (i_1, i_2 = 1, \cdots, n).$$

From this, by means of (13) and (9), we see that

$$l'_{ij} \equiv 0, \qquad \bmod p^{\alpha_{ij}} \qquad (i, j = 1, \cdots, n),$$

and consequently that $((l'_{ij}))$ satisfies (8). Hence $((l'_{ij}))$ is in S_α and, since $(\Delta', p) = 1$, it is also in G_α.

Change of nomenclature.

11. Hereafter the only classes of matrices considered will be those of S_α; and it will be convenient to regard all *congruent matrices as identical*, so that each class reduces to a single matrix, which is usually chosen so that each element is the least positive residue of its corresponding modulus. We have then in S_α a finite number of distinct matrices, one from each class, forming a closed set under composition. The law of composition will read

$$(l_{ij})(l'_{ij}) = (l''_{ij}), \qquad l''_{ik} \equiv \sum_j l_{ij} l'_{jk}, \qquad \bmod p^{\alpha_{ik}} \quad (i, k = 1, \cdots, n).$$

Subgroups G_β of the chief group G_α.

12. Every linear congruence group, whose matrices satisfy the condition $(\Delta, p) = 1$, is obviously a subgroup of the chief group. This condition also shows that its p-factors $p^{\beta_{ij}}$ $(\beta_{ij} \geqq \alpha_{ij})$ cannot all be > 1; in particular, since it contains the matrix (δ_{ij}), its axial p-factors are 1, i. e., $\beta_{ii} = 0$ $(i = 1, \cdots, n)$.

Of particular interest are the subgroups whose λ-factors λ_{ij} include all integral values, mod $p^{\alpha_{ij} - \beta_{ij}}$, that satisfy (11). In regard to them, by reference to theorems 3 and 5, we immediately derive

THEOREM 6. *If the moduli satisfy* (3) *and the p-factors satisfy* (5), *the totality G_β of the matrices whose elements, of the form* (4), *satisfy* (11), *form*

a group G_β, a subgroup of the chief group G_α, if, and only if, the p-factors also satisfy (6).

Equimodular properties.

13. If two or more rows are equimodular, i. e., have identically the same moduli, and if the corresponding columns are also equimodular, the process of interchanging rows and their corresponding columns may be employed, if necessary, to juxtapose these rows and these columns. Hereafter it will be assumed, therefore, that equimodular rows and their corresponding equimodular columns are adjacent.

14. Under this arrangement every matrix will be divided into equimodular rectangles, of which those in the axis are squares. If no two rows and the corresponding columns are equimodular, each rectangle reduces to a single element. Throughout any rectangle the p-factors of the chief group are clearly equal to one another by (7). Moreover it will be convenient hereafter to restrict the application of the notation G_β to those subgroups which not only satisfy the conditions of theorem 6, but also have their p-factors $p^{\beta_{ij}}$ equal to ane another throughout every equimodular rectangle. Therefore (§ 12) the p-factors of G_β are 1 throughout every axial square.

15. THEOREM 7. *The determinant Δ of any matrix of the set S_α is congruent, modulo p, to the product of the separate determinants of its axial squares. In particular, if no two rows and the corresponding columns are equimodular, Δ is congruent, modulo p, to the product of its axial elements.*

16. Proof. It will be sufficient to prove that if any term T of the expansion of Δ is prime to p, it is the product of elements taken entirely from the axial squares. Let $T = \pm\, l_{1,i_1} l_{2,i_2} \cdots l_{n,i_n}$, where $i_1, i_2, \cdots, i_n$ represent some permutation of the integers $1, 2, \cdots, n$. Denote any one of the cycles of this permutation by $(j_1, j_2, \cdots, j_r)$. Then

$$T = \pm\, \Pi\,(l_{j_1,j_2} l_{j_2,j_3} \cdots l_{j_r,j_1}),$$

where the product covers all the different cycles. Now if T is prime to p, equation (8) shows that $\alpha_{j_1 j_2} = \cdots = \alpha_{j_r j_1} = 0$. From this and (7) we derive the rn relations

$$0 \geqq a_{j_1,k} - a_{j_2,k}, \cdots, 0 \geqq a_{j_r,k} - a_{j_1,k} \qquad (k = 1, \cdots, n).$$

But these are all equations (and not inequalities), because of the identities

$$(a_{j_1,k} - a_{j_2,k}) + \cdots + (a_{j_r,k} - a_{j_1,k}) = 0 \qquad (k = 1, \cdots, n).$$

Therefore the (j_1)th, (j_2)th, $\cdots$ and (j_r)th rows are equimodular. Similarly the corresponding columns are also equimodular. Consequently the elements

$l_{j_1, j_2}, \cdots, l_{j_m j_1}$ are all contained in the same axial square; and all the factors of T' are contained in axial squares.

17. Corollary. In the matrices of the chief group *every axial square determinant must be prime to* p, and in particular, if no two rows and the corresponding columns are equimodular, every axial element must be prime to p.

Change of notation.

18. In order to take account of equimodular properties, it will be best to use a more explicit notation. Let the number of sets of equimodular rows and corresponding columns be r, and the number of rows in the first i sets be $n_i (i = 1, \cdots, r)$, so that the total number is $n_r = n$ and the number in the ith set is $n_i - n_{i-1}$, provided we define n_0 to be 0. Let (l_{μ_i, ν_j}) be a matrix and l_{μ_i, ν_j} an element in the (ij)th equimodular rectangle, taken mod $p^{\alpha_{ij}}$, where

$$\begin{cases} n_{i-1} < \mu_i \leqq n_i, & 1 \leqq i \leqq r, \\ n_{j-1} < \nu_j \leqq n_j, & 1 \leqq j \leqq r. \end{cases}$$

For any group G_β (as defined in § 14) let

$$l_{\mu_i, \nu_j} = p^{\beta_{ij}} \lambda_{\mu_i, \nu_j} \qquad (\mu_i, \nu_j = 1, \cdots, n). \tag{14}$$

Finally, let the equimodular sets be arranged so that

$$a_{ii} \geqq a_{i+1, i+1} \qquad (i = 1, \cdots, r-1).$$

Hereafter, unless otherwise stated, this notation will be used.

19. THEOREM 8. *The p-exponents of any group G_β or G_α expressed in the modified notation satisfy the conditions*

$$\beta_{ij} + \beta_{ji} > 0 \qquad (i, j = 1, \cdots, n; i \neq j), \tag{15}$$

i. e., if two equimodular rectangles are symmetrically situated with respect to the axis, all the elements of at least one are divisible by p in every matrix of the group.

Proof. From (5) we see that

$$\begin{matrix} \beta_{ij} \geqq a_{ik} - a_{jk}, & a_{kj} - a_{ki} \\ \beta_{ji} \geqq a_{jk} - a_{ik}, & a_{ki} - a_{kj} \end{matrix} \qquad (k = 1, \cdots, r; i, j = 1, \cdots, r).$$

Since for any given i and j $(i \neq j)$ at least one of the above $4r$ differences is $\neq 0$, the theorem follows.

The order of a group G_β.

20. THEOREM 9. *The order N_β of a linear congruence group G_β is given by the formula*

$$N_\beta = p^{\sum_{i,j=1}^{r}(n_i-n_{i-1})(n_j-n_{j-1})(a_{ij}-\beta_{ij})} \prod_{i=1}^{r} \prod_{k=1}^{n_i-n_{i-1}} \left(1-\frac{1}{p^k}\right). \tag{16}$$

Proof. The number of matrices in the set S_β is obviously equal to the product of the λ-moduli $p^{a_{ij}-\beta_{ij}}$. The order of G_β is obtained from this by the insertion of the proper factors as given by JORDAN's* formula, since each axial square forms a JORDAN matrix.

Factors of composition.

21. In G_β consider the totality K_β of matrices (l_{μ_i,ν_j}) which are $\equiv I$, mod $(p^{\delta_{ij}})$, i. e., in which

$$l_{\mu_i,\nu_j} \equiv \delta_{\mu_i,\nu_j}, \quad \bmod p^{\delta_{ij}} \qquad (\mu_i, \nu_j = 1, \cdots, n). \tag{17}$$

This means that in the matrices of K_β the elements in the axis are $\equiv 1$, mod p, the other elements in the axial squares are $\equiv 0$, mod p, and all the remaining elements are $\equiv 0$, mod 1, i. e., are unrestricted.

Now it is easy to see that K_β is an invariant subgroup of G_β. Moreover its order is a power of p, because the number of values of each element is a power of p and the different elements are independent of each other. It is therefore soluble.

The quotient-group G_β/K_β is evidently the *direct product of r axial square Jordan groups, each taken mod p*. Since the composition-factors of JORDAN groups are well known,† those of G_β, and therefore of G_α, are thus determined.

22. THEOREM 10. *K_β is the largest invariant subgroup of G_β whose order is a power of p.*

Proof. If there were a larger subgroup of that kind, K', and if K' contained K_β, it would correspond to an invariant subgroup of G_β/K_β of order a power of p, which would contradict the known properties of JORDAN groups; if K' did not contain K_β, it would clearly lead to a similar contradiction.

Soluble groups.

23. THEOREM 11. *The necessary and sufficient condition that a linear congruence group G_β (or G_α) be soluble is*

(*a*) *if $p > 3$, that no two rows and the corresponding columns are equimodular,*

* JORDAN, *Traité des Substitutions*, 1870, pp. 95–97.

† JORDAN, *Traité*, pp. 99–110.

(*b*) *if* $p = 2$ *or* 3, *that no three rows and the corresponding columns are equimodular.*

Proof. This follows from the fact that an n-ary JORDAN group, mod p, is soluble, if $n = 1$, and insoluble, if $n > 1$, except when $n = 2$ and $p = 2$ or 3.

Invariant matrices.

24. THEOREM 12. *If* $p > 2$, *the invariant matrices* $(l_{\mu_i \nu_j})$ *of any group* G_β *are characterized by the congruences*

$$\left.\begin{array}{lll} (18) & l_{\mu_i \nu_j} = 0 & (\mu_i \neq \nu_j) \\ (19) & l_{\mu_i \mu_i} \equiv l_{\nu_j \nu_j}, & \bmod p^{a_{ij} - \beta_{ij}} \end{array}\right\} \qquad (\mu_i, \nu_j = 1, \cdots, n),$$

and therefore form an axial (abelian invariant) subgroup H *of order* $p^a(p-1)$, *where* $a \geqq a_{11} - 1$.

Proof. The proof will be simplified by using the original notation of § 1. Let (l_{ij}) be an invariant matrix and (m_{ij}) be any matrix whatever of G_β, so that $(l_{ij})(m_{ij}) = (m_{ij})(l_{ij})$. Then the congruences

$$\sum_{j=1}^{n} l_{ij} m_{jk} \equiv \sum_{j=1}^{n} m_{ij} l_{jk}, \qquad \bmod p^{a_{ik}} \qquad (i, k = 1, \cdots, n)$$

must be satisfied by every matrix (m_{ij}) of the group. These congruences may be written

$$(20) \quad (l_{ii} - l_{kk}) m_{ik} + l_{ik}(m_{kk} - m_{ii}) + \sum_{j}{}' (l_{ij} m_{jk} - m_{ij} l_{jk}) \equiv 0, \bmod p^{a_{ik}} \quad (i, k = 1, \cdots, n),$$

where the $'$ on the summation sign is to indicate that $j \neq i$, $j \neq k$.

(*a*) First put $m_{ij} = 0$ $(i \neq j;\ i, j = 1, \cdots, n)$; then (m_{ij}) is axial, $m_{ii} (i = 1, \cdots, n)$ is prime to p, and (20) becomes $l_{ik}(m_{kk} - m_{ii}) \equiv 0$, mod $p^{a_{ik}} (i, k = 1, \cdots, n)$. Moreover, since $p > 2$, it is possible to make $m_{kk} - m_{ii}$ prime to p, if $i \neq k$, and therefore to derive the congruences $l_{ik} \equiv 0$, mod $p^{a_{ik}} (i \neq k;\ i, k = 1, \cdots, n)$, which are equivalent to the equations

$$(21) \qquad l_{ik} = 0 \qquad (i \neq k;\ i, k = 1, \cdots, n).$$

By means of these equations, (20) reduces to the form $(l_{ii} - l_{kk}) m_{ik} \equiv 0$, mod $p^{a_{ik}}$, which may be written $(l_{ii} - l_{kk}) p^{\beta_{ik}} \mu_{ik} \equiv 0$, mod $p^{a_{ik}}$ $(i, k = 1, \cdots, n)$.

(*b*) Now choose (m_{ij}) so that μ_{ik} is prime to p; thus we derive the congruences

$$(22) \qquad l_{ii} \equiv l_{kk}, \qquad \bmod p^{a_{ik} - \beta_{ik}} \qquad (i, k = 1, \cdots, n).$$

Moreover (21) and (22) are evidently sufficient as well as necessary conditions. By change of notation they become (18) and (19).

Corollary. Since (by § 14) $\beta_{ii} = 0$ $(i = 1, \cdots, n)$, therefore in the invariant matrices *the axial elements of any axial square are equal.* Moreover, those of the ith axial square are congruent to the others, mod $p^{a_{ij}-\beta_{ij}}$ and mod $p^{a_{ji}-\beta_{ji}}$ $(j = 1, \cdots, n)$. If we denote the greatest of the integers $a_{ij} - \beta_{ij}$, $a_{ji} - \beta_{ji}$ $(j = 1, \cdots, n)$ by a_i, then by (5) $a_i \leqq a_{ii}$. H is the direct product of a cyclic group of order $p - 1$ and an abelian group of order p^a and type $(a_{11} - 1,\ a_{22} - a_2, \cdots, a_{rr} - a_r)$. If $a_i = a_{ii}$ $(i = 2, \cdots, n)$, *all the axial elements are equal and H is cyclic of order $p^{a_{11}-1}(p-1)$.*

Examples.

25. Example 1. Let the modular matrix be $\left(\begin{smallmatrix} p & p^2 \\ 1 & p \end{smallmatrix}\right)$. Then the p-factors of S_a and of G_a are $\left(\begin{smallmatrix} 1 & p \\ 1 & 1 \end{smallmatrix}\right)$, the matrices of S_a are of the form $\left(\begin{smallmatrix} \lambda_{11} & p\lambda_{12} \\ 0 & \lambda_{22} \end{smallmatrix}\right)$, and they belong to G_a, if λ_{11} and λ_{22} are prime to p. The composition of matrices gives

$$\begin{pmatrix} \lambda_{11} & p\lambda_{12} \\ 0 & \lambda_{22} \end{pmatrix}\begin{pmatrix} \lambda'_{11} & p\lambda'_{12} \\ 0 & \lambda'_{22} \end{pmatrix} = \begin{pmatrix} \lambda_{11}\lambda'_{11} & p(\lambda_{11}\lambda'_{12} + \lambda_{12}\lambda'_{22}) \\ 0 & \lambda_{22}\lambda'_{22} \end{pmatrix}.$$

For the p-factors $(p^{\beta_{ij}}) = \left(\begin{smallmatrix} 1 & p^2 \\ 1 & 1 \end{smallmatrix}\right)$, the matrices of S_β and of G_β are of the form $\left(\begin{smallmatrix} \lambda_{11} & 0 \\ 0 & \lambda_{22} \end{smallmatrix}\right)$. The λ-moduli of G_a are $\left(\begin{smallmatrix} p & p \\ 1 & p \end{smallmatrix}\right)$ and its order is $p(p-1)^2$. K_a is of order p and its matrices are of the form $\left(\begin{smallmatrix} 1 & p\lambda_{12} \\ 0 & 1 \end{smallmatrix}\right)$. If ϵ is a primitive root of p, G_a is generated by the matrices $S = \left(\begin{smallmatrix} \epsilon & 0 \\ 0 & \epsilon \end{smallmatrix}\right)$ of order $p - 1$, $T = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \epsilon \end{smallmatrix}\right)$ of order $p - 1$, and $U = \left(\begin{smallmatrix} 1 & p \\ 0 & 1 \end{smallmatrix}\right)$ of order p, where $T^{-1}UT = U^\epsilon$ and S is invariant. Therefore, if $p > 2$, G_a is the direct product of the cyclic group $\{S\}$ and the metacyclic group $\{T, U\}$.

Example 2. Let the modular exponents a_{ij} be

$$\begin{pmatrix} 3 & 3 & 3 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix}.$$

The last two rows and the last two columns are equimodular. The matrices of G_a are of the form

$$\begin{pmatrix} \lambda_{11} & p^2\lambda_{12} & p^2\lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix},$$

where the axial square determinants λ_{11} and $\left|\begin{smallmatrix} \lambda_{22} & \lambda_{23} \\ \lambda_{32} & \lambda_{33} \end{smallmatrix}\right|$ are prime to p. It is of order $p^{11}(p-1)^2(p^2-1)$ and is insoluble if $p > 3$. Its invariant matrices are of the form

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda + p\mu & 0 \\ 0 & 0 & \lambda + p\mu \end{pmatrix},$$

if $p > 2$, and form a group generated by matrices of orders $p^2(p-1)$ and p.

Example 3. Let the modular exponents be

$$\begin{pmatrix} 6 & 9 & 6 \\ 3 & 5 & 5 \\ 0 & 4 & 2 \end{pmatrix}.$$

Then the p-exponents are

$$\begin{pmatrix} 0 & 4 & 6 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix}$$

and G_a is of order $p^{21}(p-1)^3$.

PART II.

Application to the group of isomorphisms of any abelian group.

26. The great importance of the isomorphisms of a given group is largely due to the fact that they enable us to construct new groups of which the given group is an invariant sub-group. However, very little is known about isomorphisms in general, and even when the given group belongs to the simplest and most fundamental class, viz. of abelian (commutative) groups, the group of isomorphisms has not been thoroughly studied except in a few extremely special cases.

For instance, the *i-group* (group of isomorphisms) of the cyclic group has been discussed by Burnside * and Miller †. It is itself abelian. The i-group of the abelian group of order p^m and type $(1, 1, 1, \cdots)$ was considered by Moore ‡, and was shown by him to be abstractly identical with the Jordan linear congruence group, mod p.

The type $(k, 1)$ was studied by Miller.§ The type $(2, 2)$ was treated

* Burnside, *Theory of Groups of a Finite Order* (1897), pp. 239–242.

† Miller, these Transactions, vol. 4 (1903), pp. 153–160.

‡ Moore, Bulletin of the American Mathematical Society, ser. 2, vol. 2 (1895), pp. 33–43.

§ Miller, these Transactions, vol. 2 (1901), pp. 259–264.

erroneously by O. E. GLENN.* The general type has been barely touched upon. Some general theorems have been proved by MILLER.†

In the present paper the general case of any abelian group of order p^m is considered. There is no loss of generality in this restriction on the order, because the i-group of an abelian group of any other order is the direct product of the i-groups of its SYLOW subgroups, whose orders are powers of primes.

The method used is a generalization of that used by MOORE for the type $(1, 1, 1, \cdots)$. The i-group is represented as a chief group of classes of congruent matrices (linear congruence group), whose moduli are the invariants of the abelian group. By means of this concrete representation of the group, some of its properties are easily found.

The abelian group A.

27. Let A be any abelian group of order p^m and of type

$$(\underbrace{a_1, \cdots, a_1}_{n_1}, \underbrace{a_2, \cdots, a_2}_{n_2-n_1}, \cdots, \underbrace{a_r, \cdots, a_r}_{n_r-n_{r-1}}). \tag{23}$$

Its invariants, $n = n_r$ in number, are divided into r sets of equal invariants, those of the ith set, $n_i - n_{i-1}$ in number, being $p^{a_i}(i = 1, \cdots, r;\ n_0 = 0)$. Suppose them arranged in decreasing order of magnitude, so that

$$a_i > a_{i+1} \qquad (i = 1, \cdots, r-1). \tag{24}$$

The order of A is equal to the product of its invariants, i. e.,

$$p^m = p^{\sum_{i=1}^{r} (n_i - n_{i-1}) a_i}. \tag{25}$$

Let $A_1, \cdots, A_n$ be a system of independent generators of A, those of order p^{a_i} being $A_{n_{i-1}+1}, \cdots, A_{\mu_i}, \cdots, A_{n_i}(i = 1 \cdots, r)$.

The group G of isomorphisms of A.

28. Let G be the i-group of A and let L be any isomorphism of A into itself. Then L must effect a correspondence between the above system of independent generators and another system $A'_1, \cdots, A'_n$, each of which is a product of powers of the original generators. In symbols, we have

$$L : A_{\mu_i} \sim A'_{\mu_i},\ A'_{\mu_i} = \prod_{j=1}^{r} \prod_{\nu_j=n_{j-1}+1}^{n_j} (A_{\nu_j})^{l_{\mu_i \nu_j}} \quad \begin{pmatrix} \mu_i = n_{i-1}+1, \cdots, n_i \\ i = 1, \cdots, r \end{pmatrix},$$

*GLENN, American Mathematical Monthly, vol. 12, no. 11 (1905), pp. 205–207; GLENN's condition $\Delta \not\equiv 0$, mod p^2, is too broad, and his result for the order of the i-group is too large.

†MILLER, Annals of Mathematics, ser. 2, vol. 3 (1902), pp. 183–184.

or, more briefly,

$$L : A_{\mu_i} \sim A'_{\mu_i},\ A'_{\mu_i} = \prod_{\nu_j=1}^{n} (A_{\nu_j})^{l_{\mu_i,\nu_j}} \qquad (\mu_i = 1, \cdots, n). \tag{26}$$

Representation as a linear congruence group.

29. The exponents l_{μ_i,ν_j} $(\mu_i, \nu_j = 1, \cdots, n)$ in (26) may be considered as forming a matrix (l_{μ_i,ν_j}), taken mod p^{a_j} $(j = 1, \cdots, r)$.* Moreover if L' is another isomorphism defined as follows:

$$L' : A'_{\sigma_k} \sim A''_{\sigma_k},\ A''_{\sigma_k} = \prod_{\mu_i=1}^{n} (A'_{\mu_i})^{l'_{\sigma_k,\mu_i}} \qquad (\sigma_k = 1, \cdots, n),$$

then the product $L'L$ becomes

$$L'L : A_{\sigma_k} \sim A''_{\sigma_k},\ A''_{\sigma_k} = \prod_{\nu_j=1}^{n} (A_{\nu_j})^{\sum_{\mu_i=1}^{n} l'_{\sigma_k,\mu_i} l_{\mu_i,\nu_j}} \qquad (\sigma_k = 1, \cdots, n).$$

That is, the composition of the isomorphisms L and L' takes place under exactly the same law as the composition of the matrices (l'_{μ_i,ν_j}) and (l_{μ_i,ν_j}). Therefore since the isomorphisms form a group G, the matrices representing them form a linear congruence group, mod p^{a_j}, which is *abstractly identical* with G. We shall call it G.

30. Since in this case $a_{ij} = a_j$, the moduli satisfy (3). By (7) we see that α_{ij} is the greater of the quantities 0 and $a_j - a_i$, i. e., by (24),

$$\alpha_{ij} = \begin{cases} a_j - a_i, & \text{if } i > j, \\ 0, & \text{if } i \leqq j \end{cases} \qquad (i, j = 1, \cdots, r). \tag{27}$$

Therefore the elements of the matrices of G are of the form

$$l_{\mu_i,\nu_j} = \begin{cases} p^{a_j - a_i} \lambda_{\mu_i,\nu_j}, & \text{if } i > j, \\ \lambda_{\mu_i,\nu_j}, & \text{if } i \leqq j \end{cases} \qquad (\mu_i, \nu_j = 1, \cdots, n). \tag{28}$$

This is also easily verified by means of the fact that in the isomorphism (26), A'_{μ_i} is of order at most p^{a_i}:

31. THEOREM 13. *If an abelian group A has n invariants p^{a_j} $(j = 1, \cdots, r)$, as defined in § 27, its i-group G is abstractly identical with the chief n-ary linear congruence group G_a, modulis p^{a_j}.*† That is, in (28) the λ-factors can have any integral values satisfying (11).

32. Proof. (*a*) If (26) is an isomorphism, (11) is satisfied. For, since the identical isomorphism I makes every generator A_{μ_i} correspond to itself, it is represented by the matrix (δ_{μ_i,ν_j}), whose determinant is $\equiv 1$, mod p. If the

* This makes the moduli the same throughout every column.

† This is not the only representation of G as a chief linear congruence group, but it is the most convenient one.

inverse isomorphism L^{-1} is represented by the matrix (l'_{μ_i, ν_j}), it follows from $LL^{-1} = I$ that $|l_{\mu_i, \nu_j}| \cdot |l'_{\mu_i, \nu_j}| \equiv 1$, mod p, and therefore that $|l_{\mu_i, \nu_j}|$ is prime to p.

(*b*) If (26) is not an isomorphism, (11) is not satisfied, i. e., Δ is divisible by p. For, in that case, either the generators $A'_{\mu_i}(\mu_i = 1, \cdots, n)$ are not independent, or for some value of μ_i, A'_{μ_i} is of order $< p^{a_i}$. In either case there is a relation of the form

$$\prod_{\mu_i=1}^{n} (A'_{\mu_i})^{p^{a'_i} h_{\mu_i}} = 1, \tag{29}$$

where $h_{\mu_i} \not\equiv 0$, mod p $(\mu_i = 1, \cdots, n)$, and

$$a'_i < a_i \text{ for some value of } i. \tag{30}$$

From (26), by substitution in (29), we derive

$$\prod_{\nu_j=1}^{n} (A_{\nu_j})^{\sum_{\mu_i=1}^{n} p^{a'_i} h_{\mu_i} l_{\mu_i, \nu_j}} = 1,$$

and therefore also

$$\sum_{\mu_i=1}^{n} p^{a'_i} h_{\mu_i} l_{\mu_i, \nu_j} \equiv 0, \qquad \text{mod } p^{a_j} \qquad (\nu_j = 1, \cdots, n),$$

which may be expanded, by (28), into the form

$$\sum_{\mu_i=1}^{n_{j-1}} p^{a'_i} h_{\mu_i} \lambda_{\mu_i, \nu_j} + \sum_{\mu_j=n_{j-1}+1}^{n_j} p^{a'_j} h_{\mu_i} \lambda_{\mu_j, \nu_j} + \sum_{\mu_i=n_j+1}^{n_r} p^{a_j-(a_i-a'_i)} h_{\mu_i} \lambda_{\mu_i, \nu_j} \equiv 0, \text{ mod } p^{a_j}. \tag{31}$$

Among the differences $a_i - a'_i$ $(i = 1, \cdots, r)$ there is just one, $a_j - a'_j$, which is greater than every succeeding and less than no preceding difference. That is,

$$a_j - a'_j \begin{cases} > a_i - a'_i, \text{ if } i = j+1, \cdots, r, \\ \geqq a_i - a'_i, \text{ if } i = 1, \cdots, j. \end{cases} \tag{32}$$

Then by (30),

$$a_j - a'_j > 0, \tag{33}$$

and by (24) and (32),

$$a'_i - a'_j > 0, \text{ if } i = 1, \cdots, j-1. \tag{34}$$

Therefore in (31) every term, as well as the modulus, is divisible by $p^{a'_j}$. Dividing out this factor, we obtain

$$\sum_{\mu_i=1}^{n_{j-1}} p^{a'_i - a'_j} h_{\mu_i} \lambda_{\mu_i, \nu_j} + \sum_{\mu_j=n_{j-1}+1}^{n_j} h_{\mu_j} \lambda_{\mu_j, \nu_j} + \sum_{\mu_i=n_j+1}^{n_r} p^{(a_j-a'_j)-(a_i-a'_i)} h_{\mu_i} \lambda_{\mu_i, \nu_j} \equiv 0, \text{ mod } p^{a_j-a'_j},$$

where, by (32), (33), and (34), the modulus and all the terms except those of the middle summation are divisible by p. Therefore, by reducing the modulus to p, we obtain

$$\sum_{\mu_j=n_{j-1}+1}^{n_j} h_{\mu_j}\lambda_{\mu_j,\nu_j} \equiv 0, \bmod p \qquad (\nu_j=n_{j-1}+1, \cdots, n_j),$$

which shows that the jth axial square determinant $|\lambda_{\mu_j,\nu_j}|$ is $\equiv 0$, mod p, and, by theorem 7, that the entire determinant is $\equiv 0$, mod p.

33. Evidently the following converse of theorem 13 holds:

THEOREM 14. *Every chief linear congruence group, whose moduli are the same throughout every column represents the i-group of some abelian group of order a power of p.*

Symmetry.

34. From (28) it follows that λ_{μ_i,ν_j} is taken mod p^{a_k}, where k is i or j, the greater if $i \neq j$. Hence if two equimodular rectangles are symmetrically situated with respect to the axis, the λ-factors λ_{μ_i,ν_j} and $\lambda_{\rho_j,\sigma_i}$ of their elements are *residues of the same modulus*, viz., p^{a_k}.

The order of the i-group.

35. THEOREM 15. *The order N of the i-group G of an abelian group A defined as in § 27 is given by the formula*

$$N = p^{\sum_{i=1}^{r}(n_i^2-n_{i-1}^2)a_i} \prod_{i=1}^{r}\prod_{k=1}^{n_i-n_{i-1}}\left(1-\frac{1}{p^k}\right). \tag{35}$$

Proof. This is easily derived from (16) by means of theorem 13 and § 34. Take the square common to the first i sets of rows and the first i sets of columns and subtract from it the square common to the first $i-1$ sets of rows and columns; the region remaining will contain precisely those elements, $n_i^2 - n_{i-1}^2$ in number, whose λ-factors are taken mod p^{a_i}.

Extreme cases.

36. The two opposite extremes among abelian groups A having n invariants are, (a) that in which the invariants are all distinct, (b) that in which they are all equal ($=p^a$).

(a) In this case the i-group G is soluble, because no two columns of its matrices are equimodular (although the rows are all equimodular). Its order takes the form

$$N = p^{\sum_{i=1}^{n}(2i-1)a_i}\left(1-\frac{1}{p}\right)^n. \tag{36}$$

(*b*) In this case the matrices of G reduce to a single axial square, G becomes a JORDAN group, and its order takes the form

$$N = p^{n^2 a} \prod_{k=1}^{n} \left(1 - \frac{1}{p^k}\right). \tag{37}$$

Since the order of A is p^{na}, it follows that, if p is large, the order of G is approximately equal to the nth power of the order of A.

I-groups as related to Jordan groups of the same degree.

37. As stated in the introduction to Part I, most linear congruence groups are abstractly distinct from JORDAN groups of the same degree. This will now be shown to be true of *all i-groups.*

THEOREM 16. *An i-group G of an abelian group of order p^m having n invariants, i. e., a chief n-ary linear congruence group modulis p^{a_j} $(j = 1, \cdots, r)$, is abstractly distinct from any chief n-ary Jordan group G' modulo p^a,* or subgroup thereof, if $p > n + 1$.*

Proof. A glance at the formulas for the orders of the groups shows that G cannot be abstractly identical with G' itself. If it were possible for G to be abstractly identical with a subgroup of G', its order would have to be a divisor of the order of G'; and this clearly could not happen, unless a were at least $a_r + 1$. In G consider the two matrices

$$R = \begin{bmatrix} 1 & 0 & 0 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \text{ of period } p^{a_r},$$

and

$$S = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ p^{a_1-1} & 0 & 0 & \cdots & 1 \end{bmatrix} \text{ of period } p.$$

Then we have

$$R^{-1}SR = \begin{bmatrix} 1 - p^{a_1-1} & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ p^{a_1-1} & 0 & 0 & \cdots & 1 \end{bmatrix},$$

* The trivial case $r = 1$, $a_1 = a$, $G = G'$, is left out of account.

which is different from S; hence R and S are non-commutative matrices, the product of whose periods is p^{a_r+1}. But in G' it has been shown* that for $p > n+1$ every matrix of period p^b is commutative with every matrix of period p^c, if $b + c \leqq a$. That is, if the product of the periods of two matrices in G' is p^{a_2+1}, they must be commutative. Therefore G' does not contain two non-commutative matrices of the same periods as R and S in G, and does not contain a sub-group abstractly identical with G.

If G is itself a JORDAN group, similar reasoning leads to the

Corollary. A chief n-ary JORDAN group, mod p^a, does not contain a sub-group abstractly identical with any other chief n-ary JORDAN group, mod $p^{a'}$, if $p > n+1$.

38. Since the prime factors of the order of an i-group, besides p, are factors of $p-1$, p^2-1, etc., there results immediately

THEOREM 17. (*a*) *Every abelian group has isomorphisms of even order.* (*b*) *If the invariants of an abelian group of order* 2^m *are all distinct, all its isomorphisms are of order a power of* 2. (*c*) *If an abelian group of order* p^m *has at least two equal invariants, it has isomorphisms whose order is divisible by* 3.

The largest invariant subgroup of order a power of p.

39. The results of Part I, §§ 21–24 can be applied immediately to i-groups and translated into the language of isomorphisms. Thus from §§ 21, 22 we derive

THEOREM 18. *In the i-group of an abelian group A the largest invariant subgroup K of order a power of p consists of the isomorphisms which transform every independent generator of A into itself multiplied by the product of any operator of lower order and any operator of the same order which is the p-th power of an operator of higher order; or, more generally, K consists of the isomorphisms which transform every operator into itself multiplied by the product of any operator of lower order and any operator of the same order which could not be used as an independent generator. Its order is equal to*

$$p^{\sum_{i=1}^{r}[(n_i^2-n_{i-1}^2)a_i-(n_i-n_{i-1})^2]}. \tag{38}$$

Soluble i-groups.

40. From § 23 we derive

THEOREM 19. *The necessary and sufficient condition that the i-group of A be soluble is*

(*a*) *if* $p > 3$, *that all the invariants of A are distinct,*

(*b*) *if* $p = 2$ *or* 3, *that no three of the invariants of A are equal.*

* RANUM, Bulletin of the American Mathematical Society, vol. 13 (1906–7).

Invariant isomorphisms.

41. Finally, from § 24, by identifying the a_i $(i = 2, \cdots, r)$ of that article with the a_i of this section, we derive

THEOREM 20. *In the i-group of A the invariant isomorphisms are those which transform every operator of A into the same power of itself.* They form a cyclic invariant subgroup of order $p^{a_1-1}(p-1)$.*

Examples.

42. Example 1. Let A be an abelian group of order p^3 and type $(2, 1)$. Then any isomorphism L of A into itself may be written

$$L: \begin{cases} A_1 \sim A_1', & A_1' = A_1^{\lambda_{11}} A_2^{\lambda_{12}}, \\ A_2 \sim A_2', & A_2' = A_1^{p\lambda_{21}} A_2^{\lambda_{22}}; \end{cases}$$

and thus the i-group of A may be represented as a chief group G, mod $\left(\begin{smallmatrix} p^2 & p \\ p^2 & p \end{smallmatrix}\right)$, whose matrices are of the form $\left(\begin{smallmatrix} \lambda_{11} & \lambda_{12} \\ p\lambda_{21} & \lambda_{22} \end{smallmatrix}\right)$, where λ_{11} and λ_{22} are prime to p. Its order is $p^3(p-1)^2$. K is of order p^3 and its matrices are of the form $\left(\begin{smallmatrix} 1+p\lambda & \lambda_{12} \\ p\lambda_{21} & 1 \end{smallmatrix}\right)$. If $p > 2$ and ϵ is a primitive root of p^2, we may take as generators $R = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, of order p, $S = \left(\begin{smallmatrix} 1 & 0 \\ p & 1 \end{smallmatrix}\right)$, of order p, $T = \left(\begin{smallmatrix} \epsilon & 0 \\ 0 & \epsilon \end{smallmatrix}\right)$, of order $p(p-1)$, and $U = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \epsilon \end{smallmatrix}\right)$, of order $p-1$. If ϵ is chosen so that $\epsilon^{p-1} \equiv 1 + p$, mod p^2, and if ϵ^{-1} is a root of the congruence $\epsilon x \equiv 1$, mod p^2, then the additional generational relations are $S^{-1}RS = RT^{p-1}$, $U^{-1}RU = R^{\epsilon}$, $U^{-1}SU = S^{\epsilon^{-1}}$, and T is invariant.

Since R and S are non-commutative matrices of order p, G is not abstractly identical with a subgroup of any binary JORDAN group, but, as might be expected, it is abstractly identical with the subgroup of the ternary JORDAN group modulo p, whose matrices are of the form

$$\begin{pmatrix} \alpha & \gamma & \delta \\ 0 & \beta & \epsilon \\ 0 & 0 & \alpha \end{pmatrix}.$$

Example 2. Let A be of order p^{28} and of type $(9, 9, 6, 4)$. Then the modular exponents of G are

$$\begin{bmatrix} 9 & 9 & 6 & 4 \\ 9 & 9 & 6 & 4 \\ 9 & 9 & 6 & 4 \\ 9 & 9 & 6 & 4 \end{bmatrix},$$

* This theorem was proved abstractly by MILLER, these Transactions, vol. 2 (1901), p. 260.

the p exponents are

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 5 & 5 & 2 & 0 \end{bmatrix},$$

and the exponents of p in the λ-moduli are

$$\begin{bmatrix} 9 & 9 & 6 & 4 \\ 9 & 9 & 6 & 4 \\ 6 & 6 & 6 & 4 \\ 4 & 4 & 4 & 4 \end{bmatrix},$$

which are clearly symmetrically situated with respect to the axis. The order of G is $p^{89}(p-1)^3(p^2-1)$ and the order of K is p^{88}.

A THEOREM OF ABEL AND ITS APPLICATION TO THE DEVELOPMENT OF A FUNCTION IN TERMS OF BESSEL'S FUNCTIONS*

BY

CLARA E. SMITH

§ 1. *Introduction.*

THE object of this paper is twofold: first to deduce directly from the properties of double integrals, the formula first given by ABEL,

$$(1) \qquad \frac{2u}{\pi}\int_0^1 dt \int_0^1 \frac{f''(\lambda ut)\lambda d\lambda}{\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}} = f(u) - f(0);$$

and second to apply this formula to obtain sufficient conditions under which a function is developable in a series of BESSEL's functions of multiple values of the argument.

PART I. A THEOREM OF ABEL.

§ 2. *Historical remarks.*

In a note on a mechanical problem, published in 1826,† ABEL gave the inversion formula

$$\phi(x) = \int_0^x \frac{\psi'(y)\,dy}{(x-y)^n}, \qquad \text{where} \qquad \psi(y) = \frac{\sin n\pi}{\pi}\int_0^y \frac{\phi(x)\,dx}{(y-x)^{1-n}} \quad (n<1),$$

which under sufficiently strong conditions is equivalent to

$$\phi(x) - \phi(0) = \frac{2x\sin n\pi}{\pi}\int_0^1 dy \int_0^1 \frac{y^{2n}\phi'(xyz)\,dz}{(1-y^2)^n(1-z^2)^{1-n}}.$$

Relation (1) results from setting $n = \frac{1}{2}$ in this formula. The case $n = \frac{1}{2}$ is the one to which the problem of mechanics considered by ABEL leads, and many authors refer this case alone to him. ABEL's proof which was founded on properties of the Gamma function is incomplete.

* Presented to the Society December 29, 1905 and February 24, 1906. Received for publication June 8, 1906.

† Crelle's Journal, vol. 1 (1826), p. 153, and *Oeuvres Complètes*, vol. 1, p. 97.

In 1857 SCHLÖMILCH* made use of relation (1) in determining the form of a development in terms of BESSEL's functions of multiple values of the argument. Remarking that ABEL's demonstration is not rigorous, he proceeds to give a proof founded on double integrals, but as he says nothing of any conditions, and as the reader is left to suppose that the theorem is universally true, his proof must be placed in the same category with that of ABEL. SONINE† has given a proof resting on properties of BESSEL's functions, but the work is not rigorous, and he says nothing of the conditions under which it will hold. BELTRAMI‡ discusses various forms of relation (1), and their application to SCHLÖMILCH's development, but without stating any conditions for its validity. LEVI-CIVITA§ investigates a very general inversion formula which contains ABEL's as a special case. He finds that the latter is valid for a function which is representable by FOURIER's double integral, and has an integrable derivative. VOLTERRA‖ reviews the literature of the subject, and studies more general formulas of inversion. The most recent proof of ABEL's relation seems to be that of NIELSEN.¶ His demonstration, based on properties of the Gamma function, is valid for a function $f(x)$ whose derivative $f'(x)$ can be developed in a series of functions of the form

$$f'(x) = a_1 p_1(x) + a_2 p_2(x) + a_3 p_3(x) + \cdots,$$

where the series is uniformly convergent and each $p_n(x)$ is analytic in the neighborhood of the origin.

The present proof is based on properties of multiple improper integrals as developed by Professor PIERPONT**; and aims to make rigorous the demonstration of SCHLÖMILCH already referred to. It is applicable to a more general class of functions than the proof of NIELSEN, and is much simpler than that of LEVI-CIVITA; though these latter proofs apply to ABEL's general theorem, and not merely to the case $n = \frac{1}{2}$, given in relation (1).

§ 3. *Preliminary theorem.*

THEOREM 1. *Let $f(x)$ be continuous in (a, b) except at the points of an aggregate Δ of the first species.*†† *In any subinterval of (a, b) in which $f(x)$ is not continuous, let it have limited variation. Let $f'(x)$ be integrable in (a, b), and continuous except at the points of an aggregate of the first species,*

* Zeitschrift für Mathematik und Physik, vol. 2 (1857), pp. 156, 157.

† Mathematische Annalen, vol. 16 (1880), p. 48.

‡ Rendiconti del reale Istituto Lombardo, ser. 2, vol. 13 (1880), pp. 327, 402.

§ Atti della reale Accademia delle Scienze di Torino, vol. 3 (1895), p. 25.

‖ Annali di Matematica, ser. 2, vol. 25 (1897), p. 139.

¶ *Handbuch der Theorie der Cylinderfunktionen* (1904), pp. 379–381.

** Transactions of the American Mathematical Society, vol. 7 (1906), p. 155. This paper will be referred to as *Improper Integrals*.

†† PIERPONT, *The Theory of Functions of Real Variables* (1905), § 501, 1.

where it may have finite or infinite discontinuities. Then if $f(x)$ is continuous at the end points of the interval,

$$\int_a^b f'(x)\,dx = f(b) - f(a) - D.$$

If $f(x)$ is discontinuous at the end points, then

$$\int_a^b f'(x)\,dx = f(b-0) - f(a+0) - D$$

where D is the sum of the discontinuities of $f(x)$ within the interval.

Case 1°. Suppose Δ is of order zero, i. e., $f(x)$ has only a finite number of points of discontinuity $c_1, c_2, \cdots, c_s$.

$$\int_a^b f'(x)\,dx = \int_a^{c_1-\epsilon} + \int_{c_1-\epsilon}^{c_1+\epsilon} + \int_{c_1+\epsilon}^{c_2-\epsilon} + \int_{c_2-\epsilon}^{c_2+\epsilon} + \cdots + \int_{c_s+\epsilon}^{b}$$

$$= \int_a^{c_1-\epsilon} + \int_{c_s+\epsilon}^{b} + \sum_{k=1}^{s} \int_{c_k-\epsilon}^{c_k+\epsilon} + \sum_{k=1}^{s-1} \int_{c_k+\epsilon}^{c_{k+1}-\epsilon}.$$

But *

$$\sum_{k=1}^{s-1} \int_{c_k+\epsilon}^{c_{k+1}-\epsilon} = \sum_{k=1}^{s-1} \{f(c_{k+1}-\epsilon) - f(c_k+\epsilon)\},$$

and

$$\int_a^{c_1-\epsilon} + \int_{c_s+\epsilon}^{b} = f(b) - f(a) + f(c_1-\epsilon) - f(c_s+\epsilon).$$

Hence

$$\int_a^b f'(x)\,dx = f(b) - f(a) + \sum_{k=1}^{s} \{f(c_k-\epsilon) - f(c_k+\epsilon)\} + \sum_{k=1}^{s} \int_{c_k-\epsilon}^{c_k+\epsilon} f'(x)dx.$$

Since $f'(x)$ is integrable in (a, b),

$$\lim_{\epsilon=0} \sum_{k=1}^{s} \int_{c_k-\epsilon}^{c_k+\epsilon} f'(x)\,dx = 0.$$

Hence by passing to the limit,

$$\int_a^b f'(x)\,dx = f(b) - f(a) - \lim_{\epsilon=0} \sum_{k=1}^{s} \{f(c_k+\epsilon) - f(c_k-\epsilon)\}.$$

But

$$\lim_{\epsilon=0} \{f(c_k+\epsilon) - f(c_k-\epsilon)\}$$

* De la Vallée-Poussin, Journal de Mathématiques, ser. 4, vol. 8 (1892), p. 430. See Pierpont, *Theory of Functions*, §§ 538, 605.

is the discontinuity of $f(x)$ at the point c_k, and

$$\lim_{\epsilon=0} \sum_{k=1}^{s} \{f(c_k+\epsilon)-f(c_k-\epsilon)\}$$

is the sum of the discontinuities of $f(x)$ in (a, b). Hence

$$\int_a^b f'(x)\,dx = f(b)-f(a)-D.$$

Case 2°. Suppose Δ is of order n. The case where Δ is of order zero having been already considered, we suppose the theorem is true for Δ of order $n-1$, and show that it must hold for Δ of order n. In this case $\Delta^{(n)}$ consists of a finite number of points $c_1, c_2, \cdots, c_s$. Then in the intervals $(c_i+\epsilon, c_{i+1}-\epsilon)$ the aggregate of points of discontinuity is of order $n-1$, and we have

$$\int_a^b f'(x)\,dx = \int_a^{c_1-\epsilon} + \int_{c_1-\epsilon}^{c_1+\epsilon} + \int_{c_1+\epsilon}^{c_2-\epsilon} + \cdots + \int_{c_s-\epsilon}^{c_s+\epsilon} + \int_{c_s+\epsilon}^{b}$$

$$= f(b)-f(a) + \sum_{k=1}^{s} \{f(c_k-\epsilon)-f(c_k+\epsilon)\} - \sum_{k=1}^{s+1} d_k + \sum_{k=1}^{s} \int_{c_k-\epsilon}^{c_k+\epsilon} f'(x)\,dx,$$

where d_k is the sum of the discontinuities of $f(x)$ in the interval $(c_{k-1}+\epsilon, c_k-\epsilon)$.* Since $f'(x)$ is integrable in (a, b),

$$\lim_{\epsilon=0} \sum_{k=1}^{s} \int_{c_k-\epsilon}^{c_k+\epsilon} f'(x)\,dx = 0.$$

Since $f(x)$ has limited variation in any interval when it is not continuous,

$$\lim_{\epsilon=0} \sum_{k=1}^{s+1} d_k$$

is finite, and we set

$$\lim_{\epsilon=0} \sum_{k=1}^{s+1} d_k = \sum_{k=1}^{s+1} D_k,$$

* That d_k has a meaning when the number of discontinuities in the interval $(c_{k-1}+\varepsilon, c_k-\varepsilon)$ infinite, is seen as follows: If the aggregate Δ of points of discontinuity is finite, or in other words if Δ is of order zero, d_k is obviously a well defined quantity. Hence we suppose that d_k is defined when Δ is of order $n-1$, and show that it is defined when Δ is of order n. If Δ is of order n, $\Delta^{(n)}$ consists of a finite number of points $p_1, p_2, \cdots, p_t$. Enclose these in little intervals $(p_i-\eta, p_i+\eta)$. In the remaining intervals Δ is of order $n-1$, hence d_k is defined. Let δ_{i1} be the absolute value of the first discontinuity falling in the interval $(p_i-\eta, p_i+\eta)$; let δ_{i2} be the sum of the absolute values of the first and second, etc. Then $\delta_{i1}, \delta_{i2}, \delta_{i3}, \cdots$ is an infinite increasing sequence, which, as f has limited variation, must always remain less than some fixed quantity. Then from *Theory of Functions*, §§ 109, 101, 102 the sequence $\delta_{i1}, \delta_{i2}, \delta_{i3} \cdots$ has a limit, which we define as the value of d_k in the interval $(p_i-\eta, p_i+\eta)$. Hence d_k is defined throughout the interval $(c_{k-1}+\varepsilon, c_k-\varepsilon)$. The same reasoning holds for $D_k = \lim_{\epsilon=0} d_k$.

where D_k is the sum of the discontinuities of $f(x)$ in the interval (c_{k-1}, c_k) excluding the end points.

Also since $f(x)$ has limited variation,

$$\lim_{\epsilon=0} \sum_{k=1}^{s} \{f(c_k - \epsilon) - f(c_k + \epsilon)\} = \sum_{k=1}^{s} \{f(c_k - 0) - f(c_k + 0)\}$$

exists and is finite.* Set

$$\sum_{k=1}^{s+1} D_k + \sum_{k=1}^{s} \{f(c_k - 0) - f(c_k + 0)\} = D.$$

Then D is the sum of the discontinuities of $f(x)$ in (a, b), and

$$\int_a^b f'(x)\,dx = f(b) - f(a) - D.$$

Suppose now $f(x)$ is discontinuous at $x = a$. Then obviously

$$\int_a^b f'(x)\,dx = f(b) - f(a + 0) - D.$$

Similar reasoning holds if $f(x)$ is discontinuous at $x = b$.

§ 4. *Proof of Abel's relation.*

THEOREM 2.† 1°. *Let $f(x)$ be continuous in the interval $(0, \pi)$, except at the points of an aggregate Δ of the first species. In any subinterval of $(0, \pi)$ in which $f(x)$ is not continuous let it have limited variation.*

2°. *Let $f'(x)$ be integrable in $(0, \pi)$, and continuous except at the points of an aggregate of the first species where it may have finite or infinite discontinuities.*

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* JORDAN, *Cours d'Analyse*, vol. 1, p. 55.

† Added by the author, December, 1906. In a paper presented to the American Mathematical Society in New York, December 28, 1906, Professor PIERPONT gave the following theorem: Let the infinities of $|f(x)|$ be discrete in the measurable aggregate $\mathfrak{N}$ and let

$$\int_{\mathfrak{B}} \overline{\int_{\mathfrak{C}}} |f|$$

be convergent. Then

$$\int_{\mathfrak{N}} f = \int_{\mathfrak{B}} \int_{\mathfrak{C}} f.$$

By virtue of this theorem, conditions 3°, 4° of theorem 2 may be replaced by the following: 3°. Let

$$\int_0^1 dt \int_0^1 \frac{|f'(\lambda u t)|}{\sqrt{1 - t^2} \cdot \sqrt{1 - \lambda^2}}\, d\lambda$$

be convergent. Either set of these conditions is sufficient for the validity of theorems 3 to 10.

3°. *Let the function*

$$\frac{f'(\lambda ut)}{\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}} \qquad (0 \leqq u \leqq \pi),$$

be integrable in the square S, *bounded by the lines* $\lambda = 0$, $\lambda = 1$, $t = 0$, $t = 1$.

4°. *Let the integral*

$$\int_0^1 \frac{f'(\lambda ut)}{\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}}\,d\lambda$$

be regular and integrable in* $0 \leqq t \leqq 1$. *Then*

$$(1)\quad \frac{2u}{\pi}\int_0^1 d\lambda \int_0^1 \frac{f'(\lambda ut)\lambda dt}{\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}} = \frac{2u}{\pi}\int_0^1 dt \int_0^1 \frac{f'(\lambda ut)\lambda d\lambda}{\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}}$$

$$= f(u-0) - f(0+0) - D(u),$$

where $D(u)$ *is the sum of the discontinuities of* $f(u)$ *in the interval* $(0, u)$.

Using conditions 3°, 4° we have †

$$\int_S \frac{f'(\lambda ut)\lambda dt d\lambda}{\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}} = \int_0^1 dt \int_0^1 \frac{f'(\lambda ut)\lambda d\lambda}{\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}}.$$

Since the integrand of

$$\int_0^1 \frac{f'(\lambda ut)dt}{\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}}$$

is symmetrical in λ and t, it follows from condition 4° that the integral is regular and integrable in $0 \leqq \lambda \leqq 1$. Hence

$$(2)\quad \int_S \frac{f'(\lambda ut)\lambda dt d\lambda}{\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}} = \int_0^1 dt \int_0^1 \frac{f'(\lambda ut)\lambda d\lambda}{\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}} = \int_0^1 d\lambda \int_0^1 \frac{f'(\lambda ut)\lambda dt}{\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}}.$$

Let us apply to the integral on the left of (2) the transformation

$$T: \qquad t = \frac{x}{\sqrt{x^2+y^2}}, \qquad \lambda = \frac{\sqrt{x^2+y^2}}{u}$$

whose determinant

$$J = \begin{vmatrix} \dfrac{y^2}{(x^2+y^2)^{\frac{3}{2}}} & \dfrac{-xy}{(x^2+y^2)^{\frac{3}{2}}} \\ \dfrac{x}{u\sqrt{x^2+y^2}} & \dfrac{y}{u\sqrt{x^2+y^2}} \end{vmatrix} = \frac{y}{u(x^2+y^2)}.$$

* PIERPONT, *Improper Integrals*, p. 167.

† PIERPONT, *Improper Integrals*, p. 168, theorem 27.

The image of the square S is the quadrant Q, bounded by the lines $x=0$, $y=0$, $x^2+y^2=u^2$. The transformation is regular* in S except at the points of a discrete aggregate D, which consists of the boundaries of S and a set of equilateral hyperbolas. The image of D is also discrete, consisting of the boundaries of Q and a set of lines parallel to the y axis. Hence† we have

$$(3)\qquad \int_S \frac{f'(\lambda ut)\lambda dt\, d\lambda}{\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}} = \frac{1}{u}\int_Q \frac{f'(x)\,dx\,dy}{\sqrt{u^2-x^2-y^2}}.$$

By condition 2°, $f'(x)$ is integrable in the interval $(0, u)$, $0 \leqq u \leqq \pi$, and is continuous except at the points of an aggregate of the first species. Hence since the integral

$$\int_0^{\sqrt{u^2-x^2}} \frac{f'(x)\,dy}{\sqrt{u^2-x^2-y^2}} = \frac{\pi}{2} f'(x),$$

it is regular and integrable in $0 \leqq x \leqq u$. Then‡ we have

$$(4)\qquad \int_Q \frac{f'(x)\,dx\,dy}{\sqrt{u^2-x^2-y^2}} = \int_0^u dx \int_0^{\sqrt{u^2-x^2}} \frac{f'(x)\,dy}{\sqrt{u^2-x^2-y^2}} = \frac{\pi}{2}\int_0^u f'(x)\,dx.$$

From theorem 1,

$$(5)\qquad \int_0^u f'(x)\,dx = f(u-0) - f(0+0) - D(u).$$

Then (2), (3), (4), (5) give (1).

COROLLARY 1. *If $f(x)$ is continuous in $(0, \pi)$, and conditions 2°, 3°, 4° of the preceding theorem are satisfied, then*

$$\frac{2u}{\pi}\int_0^1 dt \int_0^1 \frac{f'(\lambda ut)\lambda d\lambda\cdot}{\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}} = \frac{2u}{\pi}\int_0^1 d\lambda \int_0^1 \frac{f'(\lambda ut)\lambda dt}{\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}} = f(u)-f(0).$$

COROLLARY 2. *If $f(x)$ is continuous in $(0, \pi)$, and $f'(x)$ is limited and becomes discontinuous only at the points of an aggregate of the first species, then*

$$\frac{2u}{\pi}\int_0^1 dt \int_0^1 \frac{f'(\lambda ut)\lambda d\lambda}{\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}} = \frac{2u}{\pi}\int_0^1 d\lambda \int_0^1 \frac{f'(\lambda ut)\lambda dt}{\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}} = f(u)-f(0).$$

For, $f'(x)$ is integrable§ in $(0, \pi)$, and $f'(\lambda ut)$ is integrable § in S. Then

* PIERPONT, *Theory of Functions*, § 742.

† PIERPONT, *Improper Integrals*, p. 174, theorem 32.

‡ PIERPONT, *Improper Integrals*, p. 168, theorem 27.

§ PIERPONT, *Theory of Functions*, § 719, 2.

since $1/\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}$ is integrable in S, $f'(\lambda ut)/\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}$ is also integrable * in S. Also the integral

$$\int_0^1 \frac{f'(\lambda ut)\,d\lambda}{\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}}$$

is uniformly convergent † in $0 \leqq t \leqq 1$. Hence it is limited ‡ in the same interval, and continuous § except at the points of a discrete aggregate. It is therefore integrable in $0 \leqq t \leqq 1$, and the conditions of theorem 2 are satisfied.

It should be noted that in the general case of the theorem conditions 3°, 4° do not follow from 2°. For if $f'(x)$ has a single point of infinite discontinuity, $f'(\lambda ut)$ becomes infinite at all the points of an hyperbola, and hence has points of infinite discontinuity in common with $1/\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}$. Then $f'(\lambda ut)/\sqrt{1-\lambda^2}\cdot\sqrt{1-t^2}$ may not be integrable.

The conditions of theorem 2 obviously include those of NIELSEN ‖, since in his theorem $f'(x)$, being representable by a uniformly convergent series of continuous functions, must be a continuous function in the vicinity of the origin. His theorem, however, covers a more general set of inversion formulæ.

PART II. DEVELOPMENT OF AN ARBITRARY FUNCTION IN TERMS OF BESSEL'S FUNCTIONS.

§ 5. *Historical remarks.*

In 1857 SCHLÖMILCH ¶ in the paper already cited outlined a method for obtaining a development of the form

$$f(x) = a_0 + a_1 J_0(x) + a_2 J_0(2x) + a_3 J_0(3x) + \cdots, \tag{1}$$

where

$$\begin{aligned} a_0 &= f(0) + \frac{1}{\pi}\int_0^\pi u\,du \int_0^1 \frac{f'(uv)\,dv}{\sqrt{1-v^2}}, \\ a_n &= \frac{2}{\pi}\int_0^\pi u\cos nu\,du \int_0^1 \frac{f^n(uv)\,dv}{\sqrt{1-v^2}}, \qquad (u \neq 0), \end{aligned} \tag{2}$$

and where J_0 is a Bessel's function of order zero. By termwise differentiation of this series, and by use of the relation

$$\frac{dJ_0(x)}{dx} = -J_1(x),$$

* PIERPONT, *Improper Integrals*, p. 159.
† PIERPONT, *Theory of Functions*, § 615.
‡ PIERPONT, *Theory of Functions*, § 611, 4.
§ JORDAN, loc. cit., vol. 2, § 156.
‖ Loc. cit., pp. 379–381.
¶ Loc. cit., pp. 155–158.

he obtained also a development in terms of $J_1(x)$,

$$f(x) = b_1 J_1(x) + b_2 J_1(2x) + b_3 J_1(3x) + \cdots, \tag{3}$$

where

$$b_n = -\frac{2n}{\pi}\int_0^\pi u \cos nu\, du \int_0^1 \frac{f(uv)\,dv}{\sqrt{1-v^2}}. \tag{4}$$

He however specified no conditions sufficient for the proof of the validity of either development. BELTRAMI,* GEGENBAUER,† and VOLTERRA ‡ study developments of this and more general forms, considering the multipliers of the variable x in the respective terms as the roots of a transcendental equation. This throws the consideration of these developments back upon the very general problem studied by DINI,§ which he solved by use of the complex variable. NIELSEN ‖ gives as sufficient conditions for the development of $f(x)$ in a SCHLÖMILCH series, that $f'(x)$ shall exist and be such that $f(\alpha x) + \alpha x f'(\alpha x)$ may be developed in a FOURIER series, which shall be uniformly convergent for $-\pi < x < \pi$, $-1 \leqq \alpha \leqq 1$. It follows from this that $f(\alpha x) + \alpha x f'(\alpha x)$ is a continuous function in the interval, a condition which is not imposed in the following theorems. NIELSEN's theorem applies, however, to a more general class of developments.

§ 6. *Developments in terms of* $J_0(x)$.

THEOREM 3. *Let $f(x)$, $f'(x)$ be subject to the conditions of theorem 2. Set*

$$f(0+0) + \lambda u \int_0^1 \frac{f'(\lambda u t)\,dt}{\sqrt{1-t^2}} = F(\lambda u) \quad (0 \leqq \lambda \leqq 1,\ 0 \leqq u \leqq \pi), \tag{1}$$

Then

$$f(u-0) = \frac{2}{\pi}\int_0^1 \frac{F(\lambda u)\,d\lambda}{\sqrt{1-\lambda^2}} + D(u), \tag{2}$$

where $D(u)$ is the sum of the discontinuities of $f(u)$ in $(0, u)$.

Multiplication of (1) by $2/\pi\sqrt{1-\lambda^2}$ gives

$$\frac{2f(0+0)}{\pi\sqrt{1-\lambda^2}} + \frac{2\lambda u}{\pi}\int_0^1 \frac{f'(\lambda u t)\,dt}{\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}} = \frac{2F(\lambda u)}{\pi\sqrt{1-\lambda^2}}.$$

By condition 4° in theorem 2 this is integrable in $0 \leqq \lambda \leqq 1$. Hence, integrating, we get

* Loc. cit., pp. 327, 402.

† Wiener Sitzungsberichte, vol. 88, II (1883), p. 975.

‡ Loc. cit., p. 139.

§ *Serie di Fourier e altre rappresentazioni analitiche delle funzioni di una variabile reale*, 1880.

‖ Loc. cit., p. 348.

$$f(0+0)+\frac{2u}{\pi}\int_0^1 d\lambda\int_0^1\frac{f'(\lambda ut)\lambda dt}{\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}}=\frac{2}{\pi}\int_0^1\frac{F(\lambda u)d\lambda}{\sqrt{1-\lambda^2}};$$

whence by theorem 2,

$$f(u-0)=\frac{2}{\pi}\int_0^1\frac{F(\lambda u)d\lambda}{\sqrt{1-\lambda^2}}+D(u).$$

THEOREM 4. 1°. *Let $f(x)$, $f'(x)$ be subject to the conditions of theorem* 2. 2°. *Let the integral*

$$\int_0^1\frac{f'(tx)}{\sqrt{1-t^2}}dt$$

represent a function having limited variation in $0\leqq x\leqq\pi$. *Then* SCHLÖMILCH'S *development in terms of* $J_0(x)$ *is valid in the same interval.*

By condition 2° the function

$$F(x)=f(0+0)+x\int_0^1\frac{f'(tx)}{\sqrt{1-t^2}}dt$$

has limited variation in $(0,\pi)$, and hence can be developed in FOURIER'S series,*

$$F(x)=\frac{a_0}{2}+a_1\cos x+a_2\cos 2x+a_3\cos 3x+\cdots\qquad(0\leqq x\leqq\pi),$$

where

$$a_n=\frac{2}{\pi}\int_0^\pi F(u)\cos nu\,du. \tag{3}$$

Replacing x by $x\sin\lambda$, where $0\leqq x\leqq\pi$, $0\leqq\lambda\leqq\pi/2$, we have

$$\frac{2}{\pi}F(x\sin\lambda)=\frac{a_0}{\pi}+\frac{2a_1}{\pi}\cos(x\sin\lambda)+\frac{2a_2}{\pi}\cos(2x\sin\lambda)+\cdots.$$

By ARZELÀ'S condition † this series is integrable termwise in the interval $0\leqq\lambda\leqq\pi/2$ if it is "in general uniformly convergent by segments" and is determinate for every value of λ in the interval, if each term is integrable, and the sum of s terms is less than a finite number L for all s and all λ. Now F is continuous ‡ except, perhaps, for a set of points of the first species. These can be included in a finite number of intervals of total length ϵ, where ε is arbitrarily small. Since F is continuous in the remaining subintervals, the series converges uniformly § in each of these. Hence in the interval $(0,\pi)$ it is according to ARZELÀ ‖ "in general uniformly convergent by segments." Fur-

* JORDAN, loc. cit. vol., 2, § 231.

† *Sulle serie di funzioni*, Memorie dell' Accademia delle Scienze dell' Istituto di Bologna, (5), 8 (1900), p. 724–725.

‡ JORDAN, loc. cit., vol. 2, § 156.

§ JORDAN, loc. cit., vol. 2, § 231.

‖ Loc. cit., p. 712.

thermore each of the functions

$$\frac{2a_n}{\pi}\cos(nx\sin\lambda)$$

is integrable in $0 \leqq \lambda \leqq \pi/2$; the sum

$$\sum_{n=1}^{\infty}\frac{2a_n}{\pi}\cos(nx\sin\lambda)$$

is determinate for every value of λ in the interval $(0, \pi/2)$; and by known properties* of FOURIER's series

$$\left|\sum_{n=1}^{s}\frac{2a_n}{\pi}\cos(nx\sin\lambda)\right| < L,$$

where L is a finite number independent of s and of λ. Hence

$$\int_0^{\pi}\sum_{n=1}^{\infty}\frac{2a_n}{\pi}\cos(nx\sin\lambda)\,d\lambda = \sum_{n=1}^{\infty}\int_0^{\pi}\frac{2a_n}{\pi}\cos(nx\sin\lambda)\,d\lambda.$$

Then

$$\frac{2}{\pi}\int_0^{\pi/2}F(x\sin\lambda)\,d\lambda = \frac{a_0}{2} + \frac{2a_1}{\pi}\int_0^{\pi/2}\cos(x\sin\lambda)\,d\lambda + \frac{2a_2}{\pi}\int_0^{\pi/2}\cos(2x\sin\lambda)\,d\lambda + \cdots.$$

Substitution from the formula

$$J_0(nx) = \frac{2}{\pi}\int_0^{\pi/2}\cos(nx\sin\lambda)\,d\lambda$$

gives

$$\frac{2}{\pi}\int_0^{\pi/2}F(x\sin\lambda)\,d\lambda = a_0 + a_1J_0(x) + a_2J_0(2x) + a_3J_0(3x) + \cdots.$$

Setting λ in place of $\sin\lambda$, and substituting from (2), we have

$$(4)\qquad f(x-0) = D(x) + a_0 + a_1J_0(x) + a_2J_0(2x) + a_3J(3x) + \cdots.$$

From (1) and (3) we have

$$a_0 = f(0+0) + \frac{1}{\pi}\int_0^{\pi}u\,du\int_0^1\frac{f'(ut)\,dt}{\sqrt{1-t^2}},\qquad a_n = \frac{2}{\pi}\int_0^{\pi}u\cos nu\,du\int_0^1\frac{f'(ut)\,dt}{\sqrt{1-t^2}}$$
$$(n \neq 0).$$

In essentially the same manner as theorems 3 and 4 we may prove the following theorems:

* JORDAN, loc. cit., vol. 2, § 231.

THEOREM 5. *Let $\phi(x) \equiv xf(x)$ and $\phi'(x)$ be subject to the conditions of theorem 2. Set*

$$\int_0^1 \frac{\lambda f(\lambda x) + \lambda^2 x f'(\lambda x)}{\sqrt{1-\lambda^2}} d\lambda = F(x).$$

Then

$$f(u) = \frac{2}{\pi}\int_0^1 \frac{F(\lambda u)}{\sqrt{1-\lambda^2}} d\lambda + \frac{D(u)}{u}.$$

THEOREM 6. *Let $\phi(x) \equiv xf(x)$ and $\phi'(x)$ be subject to the conditions of theorem 2. Let the integral*

$$\int_0^1 \frac{\lambda^2 f'(\lambda x)\, d\lambda}{\sqrt{1-\lambda^2}}$$

represent a function having limited variation in $0 \leqq x \leqq \pi$. Then $f(x)$ may be developed in the form

$$(5)\quad f(x-0) = \frac{D(x)}{x} + \frac{\alpha_0}{2} + \alpha_1 J_0(x) + \alpha_2 J_0(2x) + \alpha_3 J_0(3x) + \cdots \quad (0 \leqq x \leqq \pi),$$

where

$$\alpha_n = \frac{2}{\pi}\int_0^\pi \cos nu\, du \int_0^1 \frac{\lambda f(\lambda u) + \lambda^2 u f'(\lambda u)}{\sqrt{1-\lambda^2}} d\lambda.$$

It will be noted that this development may be valid when $f(x)$ has an isolated infinity of order less than one at the origin, a fact which we are unable to affirm concerning the corresponding development of SCHLÖMILCH.

The question of the identity of these two forms of development (4), (5) is an interesting one. It seems probable that for any function which is regular at the origin they are the same, as is the case in the following examples.

Example 1. $\qquad f(x) = \sin x, \qquad f'(x) = \cos x.$

$$\alpha_n - a_n = \frac{2}{\pi}\int_0^\pi \cos nu du \int_0^1 \frac{\lambda \sin \lambda u + (\lambda^2 - 1) u \cos \lambda u}{\sqrt{1-\lambda^2}} d\lambda$$

$$= \frac{2}{\pi}\int_0^\pi \cos nu \left[\sum_{k=1}^\infty \frac{(-1)^k u^{2k-1}}{(2k-1)!}\int_0^1 \frac{(2k-1)\lambda^{2k-2} - 2k\lambda^{2k}}{\sqrt{1-\lambda^2}} d\lambda\right] du.$$

But

$$2k\int_0^1 \frac{\lambda^{2k} d\lambda}{\sqrt{1-\lambda^2}} = (2k-1)\int_0^1 \frac{\lambda^{2k-2} d\lambda}{\sqrt{1-\lambda^2}}.$$

Hence $\alpha_n - a_n = 0$, and the two developments are identical.

Example 2. $\qquad f(x) = x^m, \qquad f'(x) = mx^{m-1},$

$$\alpha_n - a_n = \frac{2}{\pi}\int_0^\pi u^m \cos nu\, du \int_0^1 \frac{m\lambda^{m-1} - (m+1)\lambda^{m+1}}{\sqrt{1-\lambda^2}} d\lambda.$$

But

$$(m+1)\int_0^1 \frac{\lambda^{m+1}}{\sqrt{1-\lambda^2}}\,d\lambda = m\int_0^1 \frac{\lambda^{m-1}}{\sqrt{1-\lambda^2}}\,d\lambda.$$

Hence $\alpha_n = a_n$, and the two developments are identical for any integral rational function of x.

§ 7. *Development in terms of* $J_1(x)$.

In the same manner as theorem 3, we may prove the following theorems.

THEOREM 7. *Let $f(x)$, $f'(x)$ be subject to the conditions of theorem 2. Set*

$$\int_0^1 \frac{f'(tx)}{\sqrt{1-t^2}}\,dt = F(x).$$

Then

$$f(u-0) = f(0+0) + \frac{2u}{\pi}\int_0^1 \frac{\lambda F(\lambda u)}{\sqrt{1-\lambda^2}}\,d\lambda + D(u).$$

THEOREM 8. *Let $\phi(x) \equiv xf(x)$ and $\phi'(x)$ be subject to the conditions of theorem 2. Set*

$$\int_0^1 \frac{f(tx) + txf'(tx)}{\sqrt{1-t^2}}\,dt = F(x).$$

Then

$$f(u) = \frac{2}{\pi}\int_0^1 \frac{\lambda F(\lambda u)}{\sqrt{1-\lambda^2}}\,d\lambda + \frac{D(u)}{u}.$$

THEOREM 9. *Let $f(x)$, $f'(x)$ be subject to the conditions of theorem 2. Let the integral*

$$\int_0^1 \frac{f'(tx)}{\sqrt{1-t^2}}\,dt$$

represent a function having limited variation in $0 \leqq x \leqq \pi$. Then $f(x)$ may be developed in the form

$$(6)\quad f(x+0) = f(0-0) + D(x) + x[b_1' J_1(x) + b_2' J_1(2x) + b_3' J_1(3x) + \cdots], \qquad (0 < x \leqq \pi),$$

where

$$b_n' = \frac{2}{\pi}\int_0^\pi \sin nu\,du \int_0^1 \frac{f'(ut)}{\sqrt{1-t^2}}\,dt.$$

Since

$$F(x) = \int_0^1 \frac{f'(tx)}{\sqrt{1-t^2}}\,dt$$

has limited variation in $0 \leqq x \leqq \pi$, it may be developed in Fourier's series,

$$F(x) = b'_1 \sin x + b'_2 \sin 2x + b'_3 \sin 3x \cdots \qquad (0 < x \leqq \pi),$$

where

$$b'_n = \frac{2}{\pi} \int_0^\pi F(u) \sin nu \, du.$$

Proceeding as in theorem 4, and substituting from the formula

$$J_1(nx) = \frac{2}{\pi} \int_0^{\pi/2} \sin(nx \sin \lambda) \sin \lambda d\lambda$$

we obtain the development (6).

In a similar manner we prove the following theorem.

THEOREM 10. *Let $\phi(x) \equiv xf(x)$ and $\phi'(x)$ be subject to the conditions of theorem 2. Let the integral*

$$\int_0^1 \frac{\lambda f'(\lambda x)}{\sqrt{1-\lambda^2}} d\lambda$$

represent a function having limited variation in $0 \leqq x \leqq \pi$. Then $f(x)$ may be developed in the form

$$(7) \quad f(x-0) = \frac{D(x)}{x} + \beta_1 J_1(x) + \beta_2 J_1(2x) + \beta_3 J_1(3x) + \cdots \qquad (0 < x \leqq \pi),$$

where

$$\beta_n = \frac{2}{\pi} \int_0^\pi \sin nu \, du \int_0^1 \frac{f(\lambda u) + \lambda u f'(\lambda u)}{\sqrt{1-\lambda^2}} d\lambda.$$

As in the case of $J_0(x)$, if SCHLÖMILCH's development given in § 5, (3) is valid, it is identical with (7) in simple cases. We note the following examples:

Example 1. $\qquad f(x) = \sin x, \qquad f'(x) = \cos x.$

$\beta_n - b_n$

$$= \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \cdot 2 \cdot 4 \cdot 6 \cdots (2k-2)}{1 \cdot 3 \cdot 5 \cdots (2k-1) \cdot (2k-1)!} \left[2k \int_0^\pi u^{2k-1} \sin nu \, du + u \int_0^\pi u^{2k} \cos nu \, du \right].$$

But

$$\int_0^\pi u^{2k} \cos nu \, du = -\frac{2k}{u} \int_0^\pi u^{2k-1} \sin nu \, du.$$

Hence $\beta_n - b_n = 0$, and the two developments are identical.

Example 2. $\qquad f(x) = x^m, \qquad f'(x) = mx^{m-1}.$

$$\beta_n - b_n = \frac{2}{\pi} \cdot \frac{2 \cdot 4 \cdot 6 \cdots (m-1)}{1 \cdot 3 \cdot 5 \cdots m} \left[(m+1) \int_0^\pi u^m \sin nu\, du + n \int_0^\pi u^{m+1} \cos nu\, du \right]$$

(m odd),

or

$$\beta_n - b_n = \frac{1 \cdot 3 \cdot 5 \cdots (m-1)}{2 \cdot 4 \cdot 6 \cdots m} \left[(m+1) \int_0^\pi u^m \sin nu\, du + n \int_0^\pi u^{m+1} \cos nu\, du \right]$$

(m even).

But

$$\int_0^\pi u^{m+1} \cos nu\, du = -\frac{m+1}{n} \int_0^\pi u^m \sin nu\, du.$$

Hence $\beta_n - b_n = 0$, and the two developments are the same for any integral rational function of x.

NORTHFORD, CONN.,
June, 1906.

IRREDUCIBLE LINEAR HOMOGENEOUS GROUPS WHOSE ORDERS ARE POWERS OF A PRIME*

BY

WILLIAM BENJAMIN FITE

The purpose of this article is to determine as far as possible the connection between the degree of an irreducible linear homogeneous group and its abstract group properties. The discussion will be limited for the most part to groups whose orders are powers of a prime. As a particular phase of the general subject there arises the question as to the circumstances under which a group can be simply isomorphic with irreducible groups of different degrees. The simple group of order 168, for example, is simply isomorphic with irreducible groups of degrees 3, 6, 7, and 8 respectively. On the other hand, I know of no group whose order is a power of a prime that is simply isomorphic with irreducible groups of different degrees. In fact, the following discussion shows that in certain irreducible groups the degree is uniquely fixed by certain abstract properties of the group; and if the degree is not thus uniquely fixed the order of the group must be greater than the seventh power of a prime.

THEOREM I. *A linear homogeneous group G, of class† k, all of whose invariant operations are similarity substitutions either is irreducible or is simply isomorphic with each of its irreducible components.*

If G is reducible, suppose that it has been put into its completely reduced form. The invariant operations will not be affected by this change. If in any operation the identity of any irreducible component were associated with a non-identical operation in the other variables, this operation would be non-invariant in G, and would therefore give at least one non-identical commutator. This commutator would also be non-invariant, and none of its successive commutators, except identity, could be invariant. But this is impossible since G is of class k. Hence every irreducible component of G is simply isomorphic with G.

THEOREM II. *A linear homogeneous group G, of order a power of a prime, that has a cyclic central either is irreducible or is simply isomorphic with at least one of its irreducible components.*

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† Cf. Transactions of the American Mathematical Society, vol. 3 (1902), p. 34[illegible].

If G is reducible, suppose that it has been put into its completely reduced form. Then at least one of its irreducible components has at least as many invariant operations as G. If G_1 be such a component, it can easily be shown that G_1 and G are simply isomorphic.

Let G be any group of finite order g that has a cyclic central (distinct from identity). It is simply isomorphic with a regular permutation group, which can be written as a linear homogeneous group. We can therefore think of G as a linear homogeneous group. The coefficients in this group are either zero or unity. The central H of G is generated by an operation h of order a and of the form (when written as a permutation)

$$(x_{1,1}x_{1,2}\cdots x_{1,a})(x_{2,1}x_{2,2}\cdots x_{2,a})\cdots(x_{g/a,1}x_{g/a,2}\cdots x_{g/a,a}).$$

The operation S,

$$y_{i,j}=\sum_{k=1}^{a}\omega^{(k-1)(j-1)}x_{i,k}\quad(i=1,2,\cdots,g/a;\ j=1,2,\cdots,a),$$

where ω is a primitive ath root of unity, transforms G into a semi-canonical form,* and in particular transforms h into its normal form. It is furthermore obvious that the roots of the characteristic equation of h are the ath roots of unity each occurring g/a times.† In the transformed form of G the variables

$$x_{1,j},\ x_{2,j},\ \cdots,\ x_{g/a,j}\qquad(j=1,2,\cdots,a)$$

are transformed into linear combinations of themselves. Let G_j be the group formed by the substitutions of G as far as they affect these g/a variables. If $j-1$ is relatively prime to a, G_j is simply isomorphic with G and its invariant operations are similarity substitutions. If we suppose that G is of class k, it follows from theorem I that G_j is simply isomorphic with each of its irreducible components. Any irreducible component of any G_j $(j=1,2,\cdots,a)$ is an irreducible representation of G. Moreover the components of no two of these G_j's are equivalent. This follows from the fact that in the simple isomorphisms between these components and G the multipliers of the operations that correspond to h are different. We have then $\phi(a)$ distinct irreducible representations of G that are simply isomorphic with it.

It follows directly from this discussion that a sufficient condition that a group of class k be simply isomorphic with an irreducible group is that its central be cyclic.‡

* BLICHFELDT, Transactions of the American Mathematical Society, vol. 5 (1904), p. 313.

† This conclusion is obviously independent of the fact that h is invariant in G. Hence if any regular permutation of order a is written as a linear homogeneous substitution the roots of its characteristic equation are the ath roots of unity, each occurring g/a times, where g is the degree of the permutation.

‡ Cf. Transactions of the American Mathematical Society, vol. 7 (1906), p. 65. The proof given above is much the simpler one.

If now G is an irreducible linear homogeneous group of order p^m (p a prime) and has p^a invariant operations, there are at least $p^{a-1}(p-1)$ distinct irreducible representations of G that are simply isomorphic with it.

Consider the formula *

$$\sum_{i=1}^{r} \chi_k^i \chi_{k'}^i = \frac{p^m}{h_k}.$$

If we let k refer to the conjugate set formed by an invariant operation of order p, then $h_k = 1$, and we have

$$\sum_{i=1}^{s} x_i p^{2n_i} + p^{m-1} = p^m,$$

where x_i is the number of irreducible representations of G of degree p^{n_i} with which G is simply isomorphic, and $\sum_{i=1}^{s} x_i$ is the total number of the irreducible representations with which G is simply isomorphic. If we denote by $\bar{H}$ the subgroup composed of the invariant operations of order p, then, inasmuch as every invariant subgroup of G contains $\bar{H}$, the terms following $x_s p^{2n_s}$ in the left member of the last equation give $\sum \chi_k^j \chi_{k'}^j$ for the group $G/\bar{H}$, where k refers to the conjugate set composed of identity, and therefore the sum of these terms is p^{m-1}.

It follows from what we have seen that $x_i \geqq p^{a-1}(p-1)$ $(i = 1, 2, \cdots, s)$. If one of the irreducible representations with which G is simply isomorphic is of degree † $p^{(m-a)/2}$, then $s = 1$ and $x_1 = p^{a-1}(p-1)$. Moreover, $\sum_{i=1}^{s} x_i$ is the difference between the number of conjugate sets of G and of $G/\bar{H}$. Now this difference is $(p-1)$ times the number of conjugate sets of $G/\bar{H}$ whose operations correspond to operations of G that give no invariant commutators besides identity. Also if every non-invariant operation of G gives an invariant commutator besides identity the degree of G is ‡ $p^{(m-a)/2}$. Hence:

THEOREM III. *A necessary and sufficient condition that an irreducible group of order p^m with p^a invariant operations be of degree $p^{(m-a)/2}$ is that every non-invariant operation give invariant commutators besides identity.*

If G contains a non-invariant operation that gives no invariant commutator besides identity, it must be of degree less than $p^{(m-a)/2}$, and therefore it must contain a non-invariant operation the sum of whose multipliers is not zero.§ Conversely, if G contains a non-invariant operation the sum of whose multipliers is not zero, its degree must be less than $p^{(m-a)/2}$, and therefore it must

* FROBENIUS, Berliner Sitzungsberichte, 1896, II, p. 1363; BURNSIDE, Proceedings of the London Mathematical Society, vol. 33 (1900), p. 153; series 2, vol. 1 (1903), p. 123.

† No one of them can be of greater degree. See Transactions of the American Mathematical Society, vol. 7 (1906), p. 67.

‡ Loc. cit., p. 67.

§ Loc. cit., p. 67.

contain a non-invariant operation that gives no invariant commutator besides identity. It should be remarked that this does not prove that in an irreducible group the sum of the multipliers of a non-invariant operation which gives no invariant commutator besides identity is necessarily different from zero. That is not yet settled.

Suppose that G, of order p^m, is an irreducible group with an abelian subgroup G_δ of index p^δ. Then G contains a subgroup $G_{\delta-1}$ of index $p^{\delta-1}$ that contains G_δ invariantly. None of the irreducible components of $G_{\delta-1}$ can be of degree greater than p. If $G_{\delta-1}$ does not coincide with G it must be invariant in a $G_{\delta-2}$ of index $p^{\delta-2}$. None of the irreducible components of $G_{\delta-2}$ can be of degree greater than * p^2. By proceeding thus with a series of subgroups each of which is invariant in the next following one and of index p under it, we see that the degree of G cannot exceed p^δ.

THEOREM IV. *If an irreducible linear homogeneous group G of order p^m contains an abelian subgroup of order $p^{m-\delta}$, the degree of G cannot exceed p^δ.*

It follows immediately that *if G, of order p^m, has a cyclic central of order p^α and contains an abelian subgroup of order greater than $p^{(m+\alpha)/2}$, it must contain non-invariant operations that give no invariant commutators besides identity.*

If G, of order p^m, is an irreducible group of degree p, it contains an abelian subgroup of order † p^{m-1}. It therefore cannot be simply isomorphic with an irreducible group of any other degree. It can now readily be verified that (as stated in the introduction) if G is of order p^m, where $m < 8$, it cannot be simply isomorphic with irreducible groups of different degrees.

If G_δ is any subgroup of G of index p^δ and with p^{α_1} invariant operations, its degree must be equal to, or less than, $p^{(m-\delta-\alpha_1)/2}$ if it is irreducible, and in case it is reducible the same must be true of each of its irreducible components. If $p^{(m-\alpha-\lambda)/2}$ is the degree of G, then $m+\delta-\alpha_1 \geqq m-\alpha-\lambda$, or $\lambda \geqq \alpha_1-\alpha-\delta$. If we know the degree of G, this inequality gives us an upper limit to the number of invariant operations of any subgroup with a given index. On the other hand, if the abstract group properties of G are known, it gives us an upper limit to the degree of G.

THEOREM V. *If G is an irreducible group of order p^m (p an odd prime) and class 3, the square of the absolute value of the sum of the multipliers of any operation of G is a power of p.*

The theorem is obvious for the invariant operations and for those non-invariant operations the sums of whose multipliers are zero. We need to consider, then, only those non-invariant operations the sums of whose multipliers are not zero.

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* This argument is based upon a selection of new variables such as is described by BLICHFELDT, loc. cit., p. 311.

† Transactions of the American Mathematical Society, vol. 7 (1906), p. 68.

Let S be such an operation. Obviously it is not in H_2.* We suppose then that it corresponds to an operation of order p^β of G/H_2. Since S can give no invariant commutator besides identity, there must be in G an operator A such that $S^{-1}AS = At$, where t is commutative with S, but is not invariant in G. Now $S^{-p^\beta}AS^{p^\beta} = At^{p^\beta}$, and therefore t^{p^β} is invariant in G. Moreover $S^{-1}A^{p^\beta}S = A^{p^\beta}t^{p^\beta}h^{[p^\beta(p^\beta-1)]/2}$, where $A^{-1}tA = th$, h being invariant in G. Since t^{p^β} is invariant, $h^{p^\beta} = 1$ and therefore $t^{p^\beta} = 1$, since otherwise S would give an invariant commutator besides identity. If t is of order $p^{\beta_1}(\beta_1 \leqq \beta)$, we put $t_1 \equiv t^{p^{\beta_1-1}}h^{p^{\beta_1-1}(p^{\beta_1-1}-1)/2}$, and $A_1 \equiv A^{p^{\beta_1-1}}$. Then $S^{-1}A_1S = A_1t_1$. Now t_1 is of order p. (If it were identity, $t^{p^{\beta_1-1}}$ would be invariant and therefore $h^{p^{\beta_1-1}} = 1$; but this would require that $t^{p^{\beta_1-1}} = 1$). If $\beta_1 > 1$, A_1 and t_1 are commutative.

Since S and t_1 are commutative, we can consider both of them written in the normal form:

$$S: \quad x_i' = \sigma_i x_i; \qquad t_1: \quad x_i' = \omega_i x_i \qquad (i=1, 2, \cdots, n).$$

Inasmuch as t_1 is not invariant, its multipliers break up into p sets of equals, those of one of the sets being unity. If A_1 and t_1 are commutative, the sum of the multipliers of S that correspond to each of these sets, except that one whose terms are all unity, is zero. If $\beta_1 = 1$ and A is not commutative with t, we shall have:

$$\sigma_i^{-1}\sigma_{p^{n-1}+i}=\omega, \quad \sigma_{p^{n-1}+i}^{-1}\sigma_{2p^{n-1}+i}=\omega^2, \quad \cdots, \quad \sigma_{(p-1)p^{n-1}+i}^{-1}\sigma_i=\omega^p=1 \quad (i=1, 2, \cdots, p^{n-1}).$$

From this it follows that

$$\sum_{i=1}^{p^n}\sigma_i = (2 + \omega + \omega^3 + \cdots + \omega^{[j(j-1)]/2} + \cdots + \omega^{[(p-1)(p-2)]/2})\sum_{i=1}^{p^{n-1}}\sigma_i.$$

If we define χ as the first of the following sums, we have

$$\begin{array}{lllll}
\chi = & & 1 + \omega + \omega^3 + & & \cdots + \omega^m, \\
\omega^{-1}\chi = & & \omega^{-1} + 1 + \omega^2 + & \cdots + \omega^{m-1}, & \\
\omega^{-3}\chi = & \omega^{-3} + \omega^{-2} + 1 + \omega^3 + & \cdots + \omega^{m-3}, & & \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\omega^{-m}\chi = & \omega^{-m} + \cdots & + 1, & &
\end{array}$$

where, for brevity, m denotes $p(p-1)/2$.

The sum of the left members of all these equations is $\chi\bar{\chi}$ ($\bar{\chi}$ being the conjugate imaginary of χ). Therefore, $\chi\bar{\chi} = p + \sigma$, where σ is the sum of the elements of all the columns on the right, except the middle column. Now the sum of the elements of the ith column to the right of the middle one is

* BURNSIDE, *Theory of groups of finite order*, p. 62.

$$\frac{\omega^{i(i+1)/2}(1-\omega^{i(p-i)})}{1-\omega^{i}}=\frac{\omega^{i^2/2}-\omega^{-i^2/2}}{\omega^{-i/2}-\omega^{i/2}}.$$

Moreover, the sum of the elements of the ith column from the right end (counting the end one as the first one) is

$$\frac{\omega^{(p-i+1)(p-i)/2}(1-\omega^{(p-i)i})}{1-\omega^{p-i}}=\frac{\omega^{i^2/2}-\omega^{-i^2/2}}{\omega^{i/2}-\omega^{-i/2}}.$$

Therefore, the sum of the elements of these two columns is zero. Since there is an even number of columns to the right of the middle one, and since the sum of the elements of any column to the left of the middle one is the conjugate imaginary of the sum of the elements of the corresponding column to the right, we have* $\chi\bar{\chi}=p$. That is, the square of the absolute value of the sum of the multipliers of S is equal to p times the square of the absolute value of the sum of the first p^{n-1} of them.

Consider now the subgroup G_1 of G that is composed of the operations that are commutative with t_1. G_1 is reducible and has p irreducible components. That part of t_1 that belongs to a certain one of these components has all its multipliers equal to unity. In this component either S is invariant or it is a non-invariant operation that gives no invariant commutator besides identity. In the former case the theorem results immediately; in the latter it is necessary to apply the same procedure again. By continuing in this way we must finally come to a component in which S is invariant. Hence the theorem.

The square of the absolute value of the sum of the multipliers of S is at most equal to p^{2n-1}. If we denote by x the number of non-invariant operations of G that give no invariant commutators besides identity, we have $p^{\alpha+2n}+xp^{2n-1}\geqq p^m$; that is, $x\geqq p^{\alpha+1}(p^{m-\alpha-2n}-1)$.

THEOREM VI. *An irreducible group of order p^m (p an odd prime), class 3, and degree $p^{(m-\alpha-\lambda)/2}$ contains at least $p^{\alpha+1}(p^{\lambda}-1)$ non-invariant operations that give no invariant commutators besides identity.*

CORNELL UNIVERSITY,
October, 1906.

* For this evaluation of $\chi\bar{\chi}$ I am indebted to PROFESSOR C. N. HASKINS.

APPLICABLE SURFACES WITH ASYMPTOTIC LINES OF ONE SURFACE CORRESPONDING TO A CONJUGATE SYSTEM OF ANOTHER.*

BY

LUTHER PFAHLER EISENHART.

Given a surface S and a surface S_1 applicable to it; the lines on S_1 corresponding to the asymptotic lines on S are called the *virtual asymptotic lines* of S_1 in its application upon S. It is our problem to determine the surfaces S which admit of one or more applicable surfaces S_1 with the virtual asymptotic lines forming a conjugate system. We find that in every case the surface S is an associate surface of a spherical surface, that is, of a surface whose gaussian curvature is a positive constant. Moreover, every surface associate to a spherical surface is of the kind sought.

In deriving the equations of condition to be satisfied by the fundamental quantities of a surface in order that it admit of the given deformation (we shall refer to such a surface as a *surface* S), we assume that the surface is referred to its asymptotic lines. Then the parametric lines on S_1 form a conjugate system. From the form of the equations of condition it is seen that, if there are more than two surfaces S_1 applicable to S with conjugate virtual asymptotic lines, there are an infinity. When there are only one or two surfaces S_1, they can be found by quadratures. But when there are an infinity of them, their determination requires the integration of a system of two linear partial differential equations of the first order in two unknowns. Certain of the equations of condition are satisfied identically when S is a quadric or a ruled surface, but all are satisfied only when S is a skew helicoid with plane director. In the latter case there are an infinity of surfaces S_1 — they are the catenoid and the surfaces of revolution applicable to it with lines of curvature in correspondence.

In § 4 we show that S is an associate of a spherical surface, Σ, and that when S_1 is known, Σ can be found by quadratures. Conversely, when Σ is

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known, S_1 can be found by quadratures. The relation between S and S_1 being reciprocal, S_1 also is an associate of a spherical surface, Σ_1, which can be found by quadratures. In § 5 we discover that Σ and Σ_1 are the Hazzidakis transforms* of one another. Hence we have a means of finding by quadratures the Hazzidakis transform of a spherical surface when a conjugate system with equal point or tangential invariants is known.†

It can be shown that when a spherical surface Σ has a conjugate system of lines with equal point and equal tangential invariants the corresponding conjugate system on the Hazzidakis transform is of the same kind. In this case these are two surfaces, S and S', associate to Σ which admit of applicable surfaces, S_1 and S_1', with conjugate virtual asymptotic lines. All four surfaces, S, S', S_1, S_1', and the surface Σ_1 associate to the last two are found by quadratures. These results are obtained in § 6 and applied to several particular cases in § 7.

In § 8 we study the spherical representation of the surfaces S_1. The equations to be satisfied by this representation are of such a form that when S admits of an infinity of applicable surfaces of the kind S_1, the representation of its asymptotic lines represents also the conjugate virtual asymptotic lines on another infinity of surfaces S. From this it follows that when a surface S is known, all the surfaces S_1 and the spherical associate surfaces Σ and Σ' can be found by quadratures.

§ 1. *General formulae.*

Let a surface S be referred to its asymptotic lines. The Codazzi equations‡ of condition reduce to

$$(1) \qquad \frac{\partial}{\partial u}\left(\frac{D'}{H}\right) = -2\left\{\begin{matrix}12\\2\end{matrix}\right\}\frac{D'}{H}, \qquad \frac{\partial}{\partial v}\left(\frac{D'}{H}\right) = -2\left\{\begin{matrix}12\\1\end{matrix}\right\}\frac{D'}{H},$$

where for the sake of brevity we have put

$$(2) \qquad H = \sqrt{EG - F^2}.$$

We seek the conditions which must be satisfied by the fundamental functions for S in order that upon one of the surfaces applicable to it the virtual asymptotic lines form a conjugate system. We assume that there is a surface applicable to S in this manner and denote it by S_1, indicating by a subscript 1 the functions pertaining to it.

* BIANCHI, *Lezioni*, II, p. 437. Also EISENHART, *Surfaces of constant curvature and their transformations*, these Transactions, vol. 6 (1905), pp. 472–485.

† L. c., where I have shown that the Hazzidakis transform can be found readily when one knows an isothermal-conjugate system.

‡ BIANCHI, *Lezioni*, I, p. 120.

Since the parametric lines on S are asymptotic, on S_1 they form a conjugate system; so that the Codazzi equations* for S_1 are

$$(3)\qquad \begin{cases} \dfrac{\partial}{\partial v}\left(\dfrac{D_1}{H}\right) + \left\{ {22 \atop 2} \right\} \dfrac{D_1}{H} + \left\{ {11 \atop 2} \right\} \dfrac{D_1''}{H} = 0, \\[2ex] \dfrac{\partial}{\partial u}\left(\dfrac{D_1''}{H}\right) + \left\{ {22 \atop 1} \right\} \dfrac{D_1}{H} + \left\{ {11 \atop 1} \right\} \dfrac{D_1''}{H} = 0. \end{cases}$$

In these equations and in equations (1) the Christoffel symbols $\left\{ {rs \atop t} \right\}$ are formed with respect to the common linear element of the two surfaces

$$(4)\qquad ds^2 = Edu^2 + 2Fdu\,dv + Gdv^2.$$

Since the gaussian curvatures of the two surfaces are equal, we have the equation

$$(4)\qquad -D'^2 = D_1 D_1'',$$

which we replace by

$$(5)\qquad D_1 = e^t D', \qquad D_1'' = -e^{-t} D',$$

where t is an auxiliary function. If these values of D_1 and D_1'' be substituted in (3) and reduction be made in accordance with equations (1), we obtain the following equations for the determination of t:

$$(6)\qquad \begin{cases} \dfrac{\partial t}{\partial v} + \left\{ {22 \atop 2} \right\} - 2\left\{ {12 \atop 1} \right\} - \left\{ {11 \atop 2} \right\} e^{-2t} = 0, \\[2ex] \dfrac{\partial t}{\partial u} - \left\{ {11 \atop 1} \right\} + 2\left\{ {12 \atop 2} \right\} + \left\{ {22 \atop 1} \right\} e^{2t} = 0. \end{cases}$$

The condition of integrability of these equations is

$$(7)\qquad \begin{aligned} & e^{2t}\left[\frac{\partial}{\partial v}\left\{ {22 \atop 1} \right\} - 2\left\{ {22 \atop 1} \right\}\left\{ {22 \atop 2} \right\} + 4\left\{ {22 \atop 1} \right\}\left\{ {12 \atop 1} \right\}\right] \\ & \quad + e^{-2t}\left[\frac{\partial}{\partial u}\left\{ {11 \atop 2} \right\} - 2\left\{ {11 \atop 2} \right\}\left\{ {11 \atop 1} \right\} + 4\left\{ {11 \atop 2} \right\}\left\{ {12 \atop 2} \right\}\right] \\ & \quad + \left[\frac{\partial}{\partial u}\left(2\left\{ {12 \atop 1} \right\} - \left\{ {22 \atop 2} \right\}\right) + \frac{\partial}{\partial v}\left(2\left\{ {12 \atop 2} \right\} - \left\{ {11 \atop 1} \right\}\right) + 4\left\{ {11 \atop 2} \right\}\left\{ {22 \atop 1} \right\}\right] = 0. \end{aligned}$$

This equation is quadratic in e^{2t}. From (5) it is seen that the two surfaces S_1 arising from a value of e^{2t} are symmetric with respect to the origin and consequently can be referred to as one surface. For convenience we write the above equation in the abbreviated form

$$(8)\qquad Ae^{2t} + Be^{-2t} + C = 0.$$

* BIANCHI, *Lezioni*, I, p. 120.

Hence unless

$$A = B = C = 0, \tag{9}$$

there are at most two surfaces S_1. In order to find the conditions to be satisfied, we solve equation (8) for e^{2t} and substitute the resulting value in equations (6). We shall find later that there are surfaces S for which the resulting equations of condition are satisfied.

When now equations (9) are satisfied, there are an infinity of surfaces applicable to S_1 with conjugate virtual asymptotic lines. It follows from (1) that

$$\frac{\partial}{\partial u}\left\{\begin{matrix}12\\1\end{matrix}\right\} = \frac{\partial}{\partial v}\left\{\begin{matrix}12\\2\end{matrix}\right\}, \tag{10}$$

so that in consequences of the identities

$$\frac{\partial}{\partial u}\log H = \left\{\begin{matrix}11\\1\end{matrix}\right\} + \left\{\begin{matrix}12\\2\end{matrix}\right\}, \qquad \frac{\partial}{\partial v}\log H = \left\{\begin{matrix}22\\2\end{matrix}\right\} + \left\{\begin{matrix}12\\1\end{matrix}\right\}, \tag{11}$$

we have also

$$\frac{\partial}{\partial u}\left\{\begin{matrix}22\\2\end{matrix}\right\} = \frac{\partial}{\partial v}\left\{\begin{matrix}11\\1\end{matrix}\right\}. \tag{12}$$

Hence the equations of condition (9) can be written thus:

$$\begin{gathered}
\frac{\partial}{\partial v}\left\{\begin{matrix}22\\1\end{matrix}\right\} - 2\left\{\begin{matrix}22\\1\end{matrix}\right\}\left\{\begin{matrix}22\\2\end{matrix}\right\} + 4\left\{\begin{matrix}22\\1\end{matrix}\right\}\left\{\begin{matrix}12\\1\end{matrix}\right\} = 0,\\
\frac{\partial}{\partial u}\left\{\begin{matrix}11\\2\end{matrix}\right\} - 2\left\{\begin{matrix}11\\2\end{matrix}\right\}\left\{\begin{matrix}11\\1\end{matrix}\right\} + 4\left\{\begin{matrix}11\\2\end{matrix}\right\}\left\{\begin{matrix}12\\2\end{matrix}\right\} = 0,\\
\frac{\partial}{\partial u}\left[2\left\{\begin{matrix}12\\1\end{matrix}\right\} - \left\{\begin{matrix}22\\2\end{matrix}\right\}\right] = \frac{\partial}{\partial v}\left[2\left\{\begin{matrix}12\\2\end{matrix}\right\} - \left\{\begin{matrix}11\\1\end{matrix}\right\}\right] = -2\left\{\begin{matrix}11\\2\end{matrix}\right\}\left\{\begin{matrix}22\\1\end{matrix}\right\}.
\end{gathered} \tag{13}$$

When S and S_1 are referred to any parametric system whatever, the necessary and sufficient condition that to asymptotic lines on S correspond a conjugate system on S_1 is

$$DD_1'' + D''D_1 - 2D'D_1' = 0.$$

The symmetrical form of this equation enables us to state that the asymptotic lines on S_1 also correspond to a conjugate system on S. Hence, when a surface is referred to a conjugate system of lines, the necessary and sufficient condition that it be capable of a deformation in which the virtual asymptotic lines are conjugate is

$$\frac{\partial}{\partial u}\log\sqrt{-K} = -2\left\{\begin{matrix}12\\2\end{matrix}\right\}, \qquad \frac{\partial}{\partial v}\log\sqrt{-K} = -2\left\{\begin{matrix}12\\1\end{matrix}\right\}, \tag{14}$$

where K denotes the total curvature of the surface and the Christoffel symbols

are formed with respect to the linear element. For equations (14) are the condition that one of the surfaces with the linear element (4) have its asymptotic lines parametric.

§ 2. *Quadrics and ruled surfaces.*

We consider the particular case

$$\left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\} = \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\} = 0\,; \tag{15}$$

that is, S is a quadric surface. Now equations (13) reduce to

$$\frac{\partial}{\partial u}\left[2\left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\} - \left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\}\right] = \frac{\partial}{\partial v}\left[2\left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\} - \left\{ \begin{matrix} 11 \\ 1 \end{matrix} \right\}\right] = 0\,. \tag{16}$$

Hence, if there is one surface S_1, there is an infinite number.

We consider first the case of central quadrics. The equations of such a surface can be put in the form

$$x = a\frac{1+uv}{u+v}, \qquad y = b\frac{u-v}{u+v}, \qquad z = c\frac{1-uv}{u+v}.$$

The constants a, b, c are real for the hyperboloid of one sheet. One finds readily

$$\left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\} = \frac{-2}{u+v},$$

$$\left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\} = \frac{a^2b^2(uv-1)(u^2+1) - 2a^2c^2(u-v)u + b^2c^2(uv+1)(u^2-1)}{(u+v)[a^2b^2(uv-1)^2 + a^2c^2(u-v)^2 + b^2c^2(vu+1)^2]},$$

which do not satisfy equation (16).

Again, the equations of the paraboloids are

$$x = a(u+v), \qquad y = b(u-v), \qquad z = uv.$$

Now

$$H = [4a^2b^2 + (a^2+b^2)(u+v) - 2(a^2-b^2)uv]^{\frac{1}{2}}$$

and

$$\left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\} = 0, \qquad \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\} = \frac{\partial}{\partial v}\log H.$$

Hence equation (16) becomes

$$\frac{\partial^2}{\partial u \partial v}\log H = 0\,,$$

which evidently is not satisfied by the above value of H. Therefore equations (15) do not furnish a solution of the problem.

We consider next the case

$$\left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\} = 0.$$

Now the lines $v = \text{const.}$ are straight and consequently S is a ruled surface. The parameters of the asymptotic lines can be so chosen that the linear element takes the form *

$$ds^2 = du^2 + 2\cos\theta\, du\, dv + (M^2u^2 + 2Nu + 1)\, dv^2,$$

where $\cos\theta$, M and N are functions of v alone. Now

$$\left\{ \begin{matrix} 11 \\ 1 \end{matrix} \right\} = 0, \qquad \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\} = -\cos\theta \frac{M^2u + N}{M^2u^2 + 2Nu + \sin^2\theta};$$

so that the first two of equations (13) are satisfied identically and the last reduces to

$$\frac{\partial}{\partial u} \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\} = \frac{\partial}{\partial u} \left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\} = \frac{\partial}{\partial v} \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\} = 0.$$

But

$$\frac{\partial}{\partial u} \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\} = \frac{\cos\theta\,[M^4u^2 + 2M^2Nu + (2N^2 - M^2\sin\theta)]}{2(M^2u^2 + 2Nu + \sin^2\theta)^2}.$$

This can vanish only when $\cos\theta = 0$, or

$$M = N = 0.$$

In the latter case the total curvature of the surface is zero, so that it may be excluded; in the former case the ruled surface is the skew helicoid with plane director, which we know is applicable to an infinity of surfaces of revolution in such a way that the lines of curvature of the latter correspond to the asymptotic lines of the former. Hence:

The helicoid with plane director is the only ruled surface which can be deformed in an infinity of ways so that the asymptotic lines become a conjugate system.

§ 3. *Spherical representation of S.*

Between the Christoffel symbols $\left\{ \begin{matrix} rs \\ t \end{matrix} \right\}'$ formed with respect to the linear element of the spherical representation of S and the symbols formed with respect to the linear element (4) the following relations obtain: †

$$(17) \qquad \begin{aligned} &\left\{ \begin{matrix} 11 \\ 1 \end{matrix} \right\} = \left\{ \begin{matrix} 11 \\ 1 \end{matrix} \right\}' - 2\left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}', \quad \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\} = -\left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}', \quad \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\} = -\left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}', \\ &\left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\} = -\left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}', \quad \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\} = -\left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}', \quad \left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\} = \left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\}' - 2\left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}'. \end{aligned}$$

In consequence of these relations equations (6) can be given the form

$$(18) \qquad \begin{aligned} &\frac{\partial t}{\partial v} + \left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\}' + \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}' e^{-2t} = 0, \\ &\frac{\partial t}{\partial u} - \left\{ \begin{matrix} 11 \\ 1 \end{matrix} \right\}' - \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}' e^{2t} = 0. \end{aligned}$$

* BIANCHI, *Lezioni*, I, p. 254.

† BIANCHI, *Lezioni*, I, p. 157.

The condition of integrability takes the form

$$
\begin{aligned}
&\left[\frac{\partial}{\partial u}\left\{\begin{matrix}22\\2\end{matrix}\right\}' + \frac{\partial}{\partial v}\left\{\begin{matrix}11\\1\end{matrix}\right\}' - 4\left\{\begin{matrix}11\\2\end{matrix}\right\}'\left\{\begin{matrix}22\\1\end{matrix}\right\}'\right] \\
&\qquad + e^{2t}\left[\frac{\partial}{\partial v}\left\{\begin{matrix}22\\1\end{matrix}\right\}' - 2\left\{\begin{matrix}22\\1\end{matrix}\right\}'\left\{\begin{matrix}22\\2\end{matrix}\right\}'\right] \\
&\qquad\qquad + e^{-2t}\left[\frac{\partial}{\partial u}\left\{\begin{matrix}11\\2\end{matrix}\right\}' - 2\left\{\begin{matrix}11\\2\end{matrix}\right\}'\left\{\begin{matrix}11\\1\end{matrix}\right\}'\right] = 0\,.
\end{aligned}
\tag{19}
$$

We have seen that, unless S is a skew helicoid with plane director, $\left\{\begin{matrix}11\\2\end{matrix}\right\} \neq 0$, $\left\{\begin{matrix}22\\1\end{matrix}\right\} \neq 0$. Hence, excluding this exceptional case, we find, for the conditions that there be an infinity of surfaces S_1,

$$
\begin{aligned}
&\frac{\partial}{\partial u}\left\{\begin{matrix}22\\2\end{matrix}\right\}' = \frac{\partial}{\partial v}\left\{\begin{matrix}11\\1\end{matrix}\right\}' = 2\left\{\begin{matrix}11\\2\end{matrix}\right\}'\left\{\begin{matrix}22\\1\end{matrix}\right\}', \\
&\frac{\partial}{\partial u}\log\left\{\begin{matrix}11\\2\end{matrix}\right\}' = 2\left\{\begin{matrix}11\\1\end{matrix}\right\}', \qquad \frac{\partial}{\partial v}\log\left\{\begin{matrix}22\\1\end{matrix}\right\}' = 2\left\{\begin{matrix}22\\2\end{matrix}\right\}'.
\end{aligned}
\tag{20}
$$

§ 4. *Surfaces associate to S.*

We have shown elsewhere* that the coördinates, x', y', z', of an associate surface of the surface S are given by the quadratures

$$
\frac{\partial x'}{\partial u} = \mu\frac{\partial x}{\partial v}, \qquad \frac{\partial x'}{\partial v} = \sigma\frac{\partial x}{\partial u},
\tag{22}
$$

where μ and σ are a pair of solutions of the equations

$$
\begin{aligned}
&\frac{\partial \mu}{\partial v} + \mu\left\{\begin{matrix}22\\2\end{matrix}\right\} - \sigma\left\{\begin{matrix}11\\2\end{matrix}\right\} = 0\,, \\
&\frac{\partial \sigma}{\partial u} + \sigma\left\{\begin{matrix}11\\1\end{matrix}\right\} - \mu\left\{\begin{matrix}22\\1\end{matrix}\right\} = 0\,.
\end{aligned}
\tag{23}
$$

On comparing these equations with (3) we remark that an associate surface, Σ, is determined by the pair of solutions

$$
\mu = \frac{D_1}{H}, \qquad \sigma = -\frac{D_1''}{H}\,.
\tag{24}
$$

If the fundamental functions for Σ be denoted by E', F', G'; Δ, Δ', Δ'', it follows from (5) and (22) that

$$
\begin{aligned}
&E' = e^{2t}G\frac{D^2}{H^2}, \qquad F' = \frac{D^2}{H^2}F, \qquad G' = e^{-2t}E\frac{D^2}{H^2}, \\
&\Delta = e^{t}\frac{D^2}{H}, \qquad \Delta' = 0\,, \qquad \Delta'' = e^{-t}\frac{D^2}{H}\,.
\end{aligned}
\tag{25}
$$

* *Associate surfaces*, Mathematische Annalen, vol. 68 (1906), p. 507.

From these values it follows that the total curvature of Σ is equal to unity. Hence Σ is a spherical surface.

Since S is referred to its asymptotic lines, the spherical representation of the parametric lines satisfies the condition

$$\frac{\partial}{\partial u}\begin{Bmatrix}12\\1\end{Bmatrix}' = \frac{\partial}{\partial v}\begin{Bmatrix}12\\2\end{Bmatrix}', \tag{26}$$

that is, the parametric system on Σ is a conjugate system with equal tangential invariants.

Conversely, given a system of lines on the sphere satisfying the condition (26), we seek the other conditions to be satisfied in order that it may represent a conjugate system on a spherical surface. Now

$$\frac{\Delta\Delta''}{\mathbf{EG}-\mathbf{F}^2} = 1,$$

where the linear element of the spherical representation of Σ is written

$$d\sigma^2 = \mathbf{E}du^2 + 2\mathbf{F}du\,dv + \mathbf{G}dv^2.$$

We replace the above equation by the two

$$\frac{\Delta}{\sqrt{\mathbf{EG}-\mathbf{F}^2}} = e^t, \qquad \frac{\Delta''}{\sqrt{\mathbf{EG}-\mathbf{F}^2}} = e^{-t}.$$

If these values be substituted in the Codazzi equations *

$$\frac{\partial}{\partial v}\left(\frac{\Delta}{\sqrt{\mathbf{EG}-\mathbf{F}^2}}\right) + \begin{Bmatrix}22\\2\end{Bmatrix}'\frac{\Delta}{\sqrt{\mathbf{EG}-\mathbf{F}^2}} + \begin{Bmatrix}11\\2\end{Bmatrix}'\frac{\Delta''}{\sqrt{\mathbf{EG}-\mathbf{F}^2}} = 0,$$

$$\frac{\partial}{\partial u}\left(\frac{\Delta''}{\sqrt{\mathbf{EG}-\mathbf{F}^2}}\right) + \begin{Bmatrix}22\\1\end{Bmatrix}'\frac{\Delta}{\sqrt{\mathbf{EG}-\mathbf{F}^2}} + \begin{Bmatrix}11\\1\end{Bmatrix}'\frac{\Delta''}{\sqrt{\mathbf{EG}-\mathbf{F}^2}} = 0,$$

we get equations (18). Hence, if we have a spherical surface referred to a conjugate system with equal tangential invariants, we have a solution of equations (18) and thus can get a surface S_1 applicable to the surface S associate to Σ and upon which the parametric lines are asymptotic. It is well known that the surface S is found by quadratures. Therefore the problem of finding surfaces capable of a deformation in which the asymptotic lines become conjugate on the new surface is equivalent to the problem of finding spherical surfaces referred to conjugate systems with equal tangential invariants. And we have the theorem:

The surfaces associate to a spherical surface can be deformed in such a way that on one of the deforms the virtual asymptotic lines form a conjugate system; and they are the only surfaces possessing this property.

* BIANCHI, *Lezioni*, I, p. 166.

It is evident from this theorem that when a surface can be deformed in the above manner in an infinity of ways it has an infinity of spherical surfaces associate to it.

§ 5. *Hazzidakis transformations.*

We proceed now to the consideration of the surface S_1. Since the parametric system is composed of virtual asymptotic lines, equation (10) is satisfied and consequently these lines form a conjugate system with equal point invariants. We have shown* that, when a surface is referred to such a family of lines, an associate surface is given by the quadratures

$$\frac{\partial \xi_1}{\partial u} = \lambda \frac{\partial x_1}{\partial u}, \qquad \frac{\partial \xi_1}{\partial v} = -\lambda \frac{\partial x_1}{\partial v}, \tag{27}$$

and similar quadratures for η_1, ζ_1, where (ξ_1, η_1, ζ_1) are the coordinates of the associate Σ_1, and the function λ is given by

$$\frac{\partial \log \lambda}{\partial v} = -2 \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}_1, \qquad \frac{\partial \log \lambda}{\partial u} = -2 \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}_1. \tag{28}$$

Since S_1 is applicable to S, it follows from (1) that†

$$\lambda = \frac{D'}{H}.$$

From these results it follows that the fundamental functions of the associate surface Σ_1, denoted by E_1', F_1', G_1'; Δ_1, Δ_1', Δ_1'', have the values

$$\begin{aligned} E_1' &= \frac{D'^2}{H^2} E_1, & F_1' &= -\frac{D'^2}{H^2} F_1, & G_1' &= \frac{D'^2}{H^2} G_1, \\ \Delta_1 &= \frac{D'^2}{H} e^t, & \Delta_1' &= 0, & \Delta_1'' &= \frac{D'^2}{H} e^{-t}. \end{aligned} \tag{29}$$

Since S and S_1 are applicable, it follows that the expression for the total curvature of Σ_1 reduces to unity, that is Σ_1 is a spherical surface.

Incidentally, we remark from (29) that if S admits of more than one applicable surface of the kind S_1 all of the corresponding associate surfaces Σ_1 will be applicable to one another with their parametric lines in correspondence.

From the form of equations (27) it follows that the parametric system on Σ_1 is conjugate and that the point invariants are equal. Moreover, if the Christoffel symbols $\left\{ \begin{matrix} rs \\ t \end{matrix} \right\}_{\Sigma_1}$ are formed with respect to the linear element of Σ_1, it follows that we must have

$$\frac{\partial \log \lambda}{\partial u} = 2 \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}_{\Sigma_1}, \qquad \frac{\partial \log \lambda}{\partial v} = 2 \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}_{\Sigma_1}. \tag{30}$$

* l. c., p. 531.

† Another surface is given by $\lambda = -D'/H$, but it is symmetric to this one with respect to the origin.

Conversely, given a surface of unit total curvature referred to a conjugate system with equal point invariants, and let ξ_1, η_1, ζ_1 be its rectangular coördinates, then the surface of coördinates x_1, y_1, z_1 determined by the quadratures (27) where λ is given by (30) is an associate of Σ_1. Between the gaussian curvatures of two surfaces associate to one another by means of equations of the form (27) the following relation exists: *

$$K_1 = -\frac{K_2}{\lambda^2}.$$

For the present case it is

$$K_1 = -\frac{1}{\lambda^2}.$$

This result and equations (28), which are the same as (30), show that the parametric system on S_1 is virtual asymptotic and from (27) it follows that it is conjugate.

We gather together the foregoing results into the following theorems:

When a surface admits of an applicable surface upon which the virtual asymptotic lines form a conjugate system, and either surface is referred to its asymptotic lines, an associate of each surface can be found by quadratures; these associates are spherical surfaces and have the same second quadratic forms.

When a spherical surface is referred to a conjugate system with equal point or tangential invariants, a surface associate to it can be found which admits of a deformation such that the virtual asymptotic lines are conjugate.

The general equation of the lines of curvature of a surface is †

$$(ED' - FD)du^2 + (ED'' - GD)\,du\,dv + (FD'' - GD')dv^2 = 0.$$

For Σ this reduces to

$$e^t F du^2 + (Ee^{-t} - Ge^t)\,du\,dv - e^{-t} F dv^2 = 0, \tag{31}$$

where E, F, G are the first coefficients of S referred to its asymptotic lines. From (29) it follows that the equation for the lines of curvature of Σ_1 is likewise (31). Since Σ and Σ_1 have their lines of curvature in correspondence and the same second quadratic forms, they are the Hazzidakis transforms of one another.‡ Since each spherical surface has a unique Hazzidakis transform, it follows that the surface Σ_1 is the same no matter which conjugate system with equal point or tangential invariants is taken for the parametric system. Hence we have the theorems:

* Loc. cit., p. 506.

† BIANCHI, *Lezioni*, I, p. 128.

‡ *Surfaces of constant curvature*, etc., these Transactions, vol. 6 (1905), p. 478.

When a spherical surface is referred to a conjugate system with equal point or tangential invariants, its Hazzidakis transform can be found by quadratures.

The surfaces S_1 into which the associates S of a spherical surface Σ are deformed so that virtual asymptotic lines are conjugate are the associate surfaces of the Hazzidakis transform of Σ.

From the general expressions for the coefficients of the spherical representation of a surface referred to a conjugate system* one deduces by means of (25) and (29) that the coefficients of the spherical representation of Σ and Σ_1 are

$$\mathbf{E} = \frac{D'^2}{H^2} E, \qquad \mathbf{F} = -\frac{D'^2}{H^2} F, \qquad \mathbf{G} = \frac{D'^2}{H^2} G,$$

$$\mathbf{E}_1 = e^{2t} \frac{D'^2}{H^2} G, \quad \mathbf{F}_1 = \frac{D'^2}{H^2} F, \quad \mathbf{G}_1 = e^{-2t} \frac{D'^2}{H^2} E.$$

Comparing these expressions with (25) and (29), we see that *each of the surfaces Σ and Σ_1 is applicable to the spherical representation of the other.*

§ 6. *Conjugate systems with equal point and tangential invariants.*

In consequence of the foregoing result, if the conjugate system on Σ corresponding to the asymptotic lines on S has equal point invariants, the corresponding system on Σ_1 has equal tangential invariants. Hence:

When there is known a spherical surface referred to a conjugate system with equal point and tangential invariants, another spherical surface can be found upon which the corresponding lines form a conjugate system of the same kind; it is the Hazzidakis transform of the given surface.

We inquire under what conditions a conjugate system is of this kind. The Codazzi equations for Σ can be written in the form

$$(32) \qquad \frac{\partial \log \Delta}{\partial v} - \begin{Bmatrix} 12 \\ 1 \end{Bmatrix}_\Sigma + \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}_\Sigma \frac{\Delta''}{\Delta} = 0, \quad \frac{\partial \log \Delta''}{\partial u} - \begin{Bmatrix} 12 \\ 2 \end{Bmatrix}_\Sigma + \begin{Bmatrix} 22 \\ 1 \end{Bmatrix}_\Sigma \frac{\Delta}{\Delta''} = 0.$$

In terms of the functions of Σ equation (26) can be put in the form

$$\frac{\partial}{\partial u}\left(\frac{\Delta''}{\Delta} \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}_\Sigma \right) = \frac{\partial}{\partial v}\left(\frac{\Delta}{\Delta''} \begin{Bmatrix} 22 \\ 1 \end{Bmatrix}_\Sigma \right).$$

Hence it follows from (32) that the necessary and sufficient condition that the point invariants be equal is

$$\frac{\partial^2}{\partial u \, \partial v} \log \frac{\Delta}{\Delta''} = 0.$$

By a suitable choice of parameters this can be replaced by

$$\Delta = \Delta''.$$

* BIANCHI, *Lezioni*, I, p. 150.

But the necessary and sufficient conditions that a conjugate system on a spherical surface be isothermal-conjugate are *

$$(33)\qquad \left\{ \begin{matrix} 11 \\ 1 \end{matrix} \right\}' + \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}' = 0, \qquad \left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\}' + \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}' = 0,$$

which are equivalent to

$$(34)\qquad \frac{\partial \mathbf{E}}{\partial u} - \frac{\partial \mathbf{G}}{\partial u} + 2\frac{\partial \mathbf{F}}{\partial v} = 0, \qquad \frac{\partial \mathbf{E}}{\partial v} - \frac{\partial \mathbf{G}}{\partial v} - 2\frac{\partial \mathbf{F}}{\partial u} = 0.$$

Eliminating $\mathbf{E} - \mathbf{G}$ from these equations, we get

$$\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) \mathbf{F} = 0,$$

so that

$$\mathbf{F} = \Phi(u + iv) + \Phi_0(u - iv),$$

where Φ is an arbitrary function and Φ_0 is its conjugate, as we are confining ourselves to the consideration of real surfaces, and

$$\mathbf{E} - \mathbf{G} = -i\left[\Phi(u + iv) - \Phi_0(u - iv)\right].$$

In order that the conditions of the problem be satisfied, the function Φ must be such that these values of $\mathbf{E}, \mathbf{F}, \mathbf{G}$ satisfy the Gauss equation for the sphere and the equation (26).

We assume that the representation of S is of this kind. Equation (19) reduces to

$$(e^t - 1)^2 \left[\frac{\partial}{\partial u} \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}' + 2 \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}' \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}' \right] = 0.$$

This is satisfied by $e^t = 1$, which is a solution of equations (18). From this it is seen that if there are more than one surface of the kind there are an infinity of them and the condition of the latter case is

$$\frac{\partial}{\partial u} \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}' = -2 \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}' \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}'.$$

Since the conjugate virtual asymptotic lines on S_1 have equal tangential invariants, one finds the coördinates of an associate S_1' of S_1 by the quadratures of the form

$$\frac{\partial x_1'}{\partial u} = \mu' \frac{\partial x_1}{\partial v}, \qquad \frac{\partial x_1'}{\partial v} = \sigma' \frac{\partial x_1}{\partial u},$$

where

$$(35)\qquad \begin{aligned} \mu' &= \rho_1 \frac{D_1}{H_1}, & \sigma' &= \rho_1 \frac{D_1''}{H_1}, \\ \frac{\partial \log \rho_1}{\partial u} &= -2 \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}_1', & \frac{\partial \log \rho_1}{\partial v} &= -2 \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}_1'. \end{aligned}$$

* BIANCHI, *Lezioni*, I, p. 168.

Moreover, we have shown* that these functions μ' and σ' are solutions of

$$\frac{\partial \mu'}{\partial v} + \mu' \left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\}_1 - \sigma' \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}_1 = 0, \qquad \frac{\partial \sigma'}{\partial u} + \sigma' \left\{ \begin{matrix} 11 \\ 1 \end{matrix} \right\}_1 - \mu' \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}_1 = 0,$$

where the Christoffel symbols $\left\{ \begin{matrix} rs \\ t \end{matrix} \right\}_1$ are formed with respect to the linear element of S_1 and consequently of S. Hence these equations are the same as equations (23), so that we can find a surface S' associate to S by means of the quadratures of the form

$$\frac{\partial x'}{\partial u} = \mu' \frac{\partial x}{\partial v}, \qquad \frac{\partial x'}{\partial v} = \sigma' \frac{\partial x}{\partial u}, \tag{36}$$

where μ' and σ' have the values (35). We see at once that the two surfaces S' and S_1' are applicable, for they have the same linear element

$$\mu'^2 G du^2 + 2\mu'\sigma' F du\, dv + \sigma'^2 E dv^2,$$

and the parametric lines are conjugate on S' and asymptotic on S_1'.

Taking $e' = 1$, we get from (5), (25), and (29),

$$\Delta = \Delta'', \qquad D_1 = -D_1'', \qquad \Delta_1 = \Delta_1''. \tag{37}$$

Moreover, the linear elements of Σ and Σ_1 reduce to

$$\frac{D'^2}{H^2}(G du^2 + 2F du\, dv + E dv^2), \qquad \frac{D'^2}{H^2}(E du^2 - 2F du\, dv + G dv^2),$$

showing that the surfaces are the Hazzidakis transforms of one another.*

If we denote by (D), (D'), (D'') the second fundamental functions of S_1', it follows from (36) that

$$(D) = \mu' D', \qquad (D') = 0, \qquad (D'') = \sigma' D'.$$

But $\mu' = -\sigma'$, hence

$$(D)\Delta'' + (D'')\Delta = 0.$$

Therefore S' and S_1' are associate surfaces of Σ and Σ_1 respectively; consequently S' can be found from Σ by making use of the fact that the conjugate parametric system has equal point invariants, and S_1' from Σ_1 from the standpoint that the conjugate system has equal tangential invariants. In both cases the determination requires only quadratures. Hence we have the theorem:

When a spherical surface has an isothermal-conjugate system with equal tangential invariants and this system is parametric, one can find by quadratures two surfaces which can be deformed in such a way that the asymptotic lines become conjugate; and these new surfaces are associate surfaces of the Hazzidakis transform of the given spherical surface.

* *Surfaces of constant curvature, etc.*, Transactions, vol. 6 (1905), p. 478.

In this connection we have been considering the case where a surface S has only one surface S_1 applicable to it with its virtual asymptotic lines conjugate. In case there are more than one, for each surface S_1 there is a spherical surface Σ associate to S.

From (37) it is seen that if we put

$$u = u_1 - v_1, \qquad v = u_1 + v_1,$$

the curves $u_1 =$ const. and $v_1 =$ const. on S_1 are asymptotic and on S conjugate. And the equation of the asymptotic lines on S is of the form

$$\lambda(du_1^2 - dv_1^2) = 0.$$

Recalling the preceding results, we see that the function t_1, by means of which S is formed from S_1 when the latter is referred to its asymptotic lines, is zero also. Consequently *when the asymptotic lines on S satisfy the conditions* (33), *a similar set of equations are satisfied by the spherical representation of the asymptotic lines of S_1*. And in all respects it can be shown that the relation between the surfaces S and S_1 and the surfaces adjoined to them is perfectly reciprocal.

§ 7. *Minimal surfaces.*

As an example of the foregoing results, we consider the case where S is a minimal surface. By a suitable choice of parameters we can put

$$\mathbf{E} = \mathbf{G} = \frac{1}{\rho}, \qquad \mathbf{F} = 0, \qquad \frac{1}{\rho^2} = -K, \tag{38}$$

where K denotes the Gaussian curvature. Equations (34) are evidently satisfied, as is also equation (26). The linear element of S is

$$ds^2 = \rho(du^2 + dv^2) \tag{39}$$

and $D' = 1$. Hence for the value $e' = 1$,

$$D_1 = -D_1'' = 1,$$

so that S_1 is the minimal surface adjoint to S. From (25) and (29) it is seen that Σ and Σ_1 are each the unit sphere and corresponding points coincide. And from (35) and (36) it follows that S' coincides with S_1 and S_1' with S.

We inquire whether there are any other surfaces S_1 applicable in the desired way to a minimal surface. Now equations (18) reduce to

$$\frac{\partial t}{\partial v} - \frac{\partial}{\partial v}\log\sqrt{\rho}(1 - e^{-2t}) = 0, \qquad \frac{\partial t}{\partial u} + \frac{\partial}{\partial u}\log\sqrt{\rho}(1 - e^{2t}) = 0, \tag{40}$$

and the equation (19) becomes

$$(e^t - e^{-t})^2 \frac{\partial^2 \rho}{\partial u \partial v} = 0.$$

Neglecting the solutions $e' = \pm 1$, which we considered above, we see that equations (40) are integrable when

$$\rho = \phi(u) + \psi(v).$$

But the functions $\phi(u)$ and $\psi(v)$ are not arbitrary, for there remains to be satisfied the Gauss equation of the sphere,* which is now

$$\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)\log\rho = \frac{2}{\rho}.$$

This is satisfied by the above value of ρ only in case ϕ or ψ is equal to a constant.† Hence all the surfaces S_1 are surfaces of revolution referred to their meridians and parallels, and the surface S_1 corresponding to the solution $t = 0$ is the catenoid. Hence S is the skew helicoid with plane director.

By a suitable choice of parameters the linear element of the skew helicoid with plane director can be reduced to the form

$$ds^2 = \frac{(e^u + a^2 e^{-u})^2}{4}(du^2 + dv^2). \tag{41}$$

Now the general integral of equations (40) is

$$e^t = \frac{2}{\sqrt{(e^u + a^2 e^{-u})^2 b + 4}}, \tag{42}$$

where b is the constant of integration. When $b = 0$, S_1 is the catenoid and for the other values of b the surfaces S_1 are the surfaces of revolution into which the catenoid can be deformed with preservation of the lines of curvature.‡

From equations (25), (41) and (42), we find the following expressions for the fundamental functions of the spherical surface Σ associate to S:

$$E' = \frac{16}{(e^u + a^2 e^{-u})^4 b + 4(e^u + a^2 e^{-u})^2}, \quad F' = 0, \quad G' = \frac{(e^u + a^2 e^{-u})^2 b + 4}{(e^u + a^2 e^{-u})^2}, \tag{43}$$

$$\Delta = \frac{8}{\sqrt{(e^u + a^2 e^{-u})^2 b + 4}\,(e^u + a^2 e^{-u})^2}, \quad \Delta' = 0, \quad \Delta'' = \frac{2\sqrt{(e^u + a^2 e^{-u})^2 b + 4}}{(e^u + a^2 e^{-u})^2}. \tag{44}$$

From these values it is found that Σ is a sphere only in case $b = 0$. We have seen that all the spherical surfaces Σ_1 associate to S_1 are applicable to one another and to the spherical representation of Σ, that is, their linear element is

$$d\sigma^2 = \frac{4}{(e^u + a^2 e^{-u})^2}(du^2 + dv^2) \tag{45}$$

and their second fundamental quantities are given by (44).

* Bianchi, *Lezioni*, I, p. 93.

† Ibid., II, p. 45.

‡ Ibid., I, p. 232.

We apply now to this case the results of the preceding section for the determination of the surfaces S' and S_1'. Since the surface Σ is applicable to the spherical representation of S_1, the last two of equations (35) can be replaced by

$$\frac{\partial \log \rho_1}{\partial u} = -2 \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}_\Sigma, \qquad \frac{\partial \log \rho_1}{\partial v} = -2 \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}_\Sigma,$$

of which the solution, without loss of generality, is

$$\rho_1 = \frac{1}{G'},$$

in consequence of (43). Hence

$$\mu' = \frac{8}{[(e^u + a^2 e^{-u})^2 b + 4]^{\frac{3}{2}}}, \qquad \sigma' = -\frac{2}{[(e^u + a^2 e^{-u})^2 b + 4]^{\frac{1}{2}}};$$

so that the linear element of the surface S' and S_1' is

$$ds'^2 = \frac{(e^u + a^2 e^{-u})^2}{(e^u + a^2 e^{-u})^2 b + 4} \left[\frac{16 du^2}{[(e^u + a^2 e^{-u})^2 b + 4]^2} + dv^2 \right],$$

and the respective second fundamental quantities are

$$(D) = \mu', \qquad (D') = 0, \qquad (D'') = \sigma',$$
$$(D_1) = 0, \qquad (D_1') = \sqrt{-\mu'\sigma'}, \qquad (D_1'') = 0.$$

From these results it is seen that S' is minimal only when $b = 0$, in which case it is the catenoid; moreover, S_1' is a minimal surface for all values of b and is applicable upon a surface of revolution.

Another solution of equations (34) is $\mathbf{E} = \mathbf{G}$; $\mathbf{F} = \text{const.}$ For the sake of convenience we put

$$\mathbf{E} = \mathbf{G} = \cosh\theta, \qquad \mathbf{F} = 1, \qquad \mathbf{EG} - \mathbf{F}^2 = \sinh^2\theta.$$

Equation (26) becomes

$$\frac{\partial}{\partial u}\left(\frac{1}{\sinh\theta}\frac{\partial\theta}{\partial u}\right) - \frac{\partial}{\partial v}\left(\frac{1}{\sinh\theta}\frac{\partial\theta}{\partial v}\right) = 0,$$

of which the general integral is

$$\tanh\frac{\theta}{2} = \phi(u+v)\cdot\psi(u-v),$$

where ϕ and ψ are arbitrary functions. The Gauss equation* for the sphere is

$$\frac{\partial^2\theta}{\partial u^2} + \frac{\partial^2\theta}{\partial v^2} + 2\sinh\theta = 0.$$

* BIANCHI, *Lezioni*, I, p. 93.

When the above value of θ is substituted, this equation becomes

$$\frac{\phi''}{\phi} + \frac{\psi''}{\psi} + 2\frac{\phi'^2\psi^2 + \psi'^2\phi^2}{1-\phi^2\psi^2} + 1 = 0, \tag{46}$$

where the primes denote differentiation with respect to the argument. If we differentiate this equation consecutively with respect to $u+v$ and $u-v$, we get

$$\frac{\phi\phi'\psi\psi'}{1-\phi^2\psi^2}\left(\frac{\phi''}{\phi} + \frac{\psi''}{\psi} + 2\frac{\phi'^2\psi^2 + \psi'^2\phi^2}{1-\phi^2\psi^2}\right) = 0.$$

Hence either ϕ' or ψ' must be zero. Let it be the latter and take $\psi = 1$; then the integral of (46) is given by the quadrature

$$\int \frac{d\phi}{\sqrt{c(\phi^2-1)^2 + (\phi^2-1)}} = u + v,$$

where c is a constant of integration, the additive constant being taken equal to zero without loss of generality.

In terms of this function ϕ the coefficients of the linear element of the spherical representation of S are given by

$$\mathbf{E} = \mathbf{G} = \frac{1+\phi^2}{1-\phi^2}, \qquad \mathbf{F} = 1.$$

From the equations

$$\frac{\partial \log \rho}{\partial u} = -2\left\{\begin{matrix}12\\2\end{matrix}\right\}', \qquad \frac{\partial \log \rho}{\partial v} = -2\left\{\begin{matrix}12\\1\end{matrix}\right\}',$$

we find

$$\rho = 1 - \phi^2,$$

and consequently

$$E = G = \rho^2\mathbf{E} = 1 - \phi^4, \qquad F = -\rho^2\mathbf{F} = (1-\phi^2)^2, \qquad D' = \rho\sqrt{\mathbf{EG} - \mathbf{F}^2} = 2\phi.$$

For the coefficients of the linear element of the spherical representation we have

$$\mathbf{E}_1 = \mathbf{G}_1 = \frac{1+\phi^2}{1-\phi^2}, \qquad \mathbf{F}_1 = -1,$$

and from (35)

$$\rho_1 = \frac{\phi^2-1}{\phi^2}, \qquad \mu' = -\sigma' = \frac{1}{\phi^2}.$$

Therefore the linear element of the surfaces S' and S_1' is

$$ds'^2 = \frac{1}{\phi^4}\left[(1-\phi^4)du^2 + 2(1-\phi^2)^2 du dv + (1-\phi^4)dv^2\right], \tag{47}$$

and the coefficients of the second quadratic forms of these surfaces are respectively

$$\frac{2}{\phi},\ 0,\ -\frac{2}{\phi};\qquad 0,\ \frac{2}{\phi},\ 0.$$

The parameters of the lines of curvature on S are given by

$$u+v=u_1,\qquad u-v=v_1.$$

In terms of these parameters the two quadratic forms of S are

$$ds^2=(1-\phi^2)\left[\phi^2 du_1^2+dv_1^2\right],$$
$$\phi(du_1^2-dv_1^2).$$

Hence S is a surface of revolution. Moreover, these are the parameters of the asymptotic lines on S_1, hence S_1 is a minimal surface. In like manner (47) becomes

$$ds'^2=\frac{1-\phi^2}{\phi^4}(du_1^2+\phi^2 dv_1^2),$$

and the curves $u_1=$ const., $v_1=$ const. are the asymptotic lines on S'. Moreover, the curves $v_1=$ const. are geodesics; consequently S' is the skew helicoid with plane director. Hence the surfaces S which we have just been considering are the surfaces S' corresponding to the case where S is the skew helicoid.

§ 8. *Spherical representation of the surfaces S_1.*

We return to the consideration of the general problem and write the linear element of the spherical representation of S_1, referred to its conjugate virtual asymptotic lines, in the form

$$d\sigma_1^2=\mathbf{E}_1 du^2+2\mathbf{F}_1 du\,dv+\mathbf{G}_1 dv^2. \tag{48}$$

Between the Christoffel symbols $\left\{ {rs \atop t} \right\}'$, formed with respect to (48) and those with respect to the linear element of S and S_1 the following relations obtain:*

$$\left\{\begin{aligned}
&\left\{ {11 \atop 1} \right\}=\frac{\partial\log D_1}{\partial u}-\left\{ {11 \atop 1} \right\}'_1, \quad \left\{ {12 \atop 1} \right\}=-\frac{D_1''}{D_1}\left\{ {11 \atop 2} \right\}'_1, \quad \left\{ {22 \atop 1} \right\}=-\frac{D_1''}{D_1}\left\{ {12 \atop 2} \right\}'_1,\\
&\left\{ {22 \atop 2} \right\}=\frac{\partial\log D_1''}{\partial v}-\left\{ {22 \atop 2} \right\}'_1, \quad \left\{ {12 \atop 2} \right\}=-\frac{D_1}{D_1''}\left\{ {22 \atop 1} \right\}'_1, \quad \left\{ {11 \atop 2} \right\}=-\frac{D_1}{D_1''}\left\{ {12 \atop 1} \right\}'_1.
\end{aligned}\right. \tag{49}$$

From (3), (5) and (6) we get

$$\left\{\begin{aligned}
&\frac{\partial\log D_1}{\partial u}=2\left\{ {11 \atop 1} \right\}-3\left\{ {12 \atop 2} \right\}-\left\{ {22 \atop 1} \right\}e^{2t},\\
&\frac{\partial\log D_1''}{\partial v}=2\left\{ {22 \atop 2} \right\}-3\left\{ {12 \atop 1} \right\}-\left\{ {11 \atop 2} \right\}e^{-2t}.
\end{aligned}\right. \tag{50}$$

* BIANCHI, *Lezioni*, I, p. 167.

By means of (49), (50), and

$$\frac{D_1}{D_1''} = -e^{2t}, \tag{51}$$

equations (6) can be put in the form

$$\begin{aligned} \frac{\partial t}{\partial v} &= -e^{-2t}\left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}_1' - \left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\}_1', \\ \frac{\partial t}{\partial u} &= e^{2t}\left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}_1' + \left\{ \begin{matrix} 11 \\ 1 \end{matrix} \right\}_1'. \end{aligned} \tag{52}$$

The condition of integrability of these equations reduces to the form

$$A_1 e^{2t} + B_1 e^{-2t} + C_1 = 0, \tag{53}$$

where for the sake of brevity we have put

$$\begin{aligned} A_1 &= \frac{\partial}{\partial v}\left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}_1' - 2\left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\}_1'\left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}_1', \\ B_1 &= \frac{\partial}{\partial u}\left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}_1' - 2\left\{ \begin{matrix} 11 \\ 1 \end{matrix} \right\}_1'\left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}_1', \\ C_1 &= \frac{\partial}{\partial v}\left\{ \begin{matrix} 11 \\ 1 \end{matrix} \right\}_1' + \frac{\partial}{\partial u}\left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\}_1' - 4\left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}_1'\left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}_1'. \end{aligned} \tag{54}$$

In terms of the symbols $\left\{ \begin{matrix} rs \\ t \end{matrix} \right\}_1'$ equations (3) and (50) have the forms

$$\frac{\partial}{\partial v}\log D_1 = \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}_1' + \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}_1' e^{-2t}, \quad \frac{\partial}{\partial u}\log D_1'' = \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}_1' + \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}_1' e^{2t}, \tag{55}$$

and

$$\begin{aligned} \frac{\partial \log D_1}{\partial u} &= \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}_1' + 2\left\{ \begin{matrix} 11 \\ 1 \end{matrix} \right\}_1' + 3\left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}_1' e^{2t}, \\ \frac{\partial \log D_1''}{\partial v} &= \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}_1' + 2\left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\}_1' + 3\left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}_1' e^{-2t}. \end{aligned} \tag{56}$$

The conditions of integrability of these expressions are reducible to

$$\begin{aligned} 3e^{2t}A_1 - e^{-2t}B_1 + 2\left(\frac{\partial}{\partial v}\left\{ \begin{matrix} 11 \\ 1 \end{matrix} \right\}_1' - 2\left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}_1'\left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}_1' \right) &= 0, \\ -e^{2t}A_1 + 3e^{-2t}B_1 + 2\left(\frac{\partial}{\partial u}\left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\}_1' - 2\left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}_1'\left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}_1' \right) &= 0, \end{aligned} \tag{57}$$

which, when added together, give equation (53).

It is evident that there is at most one value of e^{2t} which satisfies equations (53) and (57). Suppose that we have a common solution of them and that it satisfies equations (52). From (55) and (56) we find D_1, D_1'' by quadratures

and the corresponding surface S_1 is capable of a deformation in which the parametric lines become asymptotic. For the total curvature of S_1 is given by

$$K = -\frac{1}{\rho^2} = \frac{\mathbf{E}_1\mathbf{G}_1 - \mathbf{F}_1^2}{D_1 D_1''} = -\frac{\mathbf{E}_1\mathbf{G}_1 - \mathbf{F}_1^2}{D_1^2 e^{-2t}},$$

which is such that

$$\frac{\partial \log \rho}{\partial u} = 2e^{2t}\left\{\begin{matrix}22\\1\end{matrix}\right\}_1' = 2\left\{\begin{matrix}12\\2\end{matrix}\right\}, \qquad \frac{\partial \log \rho}{\partial v} = 2e^{-2t}\left\{\begin{matrix}11\\2\end{matrix}\right\}_1' = 2\left\{\begin{matrix}12\\1\end{matrix}\right\}.$$

But this is the condition that there be a surface S with the same linear element as S_1 and with asymptotic lines parametric.

By means of equations (49) and (56) the first two of equations (13) can be put in the form

$$\frac{\partial}{\partial u}\left\{\begin{matrix}12\\1\end{matrix}\right\}_1' = \frac{\partial}{\partial v}\left\{\begin{matrix}12\\2\end{matrix}\right\}_1' = 2\left\{\begin{matrix}12\\1\end{matrix}\right\}_1'\left\{\begin{matrix}12\\2\end{matrix}\right\}_1'.$$

But these are the necessary and sufficient conditions that the curves upon the sphere represent the asymptotic lines upon a surface whose total curvature is of the form

$$K' = -\frac{1}{[\phi(u) + \psi(v)]^2}. \tag{58}$$

Hence *when a surface S is applicable upon an infinity of surfaces S_1 with the virtual asymptotic lines conjugate on the latter, each of the surfaces S_1 is associate to a surface whose total curvature is of the form* (58), *when the asymptotic lines of this surface are parametric.*

From (53) and (57) it is seen that in order that there be more than one surface S_1 with the given spherical representation, it is necessary and sufficient that

$$A_1 = B_1 = C_1 = 0, \qquad \frac{\partial}{\partial v}\left\{\begin{matrix}11\\1\end{matrix}\right\}_1' = \frac{\partial}{\partial u}\left\{\begin{matrix}22\\2\end{matrix}\right\}_1'.$$

But these are equations of the same form as equations (20). Hence when a system of lines on the sphere represents the asymptotic lines on a surface S applicable to an infinity of surfaces S_1 with conjugate virtual asymptotic lines, these lines represent the conjugate lines upon an infinity of surfaces S' each of which is applicable to a surface S_1' upon which the asymptotic lines are parametric.

We consider this case and fix our attention upon one of the surfaces S'. We know that there is a spherical surface Σ' associate to it, which is referred to a conjugate system with equal point invariants. But the tangential invariants also are equal. Hence the conditions (33) must be satisfied; consequently equations (20) reduce to

$$(59)\qquad \frac{\partial}{\partial u}\log\left\{\begin{matrix}11\\2\end{matrix}\right\}' = -2\left\{\begin{matrix}22\\1\end{matrix}\right\}', \qquad \frac{\partial}{\partial v}\log\left\{\begin{matrix}22\\1\end{matrix}\right\}' = -2\left\{\begin{matrix}11\\2\end{matrix}\right\}'.$$

From these equations we find

$$\frac{\partial^2 \log}{\partial u\,\partial v}\left\{\begin{matrix}11\\2\end{matrix}\right\}' = \frac{\partial^2 \log}{\partial u\,\partial v}\left\{\begin{matrix}22\\1\end{matrix}\right\}',$$

so that

$$\left\{\begin{matrix}11\\2\end{matrix}\right\}' = \lambda V', \qquad \left\{\begin{matrix}22\\1\end{matrix}\right\}' = \lambda U',$$

where λ is an auxiliary function, U and V are functions of u and v alone and the primes indicate differentiation. Substituting in (59) and integrating, we get

$$\lambda = \frac{1}{2(U+V)}.$$

Hence equations (20) can be replaced by

$$(60)\qquad \begin{aligned} \left\{\begin{matrix}11\\1\end{matrix}\right\}' &= -\left\{\begin{matrix}22\\1\end{matrix}\right\}' = -\frac{U'}{2(U+V)},\\ \left\{\begin{matrix}22\\2\end{matrix}\right\}' &= -\left\{\begin{matrix}11\\2\end{matrix}\right\}' = -\frac{V'}{2(U+V)}.\end{aligned}$$

Equations (18) reduce to

$$(61)\qquad \begin{aligned} e^{-2t}\frac{\partial t}{\partial u} &= -(e^{-2t}-1)\frac{U'}{2(U+V)},\\ e^{2t}\frac{\partial t}{\partial v} &= (e^{2t}-1)\frac{V'}{2(U+V)}.\end{aligned}$$

Neglecting for the moment the solution

$$(62)\qquad e^{2t} = 1,$$

the integrals of these respective equations are

$$(63)\qquad e^{-2t}-1 = (U+V)\psi(v), \qquad e^{2t}-1 = (U+V)\phi(u),$$

where $\psi(v)$ and $\phi(u)$ are arbitrary functions of v and u. Since equations (61) are simultaneous, we get on dividing the above equations

$$e^{2t} = -\frac{\phi(u)}{\psi(v)}.$$

Substituting in (63), we have

$$\phi + \psi = -(U+V)\phi\psi.$$

Consequently

$$\phi = \frac{1}{U - c}, \qquad \psi = \frac{1}{V + c},$$

where c is an arbitrary constant. Hence the general integral of equations (61) is

$$e^{2t} = \frac{c + V}{c - U}. \tag{64}$$

From (5), (22), (24) and (25) we obtain the following theorem:

When a surface S is applicable to an infinity of surfaces S_1 with the virtual asymptotic lines of the latter conjugate, the fundamental coefficients of the surfaces S_1 are given without quadrature; the surface S is associate to an infinity of spherical surfaces Σ whose rectangular coördinates are given by quadratures; and the integration of the equations of the asymptotic lines on the surfaces Σ reduces to quadratures.

PRINCETON,
June, 1906.

DYNAMICAL TRAJECTORIES: THE MOTION OF A PARTICLE IN AN ARBITRARY FIELD OF FORCE*

BY

EDWARD KASNER

The object of the present paper is to work out for space of three dimensions a geometric theory analogous to that given for the two-dimensional case in an earlier paper. †

We consider a particle moving freely in space under the action of any positional force, and write its equations of motion in the form

$$(1) \qquad \ddot{x} = \phi(x, y, z), \qquad \ddot{y} = \psi(x, y, z), \qquad \ddot{z} = \chi(x, y, z),$$

where the dots indicate derivatives with respect to the time. The functions ϕ, ψ, χ are assumed to have partial derivatives of first and second order in the region of space considered. The case where the force vanishes everywhere is excluded.

The motion of the particle is determined when its initial position and initial velocity are given. By taking all possible initial conditions we obtain a definite quintuply infinite system of trajectories. Our object is to study the properties of such systems of curves with a view to obtaining a *complete geometric characterization*. The main result is stated in article 39 at the end of the paper.

The first properties derived are consequences of the elementary fact that the osculating plane at any point of a trajectory is determined by its initial direction and by the direction of the force (articles 4–11). The next set relate to osculating spheres.‡ With each lineal element there are associated ∞^1 osculating spheres whose centers lie on a straight line (articles 12–15). The straight lines corresponding to all the elements at a point form a congruence of order one and class three determined by a twisted cubic curve (articles 16–23). The properties obtained at this stage belong to a more general class of curve systems than the dynamical class. The final characterization is attained by introducing certain related systems of plane curves, here termed S-systems, which have all the

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† *The Trajectories of Dynamics*, Transactions of the American Mathematical Society, vol. 7 (1906), pp. 401–424. We shall refer to this as *Trajectories*.

‡ Properties I_1, I_2, and II of the present paper were given in a note published in the Bulletin of the American Mathematical Society, vol. 12 (1905), pp. 71–74.

properties of plane systems of trajectories (articles 29–39). The S-system in a given plane may be obtained dynamically by constructing, at each point of the plane, the component of the force (1) acting in that plane; the two-dimensional field thus derived generates the required system.

It will be seen that the spatial properties are not direct extensions of those given in the plane theory; and that the present theory requires the previous discussion of the two-dimensional case. The earlier theory is thus not absorbed by the present paper.

The logical relations of the various properties are settled by converting the main theorems, thus bringing to light general systems of curves of interest apart from their connections with dynamics. Of the incidental results we mention only the interpretation of the curl of the force given in article 21, and the special characterization of conservative forces in articles 22 and 40.*

Differential equations of the trajectories

1. In order to derive the geometrical properties of the trajectories, it is necessary to eliminate the time. We distinguish the two types of derivatives involved by means of dots and primes. Thus

$$\dot{x} = \frac{dx}{dt}, \qquad \dot{y} = \frac{dy}{dt}, \qquad \dot{z} = \frac{dz}{dt}, \text{ etc.,} \tag{2}$$
$$y' = \frac{dy}{dx}, \qquad z' = \frac{dz}{dx}, \text{ etc.}$$

The geometric derivatives may be expressed in terms of the kinematic as follows:

$$y' = \frac{\dot{y}}{\dot{x}}, \qquad z' = \frac{\dot{z}}{\dot{x}}, \qquad y'' = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3}, \qquad z'' = \frac{\dot{x}\ddot{z} - \dot{z}\ddot{x}}{\dot{x}^3}. \tag{3}$$

Combining these equations with (1), we obtain two expressions for $\dot{x}^2$, namely

$$\dot{x}^2 = \frac{\psi - y'\phi}{y''}, \qquad \dot{x}^2 = \frac{\chi - z'\phi}{z''}. \tag{4}$$

Thus one result free from the time is

$$\frac{\psi - y'\phi}{y''} = \frac{\chi - z'\phi}{z''}. \tag{5}$$

Differentiating equations (4) with respect to x, we find two other equations

*The transformation theory given at the conclusion of *Trajectories* (articles 26–29) may be extended to space, but is not considered in the present paper. In the discussion of the APPELL transformation (article 30), the formulas should be expressed, not in terms of the times t and t_1, but, as in APPELL's treatment, in terms of the differentials dt and dt_1. The finite representation of page 423 yields merely the affine transformation. The appropriate converse question was solved by PAINLEVÉ, Journal de Mathématiques (1894).

free from the time,

$$(6)\qquad (\psi - y'\phi)y''' = \begin{vmatrix} 1 & \phi_x + y'\phi_y + z'\phi_z \\ y' & \psi_x + y'\psi_y + z'\psi_z \end{vmatrix} y'' - 3\phi y''^2,$$

$$(7)\qquad (\chi - z'\phi)z''' = \begin{vmatrix} 1 & \phi_x + y'\phi_y + z'\phi_z \\ z' & \chi_x + y'\chi_y + z'\chi_z \end{vmatrix} z'' - 3\phi z''^2,$$

where the subscripts denote partial derivatives.

The quintuply infinite system of trajectories is completely represented by the equations (5) *and* (6). We write these in the form

$$(8)\qquad \begin{aligned} z'' &= Ky'', \\ y''' &= Py'' + Qy''^2, \end{aligned}$$

where

$$(8')\qquad K = \frac{\chi - z'\phi}{\psi - y'\phi}, \qquad Q = \frac{-3\phi}{\psi - y'\phi},$$

$$P = \frac{\begin{vmatrix} 1 & \phi_x + y'\phi_y + z'\phi_z \\ y' & \psi_x + y'\psi_y + z'\psi_z \end{vmatrix}}{\psi - y'\phi}.$$

All other equations between the geometric derivatives, including (7), are indeed consequences of (8). This may be seen from the fact that a trajectory is completely determined by the assignment of the initial values of x, y, z, y', z', y''; for this is equivalent, according to (4), to the assignment of initial position and initial velocity. The corresponding values of z'', y''' are given by (8), and those of z''', y^{iv}, z^{iv}, etc., are obtained by successive differentiation of (8).

2. It will be convenient, in part of the subsequent discussion, to define the force whose components are ϕ, ψ, χ by the functions

$$(9)\qquad \omega_1 = \frac{\psi}{\phi}, \qquad \omega_2 = \frac{\chi}{\phi}, \qquad \Phi = \log \phi.$$

The first two of these define the direction of the force; the equations of the lines of force are, in fact,

$$(10)\qquad dx : dy : dz = 1 : \omega_1 : \omega_2.$$

The third depends upon the intensity of the force, and completely defines it when the other two are known.

The coefficients in the fundamental equations (8) may now be written

$$(8'')\qquad \begin{aligned} K &= \frac{z' - \omega_2}{y' - \omega_1}, \qquad Q = \frac{3}{y' - \omega_1}, \\ P &= \Phi_x + y'\Phi_y + z'\Phi_2 - \frac{\omega_{1x} + y'\omega_{1y} + z'\omega_{1z}}{y' - \omega_1}. \end{aligned}$$

By differentiating the first equation (8) and combining with the second, we obtain

$$z''' = Ky''' + K'y'' = \left(P + \frac{K'}{K}\right)z'' + \frac{Q}{K}z''.$$

This may be written in the form

$$z''' = Pz'' + Qz''^2, \tag{10}$$

where

$$P = \Phi_x + y'\Phi_y + z'\Phi_z - \frac{\omega_{2x} + y'\omega_{2y} + z'\omega_{2z}}{z' - \omega_2}, \qquad Q = \frac{3}{z' - \omega_2}, \tag{10'}$$

and is, of course, equivalent to (7).

3. It may be shown, as in *Trajectories,* that if two fields ϕ, ψ, χ and ϕ_1, ψ_1, χ_1, lead to the same trajectories, that is, the same equations (8), then they can differ only by a constant factor. The system of trajectories thus determines the field.*

Osculating planes.

4. *Property* I. The osculating plane of a trajectory at a given point is determined by its initial direction and by the direction of the force acting at the given point. This familiar fact is easily verified analytically by noting that the general equation of the osculating plane is

$$\begin{vmatrix} X & Y & Z \\ 1 & y' & z' \\ 0 & y'' & z'' \end{vmatrix} = 0, \tag{11}$$

where the current coördinates X, Y, Z refer to axes passing through the given point. This is satisfied, in virtue of (5), by $X : Y : Z = \phi : \psi : \chi$.

Property I. — *The osculating planes of all trajectories passing through a given point form a pencil.*

The axis of the pencil has the direction of the force acting at the given point.

5. *Converse of* I. It is easy to determine all quintuply infinite systems of curves possessing property I. Let the direction, fixed for each point, be defined by the ratios

$$1 : \omega_1 : \omega_2,$$

where ω_1 and ω_2 are arbitrary functions of x, y, z. By hypothesis, the osculating plane (11) passes through this direction for all possible values y', z', y'', z''. Hence

$$\begin{vmatrix} 1 & \omega_1 & \omega_2 \\ 1 & y' & z' \\ 0 & y'' & z'' \end{vmatrix} = 0,$$

* A geometric construction of the field will appear in a later paper, *The inverse problem of dynamics.*

which gives

$$\frac{z''}{y''} = \frac{z' - \omega_2}{y' - \omega_1}.$$

Conversely, when this relation is fulfilled, the curves have the required property. Hence:

All quintuply infinite systems of curves possessing property I *are represented by differential equations of the form*

$$(12) \qquad \begin{aligned} z'' &= \frac{z' - \omega_2}{y' - \omega_1} y'' = Ky'', \\ y''' &= f(x, y, z, y', z', y''). \end{aligned}$$

The result thus contains two arbitrary functions ω_1, ω_2 of x, y, z and an arbitrary function f of the six arguments indicated.

6. *Property* I_1. Before proceeding to essentially new properties, we derive a few corollaries of interest. If the initial position and direction, fixed by the values of x, y, z, y', z', be given, then y'' is arbitrary and we thus have ∞^1 trajectories. For these we may state the elementary result:

The ∞^1 *trajectories corresponding to a given lineal element have (at the given point) a common osculating plane.*

7. *Property* I_2. The torsion of any curve is given by the general formula

$$(13) \qquad \frac{1}{\rho} = \frac{y''z''' - z''y'''}{(y'y'' + z'z'')^2 - (1 + y'^2 + z'^2)(y''^2 + z''^2)}.$$

For trajectories, we have

$$z'' = Ky''.$$

Hence

$$z''' = Ky''' + K'y'',$$

where

$$K' = \frac{\omega_1'(z' - \omega_2) - \omega_2'(y' - \omega_1)}{(y' - \omega_1)^2}.$$

Introducing these values in (13), we find

$$(14) \qquad \frac{1}{\rho} = \frac{K'}{(y' + Kz')^2 - (1 + y'^2 + z'^2)(1 + K^2)}.$$

This does not involve y'' or z''. Hence the result:

The ∞^1 *trajectories determined by a given lineal element have (at the given point) the same torsion.*

8. *Converse of* I_1 *and* I_2. These properties hold not only for dynamical systems, but for the more general systems (12). They are thus consequences of I and might in fact have been derived synthetically from it. The converse,

however, does not hold. There are in fact systems possessing both I_1 and I_2 (for all lineal elements) but not possessing I. We omit the discussion and state the result:

A system of space curves possesses property I_1 *when the relation between* z'' *and* y'' *is of the form*

$$z'' = k(x, y, z, y', z')y'', \tag{15}$$

where k *is arbitrary. It also possesses property* I_2 *provided the function* k *satisfies the equation*

$$k_{y'} + kk_{z'} = 0. \tag{16}$$

The function K, appearing in (8) and (12), is seen to be a special solution of (16).

8′. *Related helices.* At each point of a space curve it is possible to construct a unique helix having in common with the given curve its direction, its osculating plane, its curvature, and its torsion.* We term this the *related helix.* If the current coördinates X, Y, Z refer to axes determined by the tangent, principal normal, and binormal at the given point, then the axis of the helix is given by the formulas

$$Y = \frac{r\rho^2}{r^2 + \rho^2}, \qquad \frac{X}{Z} = -\frac{r}{\rho}, \tag{17}$$

where r and ρ are the radii of curvature and torsion respectively.

Consider now the ∞^1 trajectories passing through a given point in a given direction. These have a common osculating plane and a common torsion. The curvature varies from curve to curve. The elimination of r from (17) gives

$$Y(X^2 + Z^2) + \rho XZ = 0. \tag{18}$$

This is recognized as PLÜCKER's normal equation for a cylindroid.† Hence:

The ∞^1 *trajectories determined by a given lineal element give rise to related helices whose axes generate a cylindroid.*

The discussion shows that this holds for the more general systems obtained in article 8. The result is thus a consequence of I_1 and I_2.

9. *Distribution of torsion.* In article 7 it was shown that for each lineal element there is a definite value of ρ. If we consider the ∞^2 elements at a given point, they will subdivide into sets of ∞^1 according to the values of the parameter ρ. If ρ has a given value, say c, the corresponding elements are

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* SCHEFFERS, *Theorie der Curven*, p. 197. The contact is not of the third order unless dr/ds vanishes. Only in this case does an *osculating helix* exist.

† A cylindroid presents itself in a different connection in SCHEFFER's discussion (l. c., p. 196). There r is fixed, while ρ varies. The elimination of ρ from (17) leads to $Y(X^2 + Z^2) - rZ^2 = 0$.

found, from (14), to be

$$(20)\quad (\omega_1 z' - \omega_2 y')^2 + (y' - \omega_1)^2 + (z' - \omega_2)^2 - c \begin{vmatrix} y' - \omega_1 & \omega_{1x} + y'\omega_{1y} + z'\omega_{1z} \\ z' + \omega_2 & \omega_{2x} + y'\omega_{2y} + z'\omega_{2z} \end{vmatrix}.$$

The elements thus determine the cone

$$(20')\quad (\omega_1 Z - \omega_2 Y)^2 + (Y - \omega_1 X)^2 + (Z - \omega_2 X)^2 - c \begin{vmatrix} Y - \omega_1 X & \omega_{1x}X + \omega_{1y}Y + \omega_{1z}Z \\ Z - \omega_2 X & \omega_{2x}X + \omega_{2y}Y + \omega_{2z}Z \end{vmatrix}.$$

The trajectories which pass through a given point and have there the same torsion are arranged so that their initial directions form a quadric cone.

10. When the given torsion is zero, (20′) becomes

$$(21)\qquad \begin{vmatrix} Y - \omega_1 X & \omega_{1x}X + \omega_{1y}Y + \omega_{1z}Z \\ Z - \omega_2 X & \omega_{2x}X + \omega_{2y}Y + \omega_{2z}Z \end{vmatrix} = 0.$$

In this case the osculating plane has four, instead of three, consecutive points in common with the curve.

Of the ∞^3 trajectories through a given point there are ∞^2 with hyperosculating planes. The lineal elements of these trajectories generate the quadric cone (21).

11. By varying c in (20′) we obtain a pencil of cones. It may be shown that all pass through the element defined by the ratios

$$(22)\qquad 1 : \omega_1 : \omega_2,$$

and that along this element they have a common tangent plane

$$(23)\qquad \begin{vmatrix} 1 & \omega_1 & \omega_2 \\ 0 & \omega_{1x} + \omega_1\omega_{1y} + \omega_2\omega_{1z} & \omega_{2x} + \omega_1\omega_{2y} + \omega_2\omega_{2z} \\ X & Y & Z \end{vmatrix} = 0.$$

The element (22) has the direction of the line of force and the plane (23) is the osculating plane of the line of force.

Osculating spheres.

12. The properties thus far obtained are consequences of property I. To obtain independent properties we now consider osculating spheres. If we take the given point as origin and let X, Y, Z denote the coördinates of the center of the sphere, the equation of the sphere is

$$x^2 + y^2 + z^2 - 2Xx - 2Yy - 2Zz = 0.$$

The conditions for osculation with a given curve are

$$(24)\qquad \begin{aligned} X + y'Y + z'Z &= 0, \\ y''Y + z''Z &= 1 + y'^2 + z'^2, \\ y'''Y + z'''Z &= 3(y'y'' + z'z''). \end{aligned}$$

From these we obtain the general formulas

$$X = -\frac{(1+y'^2+z'^2)(y'z'''-z'y''')-3(y'z''-z'y'')(y'y''+z'z'')}{y''z'''-z''y'''},$$

$$(25)\quad Y = \frac{(1+y'^2+z'^2)z'''-3z''(y'y''+z'z'')}{y''z'''-z''y'''},$$

$$Z = -\frac{(1+y'^2+z'^2)y'''-3y''(y'y''+z'z'')}{y''z'''-z''y'''},$$

13. *Property* II. Consider the ∞^1 trajectories determined by a given lineal element. For these y'' is arbitrary and z'', y''', z''' are determined as functions of y'' by the fundamental differential equations (8) and the derived equation (10). If we substitute these values in (25) or, as is more convenient, in (24), we may eliminate the parameter y'' and obtain two relations between X, Y, Z. One of these is already given by the first equation (24); the other is found to be

$$(26)\qquad PY + KPZ + Q(1+y'^2+z'^2) - 3(y'+Kz') = 0.$$

The coefficients in both these equations are constants since they depend only upon the given lineal element. Since the equations are linear in X, Y, Z, we have

PROPERTY II. *The osculating spheres of the ∞^1 trajectories passing through a given point in a given direction have their centers on a straight line.*

The straight line is necessarily in the plane perpendicular to the given direction at the given point, i. e., in the common normal plane of the ∞^1 trajectories. Since the spheres all pass through the given point, it follows from the above that they pass through a common circle. Hence property II may be restated in this form:

The osculating spheres of the ∞^1 trajectories described above form a pencil.

14. *Converse of* II. It is clear that II is independent of I. In fact the proof of I depended upon only the first of the fundamental equations (8), while that of II involved also the second.

We now find all systems of curves possessing I and II. By article 5 the differential equations are necessarily of the form (12). It remains to determine the form of the function f. The hypothesis is that for each lineal element the locus of the centers (X, Y, Z) of the osculating spheres shall be a straight line. Since the centers are necessarily in the plane normal to the element, the equations of the straight line may be taken in of the form

$$X + y'Y + z'Z = 0,$$

$$A_1X + A_2Y + A_3Z + A_4 = 0,$$

where the coefficients A involve x, y, z, y', z' in any way. Substituting the general values (25) in the second of these equations, we find a relation of the form

$$(1+y'^2+z'^2)(B_1z'''-B_2y''')-3(B_1z''-B_2y'')(y'y''+z'z'')+B_3(y''z'''-z''y''')=0,$$

where the B's involve only x, y, z, y', z'. Introducing the values of z'', y''' given in (12) and the value of z''' found by differentiation, we obtain

$$f = Gy'' + Hy''^2,$$

where G and H are functions of x, y, z, y', z'.

It is easy to verify that this condition is sufficient, that is, that property II holds for all values of G and H. Hence:

The most general quintuply infinite system of space curves possessing properties I *and* II *is defined by equations of the form*

$$z'' = Ky'' = \frac{z' - \omega_2}{y' - \omega_1} y'', \qquad y''' = Gy'' + Hy''^2; \tag{27}$$

the result thus involves two arbitrary functions ω_1, ω_2 *of* x, y, z, *and two arbitrary functions* G, H *of* x, y, z, y', z'.

15. *Correspondence of centers.* We now prove a property which holds not only for dynamical systems but for the general systems just obtained.

For each of the ∞^1 curves corresponding to a given lineal element we may construct a circle of curvature and an osculating sphere. Denote their centers by O' and O'' respectively. By property II the locus of O'' is a straight line. The locus of O' is evidently also a straight line, namely, the principal normal which, by property I_1, is common to all the curves. We now prove that *the ranges described by* O' *and* O'' *are similar.*

Take the given point as origin and the given initial direction as that of the axis of abscissas. Then the distance from O' to the given point, that is, the radius of curvature, is found to be

$$r = \frac{1}{y''\sqrt{1+K^2}}. \tag{28}$$

The ordinate of O'' is

$$Z = \frac{G + Hy''}{K'y''}. \tag{29}$$

Eliminating y'', we obtain a relation of the form

$$Z = ar + b,$$

which proves the result stated. In the dynamical case the property may be expressed as follows:

If a particle is projected from a given point in a given direction, and if for each of the ∞^1 trajectories obtained by varying the initial velocity we construct the center O' of the circle of curvature and the center O'' of the osculating sphere, then the points O' and O'' describe similar ranges.

There is no difficulty in proving this synthetically as a consequence of I and II.

The congruence Ω and the cubic Γ.

16. *Equations of* Ω. We have seen that to each lineal element there corresponds a definite straight line, the locus of the centers of the osculating spheres. If the point is kept fixed and the direction of the element varied, then ∞^2 of these straight lines are obtained. What is the character of the congruence Ω thus generated?

The equations of the straight line corresponding to a given element, found in article 13, may be written

$$X + y'Y + z'Z = 0,$$

$$(30)\qquad \begin{vmatrix} 1 & \phi_x + y'\phi_y + z'\phi_z \\ y' & \psi_x + y'\psi_y + z'\psi_2 \end{vmatrix} Y + \begin{vmatrix} 1 & \phi_x + y'\phi_y + z'\phi_2 \\ z' & \chi_x + y'\chi_y + z'\chi_2 \end{vmatrix} Z = 3(\phi + y'\psi + z'\chi).$$

Here x, y, z have fixed values and y', z' are arbitrary parameters leading to the ∞^2 straight lines of Ω. The quadratic terms in y', z' may be eliminated by combining the equations, giving the result:

The congruence Ω is defined by the equations

$$(31)\qquad X + y'Y + z'Z = 0, \qquad AX + BY + CZ = 3D,$$

where

$$(31')\qquad \begin{aligned} A &= y'\phi_y + z'\phi_z, & B &= \psi_x + y'(\psi_y - \phi_x) + z'\psi_z, \\ C &= \chi_x + y'\chi_y + z'(\chi_z - \phi_x), & D &= \phi + y'\psi + z'\chi. \end{aligned}$$

17. *The cubic curve* Γ. If the values of X, Y, Z are given, then, since the equations (31) are linear in the parameters, they will determine, in general, unique values of y', z'. Hence *the congruence is of order one.*

The equations (31) may be written in the form

$$(32)\qquad X + Yy' + Zz' = 0, \qquad \alpha + \beta y' + \gamma z' = 0,$$

where

$$\alpha = \psi_x Y + \chi_x Z - 3\phi,$$

$$\beta = \phi_y X + (\psi_y - \phi_x)Y + \chi_y Z - 3\psi,$$

$$\gamma = \phi_z X + \psi_z Y + (\chi_z - \phi_x)Z - 3\chi.$$

Hence the element corresponding to a general point X, Y, Z is given by the formulas

$$(33)\qquad dx : dy : dz = \Delta_1 : \Delta_2 : \Delta_3,$$

where

$$\Delta_1 = Y\gamma - Z\beta = \phi_z XY - \phi_y XZ + \psi_z Y^2 + (\chi_z - \psi_y) YZ - 3\chi Y - \chi_y Z^2 + 3\psi Z,$$

$$\Delta_2 = Z\alpha - X\gamma = -\phi_z X^2 - \psi_z XY + (\phi_z - \chi_z) XZ + 3\chi X + \psi_x YZ + \chi_x Z^2 + 3\phi Z,$$

$$\Delta_3 = X\beta - Y\alpha = \phi_y X^2 + (\psi_y - \phi_x) XY + \chi_y XZ - 3\psi X - \psi_x Y^2 - \chi_x YZ + 3\phi Y.$$

The point X, Y, Z will be singular if the equations (33) do not determine the element. This will be the case when

$$\Delta_1 = \Delta_2 = \Delta_3 = 0. \tag{34}$$

From the identity

$$X\Delta_1 + Y\Delta_2 + Z\Delta_3 \equiv 0, \tag{35}$$

which is readily verified, it is seen that (34) represents a cubic curve.

The singular points of the congruence Ω form a twisted cubic curve.

It will be convenient to have the equations of this curve Γ in parametric form. These may be found by expressing the fact that for a singular point the equations (32) differ only by a factor λ. Hence the required parametric equations are

$$\begin{aligned} \lambda X + \psi_x Y + \chi_x Z &= 3\phi, \\ \phi_y X + (\psi_y - \phi_x + \lambda) Y + \chi_y Z &= 3\psi, \\ \phi_z X + \psi_z Y + (\chi_z - \phi_x + \lambda) Z &= 3\chi. \end{aligned} \tag{36}$$

The explicit expressions for X, Y, Z as cubic functions of the parameter λ will not be needed in the following discussion.

18. *Property* III_1. We now may prove that the congruence Ω is completely defined by the curve Γ, by showing that every straight line of Ω has two singular points, that is, is a secant of Γ.

Since any element may be made to take the direction of the axis of abscissas by revolving the axes, it will be sufficient to prove the statement for the single element $y' = 0$, $z' = 0$. The corresponding straight line, by (31), is

$$X = 0, \qquad \psi_x Y + \chi_x Z - 3\phi = 0.$$

It is easy to verify, by using either (34) or (36), that this line has two points in common with Γ.

PROPERTY III_1. *The straight lines which correspond, according to property* II, *to the ∞^2 elements at a given point, form a congruence which is composed of the secants of a twisted cubic curve.*

19. *Property* III_2. The preceding article associates with any given point a

definite cubic Γ. From the equations (34) it is seen that the curve passes through the given point, which is there taken as the origin of the X, Y, Z coördinates. In the parametric representation (36), the origin is given by the value $\lambda = \infty$. The direction of the curve at that point is found to be given by the ratios $\phi : \psi : \chi$; it is therefore the same as the direction of the force acting at the given point. This direction is defined geometrically in connection with property I. We may, therefore, state the new property as follows:

PROPERTY III_2. *The cubic curve* Γ, *associated by property* III_1 *with any point, passes through that point in the direction of the axis of the pencil of osculating planes described in property* I.

20. *Quadrics through* Γ. Consider the ∞^2 trajectories passing through a given point and having initial directions in a given plane. The lineal elements thus satisfy a linear relation

$$a\,dx + b\,dy + c\,dz = 0. \tag{37}$$

It follows, from (33), that the corresponding centres X, Y, Z satisfy the equation

$$a\Delta_1 + b\Delta_2 + c\Delta_3 = 0. \tag{38}$$

The locus of the centers of the osculating spheres of the ∞^2 *trajectories touching a given plane at a given point is a quadric surface.*

The quadrics (38) are in fact the linear system of quadrics passing through the cubic curve Γ. The result is really a corollary of the previous properties and may be derived synthetically. The straight line corresponding to a given element is obtained by constructing the plane perpendicular to that element at the given point; this plane cuts Γ in two new points; the line connecting these is the required line. If then we take the pencil of elements determined by a plane through the given point, the corresponding normal planes form a pencil; hence the corresponding straight lines form a regulus. This determines a quadric of the linear system (38).

21. *Interpretation of the curl.* To each plane (37) through the given point there corresponds a definite quadric (38). For which planes will this quadric be a rectangular hyperboloid?

The condition for such a quadric is that the sum of the coefficients of X^2, Y^2, Z^2 shall vanish. This gives

$$a(\psi_z - \chi_y) + b(\chi_x - \phi_z) + c(\phi_y - \psi_x) = 0. \tag{39}$$

This means that the plane (37) must contain the fixed direction defined by the ratios

$$\psi_z - \chi_y : \chi_x - \phi_z : \phi_y - \psi_x. \tag{40}$$

These three quantities are the components of the curl of the force ϕ, ψ, χ. Hence:

The direction of the curl of the force at any given point is defined geometrically by the fact that the quadrics corresponding, according to article 20, *to the planes containing this direction are rectangular hyperboloids. This is not the case for any other planes through the given point.*

22. *Conservative force.* The force is conservative when, and only when, the curl vanishes identically. In this case condition (39) holds for all values of a, b, c. Hence the quadric corresponding to any plane is a rectangular hyperboloid. The cubic curve Γ is then of a particular species which may also be termed rectangular. Its definitive property may be expressed in simple form: *the three asymptotic directions of the cubic are mutually orthogonal.*

Our result is thus a purely geometric characterization of the conservative case:

When, and only when, the force is conservative are the cubic curves Γ, *corresponding to all points of space in accordance with property* III_1, *of the rectangular species.*

23. *Converse of* III. Our next problem is to determine all curve systems possessing properties III_1 and III_2 in addition to I and II. It will be convenient to refer to the joint statement of III_1 and III_2 as PROPERTY III. The discussion will show incidentally that III is independent of I and II.

The most general system for which I and II hold was proved in article 14 to be represented by equations of the form (27), namely,

$$z'' = Ky'', \qquad y''' = Gy'' + Hy''^2. \tag{41}$$

The straight line corresponding to a given element is here determined by the equations*

$$X + y'Y + z'Z = 0, \qquad MY + NZ + 1 = 0, \tag{42}$$

where

$$M, N = \frac{G, KG + K'}{H(1 + y'^2 + z'^2) - 3(y' + Kz')}. \tag{42'}$$

The problem is to determine the form of the functions G and H, or, what is equivalent, of the functions M and N, so that property III shall also hold, that is, so that the congruence (42) shall be of the particular type described in III_1 and III_2.

Let the equations of the cubic associated with the given point, which we take

* In the dynamical case these reduce, of course, to equations (30).

as origin of the homogeneous cartesian coördinates X, Y, Z, T, be, in parametric form,

$$(43)\qquad \begin{aligned} X &= a_4\lambda^3 + a_0\lambda^2 + a_1\lambda + a_2, \\ Y &= b_4\lambda^3 + b_0\lambda^2 + b_1\lambda + b_2, \\ Z &= c_4\lambda^3 + c_0\lambda^2 + c_1\lambda + c_2, \\ T &= d_4\lambda^3 + d_0\lambda^2 + d_1\lambda + d_2. \end{aligned}$$

The curve is to pass through the origin by III_2. We may take the parameter so that $\lambda = \infty$ shall correspond to this point. Therefore

$$(44)\qquad a_4 = b_4 = c_4 = 0.$$

The directional property involved in III_2 requires further that

$$(45)\qquad a_0 : b_0 : c_0 = 1 : \omega_1 : \omega_2.$$

We may therefore take our cubic curve in the form

$$(46)\qquad \begin{aligned} X &= a_0\lambda^2 + a_1\lambda + a_2, \\ Y &= b_0\lambda^2 + b_1\lambda + b_2, \\ Z &= c_0\lambda^2 + c_1\lambda + c_2, \\ T &= \lambda^3 + d_0\lambda^2 + d_1\lambda + d_2. \end{aligned} \qquad (a_0 : b_0 : c_0 = 1 : \omega_1 : \omega_2).$$

The coefficients a, b, c, d may involve x, y, z in any way.

If we substitute these values in (42), we find

$$(47)\qquad L_0\lambda^2 + L_1\lambda + L_2 = 0,$$

where

$$(47')\qquad L_0 = a_0 + y'b_0 + z'c_0, \quad L_1 = a_1 + y'b_1 + z'c_1, \quad L_2 = a_2 + y'b_2 + z'c_2;$$

and

$$(48)\qquad \lambda^3 + (Mb_0 + Nc_0 + d_0)\lambda^2 + (Mb_1 + Nc_1 + d_1)\lambda + (Mb_2 + Nc_2 + d_2) = 0.$$

The condition that the line (42) shall intersect the cubic (46) in two points is that the equations (47), (48) shall have two roots λ in common. This gives two relations

$$(49)\qquad \begin{aligned} L_0(b_0L_2)M + L_0(c_0L_2)N &= L_1L_2 - L_0(d_0L_2), \\ L_0(b_1L_2)M + L_0(c_1L_2)N &= L_2^2 - L_0(d_1L_2), \end{aligned}$$

where the parentheses represent determinants

$$(49')\qquad (b_0L_2) = b_0L_2 - b_2L_0, \text{ etc.}$$

in terms of their principal diagonals.

The solution of (49) gives, after some reduction,

$$(50)\qquad \begin{aligned} M &= \frac{L_1(c_1L_2) - L_2(c_0L_2) - L_0(d_1c_2L_3)}{(abc)L_0}, \\ N &= \frac{L_2(b_0L_2) - L_1(b_1L_2) + L_0(d_1b_2L_3)}{(abc)L_0}. \end{aligned}$$

Arranging these with respect to y', z', we find that the functions M and N are necessarily of the form

$$(51)\qquad \begin{aligned} M &= \frac{m_1 + m_2y' + m_3z' + y'(l_1y' + l_2z')}{3(1 + \omega_1y' + \omega_2z')}, \\ N &= \frac{n_1 + n_2y' + n_3z' + z'(l_1y' + l_2z')}{3(1 + \omega_1y' + \omega_2z')}, \end{aligned}$$

where the coefficients l, m, n involve only x, y, z.

Conversely, any functions of this form lead to a system (41) with the required properties. The congruence (42) may, in fact, for these values of M and N, be put into the form

$$(52)\qquad \begin{aligned} X + y'Y + z'Z &= 0, \\ -(l_1y' + l_2z')X + (m_1 + m_2y' + m_3z')Y + (n_1 + n_2y' + n_3z')Z \\ &= 3(1 + \omega_1y' + \omega_2z'). \end{aligned}$$

A discussion entirely analogous to that given in articles 17 and 18 shows that the congruence is defined by the cubic curve

$$(53)\qquad \frac{m_1Y + n_1Z + 3}{X} = \frac{m_2Y + n_2Z - l_1X + 3\omega_1}{Y} = \frac{m_3Y + n_3Z - l_2X + 3\omega_2}{Z}.$$

This passes through the origin in the direction $1 : \omega_1 : \omega_2$, hence property III holds in its entirety. Our result may be stated as follows:

The most general quintuply infinite system of curves with properties I, II *and* III *is represented by equations of the form*

$$(54)\qquad z'' = Ky'', \qquad y''' = Gy'' + Hy''^2,$$

where

$$(54')\qquad K = \frac{z' - \omega_2}{y' - \omega_1},$$

and G and H are defined by the relations

$$(54'')\qquad \begin{aligned} G : KG + K' : H(1 + y'^2 + z'^2) - 3(y' + Kz') &= m_1 + m_2y' + m_3z' \\ + y'(l_1y' + l_2z') : n_1 + n_2y' + n_3z' + z'(l_1y' + l_2z') &: 3(1 + \omega_1y' + \omega_2z'). \end{aligned}$$

The system thus involves ten arbitrary functions of x, y, z, namely,

$$(55)\qquad \omega_1,\ \omega_2,\ l_1,\ l_2,\ m_1,\ m_2,\ m_3,\ n_1,\ n_2,\ n_3.$$

Relations between the ten coefficients.

24. The dynamical systems, defined by equations of the form (8) in connection with (8′) or (8″), involve only three arbitrary functions ϕ, ψ, χ and thus constitute a special case of the systems obtained in the preceding article. The question then arises, what is the special nature of the ten functions (55), which appear as coefficients in the equations (54″), if the system (54) is to be of the dynamical type?

Comparing the equations (52) with the corresponding equations (31), (31′) of the dynamical case, and introducing the functions ω_1, ω_2, Φ defined by (9), we find that

$$
\begin{aligned}
&l_1 = \Phi_y, && l_2 = \Phi_z, && \\
(56)\quad &m_1 = -\omega_1 \Phi_x - \omega_{1x}, && m_2 = \Phi_x - \omega_{1y} - \omega_1 \Phi_y, && m_3 = -\omega_{1z} - \omega_1 \Phi_z, \\
&n_1 = -\omega_2 \Phi_x - \omega_{2x}, && n_2 = -\omega_{2y} - \omega_2 \Phi_y, && n_3 = \Phi_x - \omega_{2z} - \omega_2 \Phi_z.
\end{aligned}
$$

In order that the ten functions (55) shall belong to a dynamical system it is necessary and sufficient that they shall be expressible in terms of three functions ω_1, ω_2, Φ according to (56).

25. The explicit conditions may be obtained by eliminating Φ from (56). The algebraic elimination of Φ_x, Φ_y, Φ_z yields five relations, namely:

$$
(57)\quad
\begin{aligned}
&m_3 + \omega_{1z} + \omega_1 l_2 = 0, \qquad n_2 + \omega_{2y} + \omega_2 l_1 = 0, \\
&m_2 + \omega_{1y} + \omega_1 l_1 = n_3 + \omega_{2z} + \omega_2 l_2 = -\frac{m_1 + \omega_{1x}}{\omega_1} = -\frac{n_1 + \omega_{2x}}{\omega_2}.
\end{aligned}
$$

Furthermore, from (56) we obtain the equations

$$
(58)\qquad \Phi_x = -\frac{m_1 + \omega_{1x}}{\omega_1} = -\frac{n_1 + \omega_{2x}}{\omega_2}, \qquad \Phi_y = l_1, \qquad \Phi_z = l_2,
$$

for which the conditions of integrability may be written

$$
(59)\qquad l_{1z} - l_{2y} = 0, \qquad l_{2x} + \left(\frac{n_1 + \omega_{2x}}{\omega_2}\right)_z = 0, \qquad l_{1x} + \left(\frac{m_1 + \omega_{1x}}{\omega_1}\right)_y = 0.
$$

26. We have thus obtained eight necessary relations (57), (59). We now prove that these are also sufficient. Suppose that any ten functions satisfying these relations are given. Then, on account of (59), the equations (58) will be consistent; hence a function Φ can be determined.* It is then easily verified that, on account of the relations (57), the equations (56) hold.

*The integration constant involved enters additively in Φ. Hence, from (9), the force ϕ, ψ, χ is determined except for a constant factor. This agrees with the result stated in article 3.

In order that a system of curves with properties I, II, *and* III, *that is, a system defined by equations of the form* (54), (54′), (54″), *shall be of the dynamical type, it is necessary and sufficient that the ten functions* (55) *shall satisfy the relations* (57) *and* (59).

27. *Preliminary interpretation.* It remains to interpret the relations (57), (59) geometrically. For any system of curves with properties I, II, III there is associated with each point x, y, z a definite cubic curve Γ, whose equations are given by (53). The association of point and corresponding cubic is fixed by the functions (55). Any relation between the functions is thus equivalent to a restriction on this association. It would in this way be possible to derive a direct geometrical interpretation of the relations (57), (59) which express the character of the association in the dynamical case. However, the result thus obtained is quite complicated and we shall not state it. A simple geometric interpretation will be obtained indirectly in the subsequent discussion of associated S-systems.

28. We derive first a property which is equivalent to two of the eight relations (57). According to article 13 there corresponds to any element (y', z') at a given point a definite straight line. Consider now the particular element (ω_1, ω_2) whose direction is that of the force acting at the point. By taking the axis of abscissas in this direction, we find that the corresponding line is represented by

$$X = 0, \qquad \omega_{1x} Y + \omega_{2x} Z = 3. \tag{60}$$

The perpendicular distance from the given point, here taken as the origin of the X, Y, Z system, to this line is

$$\frac{3}{\sqrt{\omega_{1x}^2 + \omega_{2x}^2}}.$$

On the other hand the radius of curvature of the line of force (10) passing through the given point is, under the assumed conditions,

$$\frac{1}{\sqrt{\omega_{1x}^2 + \omega_{2x}^2}};$$

that is, just one third of the perpendicular distance.

The direction of the straight line (60) is also simply related to the line of force. The osculating plane of the latter is

$$\omega_{2x} Y - \omega_{1x} Z = 0.$$

It is thus perpendicular to the line (60). The results may be stated as follows:

The straight line which corresponds, in accordance with property II, *to a lineal element belonging to a line of force is parallel to the binormal of the*

line of force; and its perpendicular distance from the given point is three times the radius of curvature of the line of force.

This may be stated in purely geometric form by replacing the direction of the force by that of the axis of the pencil considered in property I. It is found that the results hold for all systems in which the functions (55) satisfy the conditions

$$
\begin{aligned}
&\{m_1 + m_2\omega_1 + m_3\omega_2 + \omega_1(l_1\omega_1 + l_2\omega_2)\}^2 + \{n_1 + n_2\omega_1 + n_3\omega_2 + \omega_2(l_1\omega_1 + l_2\omega_2)\}^2 \\
&\qquad = (\omega_{1x} + \omega_1\omega_{1y} + \omega_2\omega_{1z})^2 + (\omega_{2x} + \omega_1\omega_{2y} + \omega_2\omega_{2z})^2 \\
&\qquad\qquad - \frac{1}{1 + \omega_1^2 + \omega_2^2}\begin{vmatrix} \omega_1 & \omega_{1x} + \omega_1\omega_{1y} + \omega_2\omega_{1z} \\ \omega_2 & \omega_{2x} + \omega_1\omega_{2y} + \omega_2\omega_{2z} \end{vmatrix},
\end{aligned}
\tag{61}
$$

$$
\begin{vmatrix} \omega_{1x} + \omega_1\omega_{1y} + \omega_2\omega_{1z} & m_1 + m_2\omega_1 + m_3\omega_2 + \omega_1(l_1\omega_1 + l_2\omega_2) \\ \omega_{1x} + \omega_1\omega_{2y} + \omega_2\omega_{2z} & n_1 + n_2\omega_1 + n_3\omega_2 + \omega_2(l_1\omega_1 + l_2\omega_2) \end{vmatrix} = 0. \tag{62}
$$

These are, in fact, consequences of the relations (57), though of course the converse is not true. The property stated in this article thus holds not only for dynamical but for more general systems. Its addition to I, II, and III will therefore not yield a complete characterization of the dynamical type.

The associated plane systems S.

29. *Definition.* To reach the desired characterization we introduce certain plane systems of curves which may be associated with any quintuply infinite system of space curves.

Consider any plane π. Through each point of π there pass ∞^2 curves of the given system which are tangent to π. Project the differential elements of the third order belonging to these space curves orthogonally upon π, thus obtaining ∞^2 plane differential elements of the third order at the selected point. Applying this process to all points of π, we have a definite system of ∞^4 differential elements of the third order. These elements define a certain differential equation of the third order and thus determine ∞^3 integral curves. These form what we shall term the *associated S-system* in the plane π.

The system S is easily determined analytically. Let the given plane π be taken as the xy plane, and let the equations of the arbitrary quintuply infinite system be

$$
\begin{aligned}
z'' &= g(x, y, z, y', z', y''), \\
y''' &= f(x, y, z, y', z', y'').
\end{aligned}
\tag{63}
$$

Then for the lineal elements contained in π we have

$$z = 0, \qquad z' = 0.$$

Hence the differential equation of S is

$$y''' = f(x, y, 0, y', 0, y'') = f_0(x, y, y', y''). \tag{64}$$

There will be in all ∞^3 of these triply infinite systems S, namely, one for each plane of space. Any geometric property of these systems is at the same time a property of the original space system; for the former are determined geometrically by the latter.

30. We apply this notion to the dynamical systems and obtain:

PROPERTY IV. *The plane systems S associated with any quintuply infinite system of dynamical trajectories are of the (two-dimensional) dynamical type.*

To prove this it will be sufficient to consider the S-system associated with the xy plane. Applying the method given in the preceding article to the equations (8), or, as is equivalent, to the equations (5) and (6), we find that the system is defined by

$$(\bar{\psi} - y'\bar{\phi})y''' = \begin{vmatrix} 1 & \bar{\phi}_x + y'\bar{\phi}_y \\ y' & \bar{\psi}_x + y'\bar{\psi}_y \end{vmatrix} - 3\bar{\phi}y''^2, \tag{65}$$

where $\bar{\phi}$, $\bar{\psi}$ are functions of x, y obtained from ϕ, ψ by substituting 0 for z This is precisely the equation of the trajectories generated by the plane field of force whose components are $\bar{\phi}$, $\bar{\psi}$.* The force acting in the plane is seen to be derivable by orthogonal projection from the given force (1).

31. The geometrical properties of plane dynamical systems are given in the paper already cited. We shall number the properties as in that paper, but distinguish them from the spatial properties already obtained by attaching a subscript p. As the characteristic set we may take †

$$\mathrm{I}_p,\ \mathrm{II}_p,\ \mathrm{III}_p,\ \mathrm{V}_p,\ \mathrm{VI}_p. \tag{67}$$

Property IV, given in article 30, may thus be restated in purely geometric form as follows:

PROPERTY IV. *The systems S associated with any quintuply infinite system of dynamical trajectories possess the plane properties* I_p, II_p, III_p, V_p, VI_p.

It thus may be broken up into five distinct statements, of which the first, for example, would be: Every S-system is such that the osculating parabolas (constructed at the common point) of the ∞^1 curves passing through a given point in a given direction have their foci located on a circle passing through the given point.

* Cf. *Trajectories*, p. 403, formula (5).

† Loc. cit., p. 417.

32. *Converse of* I_p. The object of the remaining discussion is to show that the addition of property IV to properties I, II, III gives a set which completely characterizes the dynamical type in space. We must therefore examine the effect of adding successively the properties (67) to I, II, III. The first result is this:

If any quintuply infinite system of space curves possesses properties I, II, *then the associated S-systems necessarily possess property* I_p.

This is proved by noting that the condition for I_p is that the differential equation of the plane system shall be of the form *

$$y''' = G(x, y, y')y'' + \bar{H}(x, y, y',)y''^2. \tag{68}$$

That the S-system is of this type is seen from the equations (27), which represent the most general system with properties I and II.

It is thus shown that I_p is redundant, in the sense that it is a consequence of preceding spatial properties.

33. *Converse of* II_p, III_p. In order that the system (68) shall have properties II_p, III_p, it is necessary that the $\bar{G}$ and $\bar{H}$ shall have the special forms †

$$\bar{G} = \frac{\lambda y'^2 + \mu y' + \nu}{y' - \omega}, \qquad \bar{H} = \frac{3}{y' - \omega}, \tag{69}$$

where λ, μ, ν, ω involve x, y only.

We now determine under what conditions a system with properties I, II, and III, that is, a system whose differential equations are of the type defined by (54), (54′), (54″), will have associated S-systems possessing properties II_p, III_p. This is to be the case, of course, for the S-system in every plane.

Consider first the planes parallel to the xy plane. The systems S are then found by substituting 0 for z' and an arbitrary constant for z. By making use of forms (69) in connection with (54″), we find

$$\begin{aligned} m_1 + m_2 y' + l_1 y'^2 &= \lambda y'^2 + \mu y' + \nu, \\ n_1 + n_2 y' &= -\frac{\omega_2(\lambda y'^2 + \mu y' + \nu) + (y' - \omega_1)\omega_2' + \omega_2\omega_1'}{y' - \omega_1}. \end{aligned} \tag{70}$$

The elimination of λ, μ, ν then gives

$$\begin{vmatrix} y' - \omega_1 & m_1 + \omega_{1x} + (m_2 + \omega_{1y})y' + l_1 y'^2 \\ -\omega_2 & n_1 + \omega_{2x} + (n_2 + \omega_{2y})y' \end{vmatrix} = 0.$$

* *Trajectories*, article 13.

† Loc. cit., articles 14, 15.

This is to be satisfied identically; hence

$$
\begin{aligned}
& n_2 + \omega_{2y} + l_1\omega_2 = 0\,, \\
(71) \qquad & \omega_1(n_1 + \omega_{2x}) - \omega_2(m_1 + \omega_{1x}) = 0\,, \\
& n_1 + \omega_{2x} - \omega_1(n_2 + \omega_{2y}) + \omega_2(m_2 + \omega_{1y}) = 0\,.
\end{aligned}
$$

An analogous discussion for the systems S in the planes parallel to the xz plane yields the relations

$$
\begin{aligned}
& m_3 + \omega_{1z} + l_2\omega_1 = 0\,, \\
(72) \qquad & \omega_1(n_1 + \omega_{2x}) - \omega_2(m_1 + \omega_{1x}) = 0\,, \\
& m_1 + \omega_{1x} + \omega_1(n_3 + \omega_{2z}) - \omega_2(m_3 + \omega_{1z}) = 0\,.
\end{aligned}
$$

No new relations are found by considering other planes. The relations (71), (72) are seen to reduce to five and to be equivalent to the relations (57).

The only systems of curves with properties I, II, III, I_p, II_p, III_p *are those systems of type* (54), (54′), (54″) *in which the functions* (55) *satisfy the relations* (57).

34. It has been proved incidentally that when II_p, III_p hold for the systems S in two series of parallel planes, say the planes parallel to two of the coördinate planes, they then necessarily hold for all planes of space. Of course it is assumed here, as in the following, that I, II, III hold for the original space system.

35. *Converse of* V_p. It will now be shown that property V_p imposes no additional restrictions. The condition that a plane system defined by the forms (69) shall have this property is*

$$(73) \qquad \lambda\omega^2 + \mu\omega + \nu + \omega_x + \omega\omega_y = 0\,.$$

Applying this to the S-systems in the planes parallel to the xy plane, for which, according to (70),

$$(74) \qquad \lambda = l_1, \qquad \mu = m_2, \qquad \nu = m_1, \qquad \omega = \omega_1,$$

we find

$$(75) \qquad l_1\omega_1^2 + m_2\omega_1 + m_1 + \omega_{1x} + \omega_1\omega_{1y} = 0\,.$$

This, however, is seen to be a consequence of the relations (57).

Property V_p *is a consequence of* II_p, III_p *in connection with* I, II, III.

36. *Converse of* VI_p. It will now be shown that property VI_p does impose additional restrictions and that these are precisely equivalent to the relations (59) obtained in article (25).

* *Trajectories*, article 22.

The condition for VI_p is*

$$\lambda_x + \left(\frac{\nu + \omega_x}{\omega}\right)_y = 0. \tag{76}$$

We begin by applying this to the S-systems in the planes parallel to the xy plane. In this case we may use the values given in (74); hence

$$l_{1x} + \left(\frac{m_1 + \omega_{1x}}{\omega_1}\right)_y = 0;$$

which is the third of the equations (59). The corresponding condition for the planes parallel to the xz plane is obtained by replacing y by z, ω_1 by ω_2, l_1 by l_2 and m_1 by n_1. This gives the second of the equations (59).

The discussion for the S-systems in planes parallel to the yz plane is not so simple; for here we can no longer employ x as the independent variable. It is necessary to transform the second of the equations (54) so that say y shall be the independent variable. The result is

$$\frac{d^3z}{dy^3} = \frac{\lambda\left(\frac{dz}{dy}\right)^2 + \mu\frac{dz}{dy} + \nu}{\frac{dz}{dy} - \omega}\,\frac{d^2z}{dy^2} + \frac{3}{\frac{dz}{dy} - \omega}\left(\frac{d^2z}{dy^2}\right)^2, \tag{77}$$

where

$$\lambda = -\frac{m_3}{\omega_1}, \qquad \mu = \frac{n_3 - m_2 - m_1\frac{dx}{dy}}{\omega_1}, \qquad \nu = \frac{n_2 + n_1\frac{dx}{dy}}{\omega_1}, \qquad \omega = \frac{\omega_2}{\omega_1}. \tag{77'}$$

For the S-systems considered, x is an arbitrary constant and $dx/dy = 0$. Thus the condition (76) gives

$$\frac{\partial}{\partial y}\left(-\frac{m_3}{\omega_1}\right) + \frac{\partial}{\partial z}\left(\frac{\frac{n_2}{\omega_1} + \left(\frac{\omega_2}{\omega_1}\right)_y}{\frac{\omega_2}{\omega_1}}\right) = 0.$$

By using the relations (57), it may be shown that this takes the simple form

$$l_{1z} - l_{2y} = 0,$$

which is the first of the relations (59).

We may avoid the calculations that would be introduced by the direct consideration of other planes as follows. It has been shown that when the S-systems in planes parallel to the coördinate planes possess property VI_p, then relations (59) are necessarily fulfilled. The relations (57) are already fulfilled on account of properties II_p, III_p. From the theorem of article 25 it then follows that the

* *Trajectories*, p. 415.

original system of space curves is of dynamical type. Hence, by article 30, the systems S for all the planes of space are of dynamical type and hence possess VI_p.

In order that a system with properties I, II, III, II_p, III_p *shall also possess property* VI_p *it is necessary and sufficient that the relations* (59) *shall be fulfilled.*

37. It has been shown incidentally that when VI_p holds for the S-systems constructed in a triply orthogonal series of planes, it then holds for all S-systems.

38. *Restatement of* IV. It will be recalled that the plane systems S were obtained, not by the orthogonal projection of the space curves, but by the orthogonal projection of the differential elements of the third order belonging to these curves. If we consider the ∞^4 space curves of the original quintuply infinite system which touch a given plane π, their orthogonal projections upon that plane would give rise to a system of ∞^4 plane curves. The relation between these curves and those belonging to the system S in the same plane is as follows: the projections of the ∞^2 space curves touching π at a common point O have, at that point, contact of the third order with the ∞^2 S-curves passing through O.

Properties I_p, II_p, III_p have reference to the curves passing through a given point; properties V_p, VI_p relate to certain variations produced by changing that point. Neither set involves differential elements of higher than the third order. Hence it is possible to replace property IV by the following equivalent:

If the ∞^2 *trajectories touching a given plane* π *at a common point* O *are projected orthogonally upon* π, *the plane curves thus obtained possess the properties* I_p, II_p, III_p *and hence also the derived property* IV_p *as stated in the two-dimensional theory.** *When the point* O *varies in the plane* π, *the direction associated with it by* II_p *and the conic associated with it by* IV_p *vary in accordance with the restrictions expressed in* V_p *and* VI_p.*

Complete characterization.

39. The main result of our entire discussion may be stated as follows:

In order that a system of ∞^5 *space curves, of which* ∞^1 *pass through each point in each direction, shall be identifiable with the system of trajectories due to a (positional) field of force*

$$\frac{d^2x}{dt^2} = \phi(x, y, z), \qquad \frac{d^2y}{dt^2} = \psi(x, y, z), \qquad \frac{d^2z}{dt^2} = \chi(x, y, z),$$

it is necessary and sufficient that it shall have the following purely geometric properties:

I. *The osculating planes of the* ∞^2 *curves passing through a given point form a pencil; that is, all the planes pass through a fixed direction.*

* See *Trajectories.*

II. *The osculating spheres of the* ∞^1 *curves passing through a given point in a given direction form a pencil; their centers thus lie on a straight line.*

III. *The straight lines which correspond, in accordance with* II, *to all the directions at a given point, form a congruence, of order one and class three, consisting of the secants of a twisted cubic curve; which cubic passes through the given point in the direction fixed by property* I.

IV. *The associated systems* S, *constructed as in article* 29, *have the characteristic properties of plane dynamical systems.*

These four properties are independent in the sense that no one can be derived from those which precede it.* The last one, IV, may be replaced by weaker requirements: of the component parts it is sufficient to retain II_p, III_p, VI_p, and of all S-systems it is sufficient to consider those constructed in a triply orthogonal series of planes.

40. From article 22 it is seen that a complete geometric characterization of the trajectories produced by a *conservative force* is obtained by adding to III in the preceding set the requirement that, for every point of space, the corresponding cubic curve shall be of the rectangular species.

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* The question of absolute independence is left open. It may be that the other properties are consequences of IV.

A CLASS OF PERIODIC ORBITS OF AN INFINITESIMAL BODY SUBJECT TO THE ATTRACTION OF n FINITE BODIES*

BY

WILLIAM RAYMOND LONGLEY

Introduction.

The question of periodic solutions of a set of differential equations with application to the problem of three bodies has been treated by POINCARÉ,† and the idea of analytic continuation which he developed will be employed in this paper to prove the existence of the solutions under consideration.

Suppose that n finite bodies move in one plane and that their coördinates with reference to one of the bodies, M, as origin are known functions of the time. At any epoch t_0 the bodies form a certain geometrical figure and have certain relative components of velocity. Suppose that at the epoch $t_0 + T_1$ the bodies form again the same geometrical figure and have the same relative components of velocity; it is said that the motion is periodic with the period T_1.

A particle P is supposed to move in the plane of the motion of the finite bodies, subject to their newtonian attraction. In the differential equations of motion of the particle it is convenient to divide the terms into two classes: (1) those which are due to the attraction of M alone, and, (2) those which are due to the attraction of the other bodies. A parameter μ is introduced as a factor of all the terms of the second class, so that when $\mu = 0$ the differential equations represent the motion of P when subject to the attraction of only one body, M. In this case the solution is known and the conic in which the particle moves is called the *undisturbed orbit.*

If there exists a periodic solution of the differential equations when $\mu \neq 0$,‡ the period T of the solution must be a multiple of T_1, $T = qT_1$. Suppose the period is assigned in advance. When $\mu = 0$ a periodic solution is known. The initial conditions may be determined so that the period T_2 of this solution is a submultiple of the assigned period, that is, $pT_2 = qT_1 = T$, where p and q are

* Presented to the Society April 14, 1906. Received for publication June 15, 1906.

† Bulletin Astronomique, vol. 1 (1884), p. 65; Acta Mathematica, vol. 13 (1890), p. 5; *Les méthodes nouvelles de la mécanique céleste*, vol. 1, chap. 3 (1892).

‡ The analysis employed in this paper is valid for sufficiently small values of the parameters involved. All statements concerning the properties of the solutions for $\mu \neq 0$ refer only to values of μ so small that the solutions are known to be valid.

integers. There exists, therefore, a solution of the differential equations for $\mu = 0$, which has the assigned period. Does this solution persist, in the sense of analytic continuation, for values of μ different from zero but sufficiently small? This question is here investigated by the method of POINCARÉ. Three cases present themselves for consideration:

I. When the orbit of the particle is symmetrical with respect to a line.

II. When there exists a uniform integral of the differential equations of motion.

III. When no uniform integral exists.

I. The first case may occur as a particular sub-case of either the second or the third; but, if the differential equations admit a symmetrical solution, the proof of the existence of a symmetrical periodic orbit does not depend upon the consideration of the integral, and may be most conveniently treated as a distinct case.

If a particle is subject to the attraction of two finite bodies, there are always initial conditions under which the orbit is symmetrical with respect to the line joining the finite bodies, and with respect to the time of crossing this line. If the body M_1 revolves about M in a circle, the condition for a symmetrical orbit of the particle P is that P shall cross the radius vector of M_1 orthogonally at any time. The orbits of HILL,* DARWIN,† and MOULTON ‡ are of this type. If the orbit of M_1 is an ellipse, the condition is that P shall cross the radius vector orthogonally when M_1 is at an apse.§ The property of the symmetry of the orbit of P with respect to a line rotating with an angular velocity which is a function of t^2, holds for some cases in which the particle is subject to the attraction of more than two finite bodies. When the motion of n finite bodies is such that they form a geometrical figure having always the same shape, in which equal masses are symmetrically situated with respect to a line of the figure, the particle may be started so that its orbit will be symmetrical.|| An example of another type of the motion of the finite bodies, which is such that the orbit of the particle may be symmetrical, is given in the last section of this paper in the third numerical problem.

II. The case where a uniform integral exists occurs when the finite bodies move in circles about M with the same angular velocity N. The bodies form a fixed geometrical configuration in a plane which rotates about the origin with

* American Journal of Mathematics, vol. 1 (1878), p. 245.

† Acta Mathematica, vol. 21 (1897), p. 99.

‡ Transactions of the American Mathematical Society, vol. 7 (1906), p. 537.

§ This case has been treated in an unpublished paper by W. D. MACMILLAN presented to the Society, April 14, 1906.

|| Examples are furnished by the equilateral triangular solution of Lagrange, when two of the bodies are equal, and by many particular solutions of the problem of n bodies. See W. R. LONGLEY, *Some particular solutions in the problem of n bodies*, Bulletin of the American Mathematical Society, vol. 13 (1907), p. 324.

the angular velocity N. When the coördinates of the particle are referred to axes in the rotating plane, the differential equations of motion do not contain the time explicitly, and the period of the solution may be chosen arbitrarily except for limitations which are necessary to assure the convergence of the series employed.

III. The third case occurs when the finite bodies do not move in circular orbits about M. The parameter μ is introduced into the differential equations in such a way that the proof of the existence of periodic orbits of the particle is made to depend upon the terms which are due to the attraction of only one of the perturbing bodies, M_1. The motion of M_1 is subject to mild restrictions, namely, that the radius vector shall be given by an expression involving only cosines of multiples of the time, and that the expressions for the longitude shall involve only sines of multiples of the time. The only restriction upon the motion of the other bodies is that it shall be periodic. The treatment is sufficiently general to permit applications in the problems presented by the motions of the solar system. For example, suppose that P is a satellite of one of the planets M, and that M_1 is the sun. The conditions upon the motion of M_1 are fulfilled if we neglect the perturbations of the other planets upon M. If we neglect the inclinations of the orbits of the other planets, and suppose that their motion is periodic, it is possible by the methods given below to treat the periodic motion of the satellite in the plane of the planetary orbit when subject to the attraction of the sun and all the planets.

It is shown that when the period of the motion of the particle is assigned, there exist two and only two real orbits of this analytic type having the prescribed period. In one of the orbits the motion is direct, and in the other it is retrograde. If symmetrical orbits exist, there are no unsymmetrical periodic orbits. It is shown that, in the analysis employed, the eccentricity of the undisturbed orbit must be zero. A method is given for constructing the periodic solutions to any desired degree of accuracy by processes which involve only algebraic computation.

For convenience and clearness the details of the analysis are carried out for a special example of the motion of the finite bodies, namely, for three finite bodies which move according to the equilateral triangular solution of Lagrange. This example illustrates the three cases which may occur, and only slight changes in the details are necessary in order to apply the treatment to the more general examples previously mentioned.

Some numerical examples are given at the end of this paper with drawings of the orbits which have been computed.

Existence of Periodic Orbits.

The differential equations of motion. Let M, M_1, and M_2 denote the masses of three finite bodies moving according to the lagrangian equilateral triangular solution. With reference to M as origin and an axis having a fixed direction in space, let the coördinates of M_1, M_2, and the particle P be respectively (R_1, V_1), $(R_2\ V_2)$, and (r, v). The coördinates of the bodies are expressed in terms of the time as follows:

$$(1)\qquad \begin{aligned} R_1 = R_2 = AR &= A\left\{1 - \epsilon\cos Nt - \frac{\epsilon^2}{2}(\cos 2Nt - 1) + \cdots\right\}, \\ V_1 - \varpi_1 = V_2 - \varpi_2 = V &= Nt + 2\epsilon\sin Nt + \tfrac{5}{4}\epsilon^2\sin 2Nt + \cdots, \end{aligned}$$

where

$$N^2A^3 = k^2(M + M_1 + M_2).$$

The finite bodies are at apses of their respective orbits at $t = 0$; N is the mean angular motion, A the major semi-axis, ϵ the eccentricity, k is a constant depending upon the units employed; and ϖ_1, ϖ_2 are constants such that $\varpi_2 - \varpi_1 = \pi/3$.

The differential equations of motion of P are

$$(2)\qquad \begin{aligned} &\frac{d^2r}{dt^2} - r\left(\frac{dv}{dt}\right)^2 + \frac{k^2M}{r^2} = \frac{\partial\Omega}{\partial r}, \\ &r\frac{d^2v}{dt^2} + 2\frac{dr}{dt}\frac{dv}{dt} = \frac{1}{r}\frac{\partial\Omega}{\partial v}, \end{aligned}$$

where

$$\Omega = k^2\left[\frac{M_1}{r_1} + \frac{M_2}{r_2} - \frac{M_1}{R_1^2}r\cos(v - V_1) - \frac{M_2}{R_2^2}r\cos(v - V_2)\right],$$

$$(3)\qquad \begin{aligned} r_1 &= \sqrt{r^2 + R_1^2 - 2rR_1\cos(v - V_1)}, \\ r_2 &= \sqrt{r^2 + R_2^2 - 2rR_2\cos(v - V_2)}. \end{aligned}$$

Let us define m and a by the relations

$$(4)\qquad m\nu = N, \qquad \nu^2a^3 = k^2M,$$

where ν is the mean angular velocity of P in the undisturbed orbit.

The period of the motion of the particle in the undisturbed orbit is $2\pi/\nu$. The period of the motion of the finite bodies is $2\pi/N$. In order that the periods shall be commensurable, ν is determined by the relation $2p\pi/\nu = 2q\pi/N$, where p and q are integers. This restriction is not necessary, however, when $\epsilon = 0$.

The next step in obtaining the final form of the differential equations is to refer the motion to a plane rotating with the angular velocity of the finite bodies. If $\epsilon = 0$, the coördinates of M_1 and M_2 become constants, and ν may be

chosen arbitrarily. The motion is referred to an axis rotating with the angular velocity of the finite bodies by the substitution $v = w + V$. It is convenient to eliminate the factors depending upon the units employed by, the substitution

$$r = a\rho, \qquad \nu t = \tau.$$

Hence Nt becomes $m\tau$. We may expand Ω as a power series in $a\rho/AR$ as follows:

$$\Omega = \frac{k^2 M_1}{AR}\left[1 + \tfrac{1}{4}\left(\frac{a\rho}{AR}\right)^2\left\{1 + 3\cos 2(w - \varpi_1)\right\}\right.$$
$$\left. + \tfrac{1}{8}\left(\frac{a\rho}{AR}\right)^3\left\{3\cos(w - \varpi_1) + 5\cos 3(w - \varpi_1)\right\} + \cdots\right]$$
$$+ \frac{k^2 M_2}{AR}\left[1 + \tfrac{1}{4}\left(\frac{a\rho}{AR}\right)^2\left\{1 + 3\cos 2(w - \varpi_2)\right\}\right.$$
$$\left. + \tfrac{1}{8}\left(\frac{a\rho}{AR}\right)^3\left\{3\cos(w - \varpi_2) + 5\cos 3(w - \varpi_2)\right\} + \cdots\right].$$

Let λ_1 and λ_2 be defined by the relations

$$M_1 = \lambda_1(M_1 + M_2), \qquad M_2 = \lambda_2(M_1 + M_2). \tag{5}$$

From equations (1) we have

$$\frac{k^2(M_1 + M_2)}{A^3} = \frac{N^2}{1 + \dfrac{M}{M_1 + M_2}}.$$

On setting

$$\frac{1}{1 + \dfrac{M}{M_1 + M_2}} = K,$$

it follows that

$$\frac{1}{a}\frac{\partial\Omega}{\partial(a\rho)} = \frac{Km^2\rho}{R^3}\left[\frac{\lambda_1}{2}\left\{1 + 3\cos 2(w - \varpi_1)\right\}\right.$$
$$+ \frac{3\lambda_1}{8}\left(\frac{a\rho}{AR}\right)\left\{3\cos(w - \varpi_1) + 5\cos 3(w - \varpi_1)\right\} + \cdots$$
$$+ \frac{\lambda_2}{2}\left\{1 + 3\cos 2(w - \varpi_2)\right\} \tag{6}$$
$$\left. + \frac{3\lambda_2}{8}\left(\frac{a\rho}{AR}\right)\left\{3\cos(w - \varpi_2) + 5\cos 3(w - \varpi_2)\right\} + \cdots\right],$$

$$
(6)\qquad
\begin{aligned}
\frac{1}{a^2\rho}\frac{\partial\Omega}{\partial w} = -\frac{Km^2\rho}{R^3}\Bigg[& \frac{3\lambda_1}{2}\sin 2(w-\varpi_1) \\
& + \frac{3\lambda_1}{8}\left(\frac{a\rho}{AR}\right)\left\{\sin(w-\varpi_1)+5\sin 3(w-\varpi_1)\right\}+\cdots \\
& + \frac{3\lambda_2}{2}\sin 2(w-\varpi_2) \\
& + \frac{3\lambda_2}{8}\left(\frac{a\rho}{AR}\right)\left\{\sin(w-\varpi_2)+5\sin 3(w-\varpi_2)\right\}+\cdots\Bigg].
\end{aligned}
$$

The differential equations of motion become

$$
(7)\qquad
\begin{aligned}
& \frac{d^2\rho}{d\tau^2}-\rho\left(\frac{dw}{d\tau}+\frac{dV}{d\tau}\right)^2+\frac{1}{\rho^2}=\frac{1}{a}\frac{\partial\Omega}{\partial(a\rho)}, \\
& \rho\left(\frac{d^2w}{d\tau^2}+\frac{d^2V}{d\tau^2}\right)+2\frac{d\rho}{d\tau}\left(\frac{dw}{d\tau}+\frac{dV}{d\tau}\right)=\frac{1}{a^2\rho}\frac{\partial\Omega}{\partial w}.
\end{aligned}
$$

Case I. Symmetrical solutions. Suppose the masses M_1 and M_2 are equal and that the w-axis passes through the center of gravity of the system. These conditions are expressed by the relations,

$$
(8)\qquad \lambda_1=\lambda_2, \qquad \varpi_1=-\frac{\pi}{6}, \qquad \varpi_2=\frac{\pi}{6}.
$$

It follows that the first of equations (7) involves only *cosines* of multiples of w, and the second involves only *sines* of multiples of w. Since $dV/d\tau$ and R are even functions of τ, it follows that the equations are unchanged if ρ is left unchanged while w and τ are changed into $-w$ and $-\tau$ respectively.* When subject to the restrictions (8), let the differential equations be denoted by (7_{I}).†

Suppose that

$$
(9)\qquad \rho=\psi_1(\tau), \qquad w=\psi_2(\tau),
$$

is a solution of (7_{I}) such that

$$
(10)\qquad \left.\frac{d\rho}{d\tau}\right|_{\tau=p\pi}=0, \qquad w(p\pi)-p\pi=0.
$$

The differential equations are periodic in w and τ, and M_1 and M_2 are at apses at $\tau=p\pi$. It follows from the form of the differential equations that ψ_1 is an even function and ψ_2 is an odd function of τ. Hence if the particle crosses the w-axis when the finite bodies are at apses of their respective orbits, the orbit of P is symmetrical with respect to the axis and the time of crossing. If the orbit is symmetrical with respect to two epochs, for example, $\tau_0=0$ and $\tau_1=p\pi$, it is periodic with the period $2p\pi$. If the solution (9) is symmetrical with

* In the more general cases in which symmetrical solutions occur it may happen that the first equation contains also sines of multiples of w multiplied by odd functions of τ, and that the second equation contains also cosines of multiples of w multiplied by odd functions of τ.

† Throughout this paper the subscripts I, II, III refer to the case I, II, III, respectively.

respect to the epoch $\tau = 0$, the conditions (10) are necessary and sufficient for it to be periodic with the period $2p\pi$.

We will now investigate the existence of symmetrical periodic orbits which have the period $2p\pi$. The terms of equations (7_{I}) are periodic in τ and analytic in m; but, when expanded in this parameter, the coefficients of the various powers of m contain non-periodic terms. It is convenient in this discussion of periodic solutions to avoid such terms in the differential equations, and this is accomplished by a generalization of the parameter m.* Consider the set of differential equations,

$$
(11)\qquad
\begin{aligned}
&\frac{d^2\rho}{d\tau^2} - \rho\left(\frac{dw}{d\tau} + \mu\frac{dU}{d\tau}\right)^2 + \frac{1}{\rho^2} = \mu^2 f_{\mathrm{I}},\\
&\rho\left(\frac{d^2w}{d\tau^2} + \mu\frac{d^2U}{d\tau^2}\right) + 2\frac{d\rho}{d\tau}\left(\frac{dw}{d\tau} + \mu\frac{dU}{d\tau}\right) = \mu^2 g_{\mathrm{I}},
\end{aligned}
$$

where

$$
U = \frac{\mu}{m}\,V = \frac{\mu}{m}\left\{m\tau + 2\epsilon\sin m\tau + \tfrac{5}{4}\epsilon^2\sin 2m\tau + \cdots\right\}
$$

and $\mu^2 f_{\mathrm{I}}$ and $\mu^2 g_{\mathrm{I}}$ are obtained from $1/a\cdot\partial\Omega/\partial(a\rho)$ and $1/a^2\rho\cdot\partial\Omega/\partial w$ respectively by writing μ^2 for the factor m^2, and by making the substitution $a/A = \eta\mu$, where η is a numerical constant.

Equations (11) do not represent any physical problem except for a special value of μ, namely, $\mu = m$, and are to be studied from a mathematical point of view. They may be solved for $\rho - 1$ and w as power series in μ. For a preassigned arbitrary interval for τ the series are convergent for all values of μ numerically less than $\bar{\mu}$,† and satisfy the differential equations identically in μ. Therefore, if in the solution μ is given a value numerically equal to $m\,(m < \mu)$, the solution satisfies the differential equations (7_{I}) and represents the physical problem under consideration. It follows from the form in which m enters the differential equations that $\bar{\mu}$ is independent of m.

When $\mu = 0$ the equations are of the form occurring in the problem of two bodies, and a symmetrical solution having the required period is known, namely,

$$
(12)\qquad
\begin{aligned}
&\rho = 1 - \bar{e}\cos E,\\
&w = \arccos\left(\frac{\cos E - \bar{e}}{1 - \bar{e}\cos E}\right) = \arcsin\left(\frac{\sqrt{1-\bar{e}^2}\sin E}{1 - \bar{e}\cos E}\right),\ ‡
\end{aligned}
$$

where E is defined by the relation

$$
\tau = E - \bar{e}\sin E,
$$

* F. R. Moulton, *On certain rigorous methods of treating problems in celestial mechanics.* The Decennial Publications of The University of Chicago, vol. 3 (1902), p. 126.

† Poincaré, *Les méthodes nouvelles de la mécanique céleste*, vol. 1 (1892), p. 58.

‡ The positive sign is to be taken with the radical.

and $\bar{e}$ is an arbitrary constant less than unity. The initial conditions for $\tau = 0$ are

$$\rho = 1 - \bar{e}, \qquad \frac{d\rho}{d\tau} = 0, \qquad w = 0, \qquad \frac{dw}{d\tau} = \frac{\sqrt{1-\bar{e}^2}}{(1-\bar{e})^2}. \tag{13}$$

Consider the solution for values of μ different from zero but sufficiently small, and let the initial conditions for $\tau = 0$ be

$$\begin{gathered} \rho = 1 - \bar{e} + \beta = (1+\alpha)[1-(\bar{e}+e)],^* \qquad \frac{d\rho}{d\tau} = 0, \qquad w = 0, \\ \frac{dw}{d\tau} = \frac{\sqrt{1-\bar{e}^2}}{(1-\bar{e})^2} + \gamma = \frac{\sqrt{1-(\bar{e}+e)^2}}{[1-(\bar{e}+e)]^2} - \mu \left.\frac{dU}{d\tau}\right|_{\tau=0}. \end{gathered} \tag{14}$$

The solution is symmetrical with respect to the epoch $\tau = 0$, and may be expressed as power series in α, e, and μ, which are convergent for an interval in τ including the interval 0 to $p\pi$, if the parameters are sufficiently small. If α and e can be determined in terms of μ, vanishing with μ, so that the conditions (10) are satisfied, then the solution will be periodic with the period $2p\pi$. All terms of the solution which are independent of μ^2 may be obtained from the two body problem by making the substitution $w = u - \mu U$.

These terms are given in finite form by the expressions

$$\rho = (1+\alpha)[1-(\bar{e}+e)\cos E],$$

$$u = \arccos\left(\frac{\cos E - (\bar{e}+e)}{1-(\bar{e}+e)\cos E}\right) = \arcsin\left(\frac{\sqrt{1-(\bar{e}+e)^2}\sin E}{1-(\bar{e}+e)\cos E}\right),$$

where E is defined by the relation

$$\frac{\tau}{(1+\alpha)^{\frac{3}{2}}} = E - (\bar{e}+e)\sin E.$$

Returning to the variable w, writing the terms in α and e as power series by Taylor's expansion, and applying the conditions (10), we obtain the equations

$$\begin{gathered} 0 = -\tfrac{3}{2}p\pi\frac{\bar{e}}{(1-\bar{e})^2}\alpha - \tfrac{3}{4}p\pi\frac{1+\bar{e}}{(1-\bar{e})^3}\alpha e + \cdots, \\ 0 = -\tfrac{3}{2}p\pi\frac{\sqrt{1-\bar{e}^2}}{(1-\bar{e})^2}\alpha - p\pi\mu + \cdots. \end{gathered} \tag{15}$$

It follows from the known properties of the series that there are no terms in e alone, and there are no terms involving μ to the first degree except the term

* The introduction of the very convenient parameters α and e, instead of the additive increments β and γ, is due to Professor F. R. MOULTON, these Transactions, loc. cit., from whom I have received many suggestions and much valuable criticism.

$-p\pi\mu$. Hence the second of equations (15) may be solved for α as a power series in e and μ in which μ is contained as a factor:

$$\alpha = \mu\left\{-\tfrac{2}{3}\frac{(1-\bar{e})^2}{\sqrt{1-\bar{e}^2}} + \cdots\right\}. \tag{16}$$

When this value of α is substituted in the first equation, a factor μ may be divided out, leaving

$$0 = p\pi\frac{\bar{e}}{\sqrt{1-\bar{e}^2}} + \tfrac{1}{2}p\pi\frac{1+\bar{e}}{(1-\bar{e})\sqrt{1-\bar{e}^2}}e + \cdots.$$

This equation may be solved for e as a power series in μ, which vanishes with μ, if and only if $\bar{e}=0$, and the result may then be used to eliminate e from equation (16). Since only those solutions are under consideration which are the analytic continuations with respect to μ of those for $\mu = 0$, the condition $\bar{e} = 0$ must be imposed. *The condition $\bar{e} = 0$ means that the undisturbed orbit must be circular.*

The initial conditions (14) have been determined uniquely as power series in μ, convergent for sufficiently small values of μ, so that the conditions of periodicity are satisfied. When the values of α and e thus found are substituted in the solution, we have $\rho - 1$ and w represented as power series in μ alone, and the solution is periodic with the period $2p\pi$.

Suppose $\mu = m$ and consider the physical interpretation of the solution. The period of the motion of the finite bodies is $2\pi/m = 2p\pi/q$, where p and q are integers which may be chosen arbitrarily except for the condition that q/p must be sufficiently small. The period of the solution is $2p\pi$. Hence during this period the finite bodies make q revolutions, the angle w is increased by $2p\pi$, and therefore the particle makes $q + p$ revolutions with reference to fixed axes. If p and q have the same signs, that is, if m is positive, the motion of the particle is direct. If m is negative, the motion is retrograde. For a given value of m there exists one and only one real symmetrical periodic solution of the differential equations. Hence for a given period there exist two and only two real symmetrical orbits of the particle with the required period. In one the motion is direct, and in the other it is retrograde. All distinct orbits are obtained when p and q are prime to each other.

The initial conditions were limited so that the solutions should be symmetrical. When this condition is not imposed, and whether or not the differential equations admit symmetrical solutions, it will be shown in the next two sections that, for a preassigned period, there exist two and only two real periodic orbits, in one of which the motion is direct and in the other retrograde. Hence, if symmetrical periodic orbits exist, there are no unsymmetrical periodic orbits of this type.

The existence proof was made for a particular example, but did not depend upon the right hand members of the differential equations and consequently holds for any case in which the orbits of the perturbing bodies are such that the differential equations admit symmetrical solutions. The only variations which it is necessary to make in the details for the other cases, are in the particular manner in which the parameter μ is introduced, and in the choice of the rotating axis. It may happen in some cases that there exist several possible axes of symmetry.

Case II. *When a uniform integral exists.* Suppose that M_1 and M_2 are not in general equal and that they move in circles about M with the w-axis passing through M_1. These conditions are expressed by the relations

$$\epsilon = 0, \qquad \varpi_1 = 0, \qquad \varpi_2 = \frac{\pi}{3}. \tag{17}$$

When subject to the conditions (17) let the differential equations of motion be denoted by (7_{II}).

In this case it is convenient to introduce the parameter μ by the relations

$$m = \mu, \qquad \lambda_2 = \lambda\mu, \qquad \frac{a}{A} = \eta\mu, \tag{18}$$

for all powers of a/A higher than the *first*. Here it is not necessary to generalize the parameter m since R and $dV/d\tau$ are constants. By relating λ_2 and μ, the existence proof is made to depend only upon those terms of the differential equations which involve λ_1. The generalization of the parameter a/A is merely for convenience in having finite expressions in the equations which determine the coefficients at the various steps in the solution.

The differential equations become

$$\begin{aligned} &\frac{d^2\rho}{d\tau^2} - \rho\left(\frac{dw}{d\tau} + \mu\right)^2 + \frac{1}{\rho^2} = \mu^2 f_{\text{II}}, \\ &\rho\frac{d^2 w}{d\tau^2} + 2\frac{d\rho}{d\tau}\left(\frac{dw}{d\tau} + \mu\right) = \mu^2 g_{\text{II}}, \end{aligned} \tag{19}$$

where

$$f_{\text{II}} = K\rho\left[\lambda_1\left\{\tfrac{1}{2}(1 + 3\cos 2w) + \tfrac{3}{8}\frac{a}{A}\rho(3\cos w + 5\cos 3w)\right\} + \cdots \mu\,\text{func}\,(\rho, \mu, \cos jw, \sin jw)\right],$$

$$g_{\text{II}} = -K\rho\left[\lambda_1\left\{\tfrac{3}{2}\sin 2w + \tfrac{3}{8}\frac{a}{A}\rho(\sin w + 5\sin 3w)\right\} + \cdots \mu\,\text{func}\,(\rho, \mu, \sin jw, \cos jw)\right].$$

Equations (19) are periodic in w with period 2π and do not involve τ explicitly. Suppose that

$$\rho = \psi_1(\tau), \qquad w = \psi_2(\tau),$$

is a solution. The necessary and sufficient conditions that the solution shall be periodic with the period $2p\pi$ are

$$(20) \qquad \begin{aligned} \psi_1(2p\pi) &= \psi_1(0), & \left.\frac{d\psi_1}{d\tau}\right|_{2p\pi} &= \left.\frac{d\psi_1}{d\tau}\right|_0, \\ \psi_2(2p\pi) - 2p\pi &= \psi_2(0), & \left.\frac{d\psi_2}{d\tau}\right|_{2p\pi} &= \left.\frac{d\psi_2}{d\tau}\right|_0. \end{aligned}$$

When $\mu = 0$ a periodic solution is known,* namely,

$$\rho = 1, \qquad w = \tau.$$

The initial conditions for $\tau = 0$ are

$$\rho = 1, \qquad \frac{d\rho}{d\tau} = 0, \qquad w = 0, \qquad \frac{dw}{d\tau} = 1.$$

For μ different from zero, but sufficiently small, consider the solution of equations (19) subject to the initial conditions

$$(21) \qquad \begin{aligned} \rho &= 1 + \beta_1 = (1+\alpha)(1 - e\cos\theta), \\ \frac{d\rho}{d\tau} &= \beta_2 = \frac{e\sin\theta}{\sqrt{1+\alpha}(1 - e\cos\theta)}, \\ w &= \beta_3 = \arccos\left[\frac{\cos\theta - e}{1 - e\cos\theta}\right] - \arccos\left[\frac{\cos(\theta-\phi) - e}{1 - e\cos(\theta-\phi)}\right], \\ \frac{dw}{d\tau} &= 1 + \beta_4 = \frac{\sqrt{1-e^2}}{(1+\alpha)^{\frac{3}{2}}(1 - e\cos\theta)^2} - \mu. \end{aligned}$$

The parameters, β_1, β_2, β_3, β_4, which are additive increments to the initial conditions of undisturbed motion, are inconvenient in the discussion which follows. In terms of the parameters α, e, θ, ϕ, the properties of the solution are well known, and the conditions of periodicity can be easily discussed. The geometrical meaning of the angles θ and ϕ is shown in Fig. 1.

We may now write, in finite form, those terms of the solution which do not contain μ, namely,

$$(22) \qquad \begin{aligned} r &= (1+\alpha)(1 - e\cos E), \\ \cos(w + W_1 - W_0) &= \frac{\cos E - e}{1 - e\cos E}, \\ \sin(w + W_1 - W_0) &= \frac{\sqrt{1-e^2}\sin E}{1 - e\cos E}, \end{aligned}$$

* As in the case of symmetrical orbits, no greater generality is obtained by assuming that the undisturbed orbit has an eccentricity different from zero. Therefore we will assume here that $\bar{e} = 0$.

where E is defined by the relation

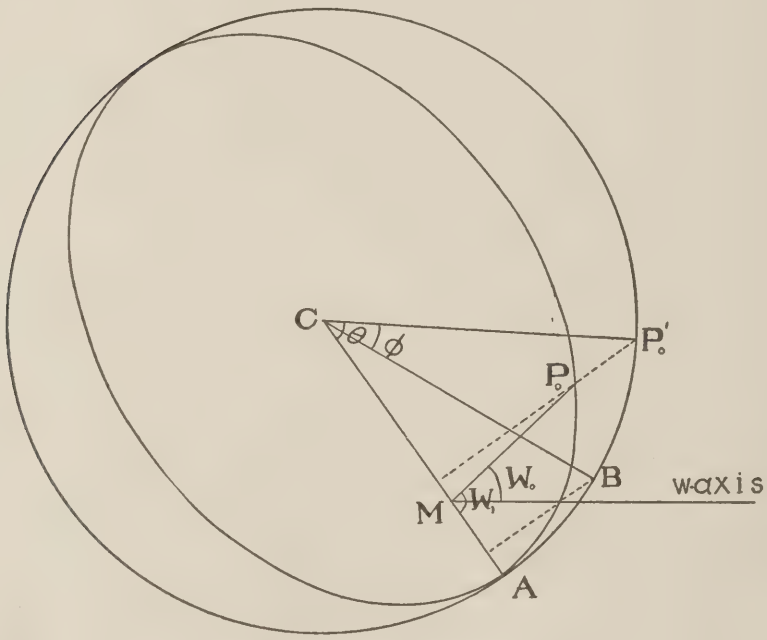
$$\frac{\tau}{(1+\alpha)^{\frac{3}{2}}} + \theta - e \sin \theta = E - e \sin E.$$

The only term of the solution which involves μ to the first degree is $-\tau\mu$, which occurs in the expression for w. To find the terms in μ^2 and $\mu^2\phi$ we may write

$$\rho = 1 + \rho_2 \mu^2 + \bar{\rho}\mu^2\phi + \cdots,$$

$$w = \tau - \tau\mu + \phi + w_2\mu^2 + \bar{w}\mu^2\phi + \cdots.$$

FIG. 1.



P_0 is the position of the particle at $\tau = 0$.
W_0 is the longitude of the particle at $\tau = 0$.
$W_1 - W_0$ is the longitude of perihelion.

On substituting these expressions in the differential equations, there results for the determination of ρ_2 and w_2 the following set of equations,

$$\frac{d^2\rho_2}{d\tau^2} - 2\frac{dw_2}{d\tau} - 3\rho_2 = \frac{K\lambda_1}{2}(1 + 3\cos 2\tau) + \frac{3K\lambda_1}{8}\frac{a}{A}(3\cos\tau + 5\cos 3\tau),$$

$$\frac{d^2 w_2}{d\tau^2} + 2\frac{d\rho_2}{d\tau} = -\frac{3K\lambda_1}{2}\sin 2\tau - \frac{3K\lambda_1}{8}\frac{a}{A}(\sin\tau + 5\sin 3\tau).$$

It follows that

$$\rho_2 = K\lambda_1 \left[-1 - 2\frac{a}{A} + \left(2 + \tfrac{153}{64}\frac{a}{A}\right)\cos\tau \right.$$
$$\left. + \tfrac{15}{16}\frac{a}{A}\tau\sin\tau - \cos 2\tau - \tfrac{25}{64}\frac{a}{A}\cos 3\tau \right],$$

$$w_2 = K\lambda_1 \left[\left(\tfrac{5}{4} + 3\frac{a}{A}\right)\tau - \left(4 + \tfrac{201}{32}\frac{a}{A}\right)\sin\tau \right.$$
$$\left. + \tfrac{15}{8}\frac{a}{A}\tau\cos\tau + \tfrac{11}{8}\sin 2\tau + \tfrac{135}{32}\frac{a}{A}\sin 3\tau \right].$$

By a similar computation it is shown that

$$\bar{\rho} = K\lambda_1 \left[-\left(4 + \tfrac{285}{64}\frac{a}{A}\right)\sin\tau + \tfrac{15}{16}\frac{a}{A}\tau\cos\tau + 2\sin 2\tau + \tfrac{75}{64}\frac{a}{A}\sin 3\tau \right],$$

$$\bar{w} = K\lambda_1 \left[\tfrac{21}{4} + \tfrac{288}{32}\frac{a}{A} - \left(8 + \tfrac{333}{32}\frac{a}{A}\right)\cos\tau \right.$$
$$\left. - \tfrac{15}{8}\frac{a}{A}\tau\sin\tau + \tfrac{11}{4}\cos 2\tau + \tfrac{45}{32}\frac{a}{A}\cos 3\tau \right].$$

The terms independent of μ are obtained from equations (22) by Taylor's expansion, and the solution becomes

$$(23)\quad \begin{aligned} \rho &= 1 + \alpha - e\cos\tau - \alpha e(\cos\tau - \tfrac{3}{2}\tau\sin\tau) + e\theta\sin\tau \\ &\qquad + \alpha e\theta(\sin\tau - \tfrac{3}{2}\tau\cos\tau) + \rho_2\mu^2 + \bar{\rho}\mu^2\phi + \cdots, \\ w &= \tau - \tfrac{3}{2}\tau\alpha + 2e\sin\tau + \phi - \tau\mu - 3\alpha e\tau\cos\tau - e\theta(1 - 2\cos\tau) \\ &\qquad + e\phi + \alpha e\theta(5\cos\tau + 3\tau\sin\tau) + w_2\mu^2 + \bar{w}\mu^2\phi + \cdots. \end{aligned}$$

Applying the conditions (20) that the solution shall be periodic, we have

$$(24)\quad \begin{aligned} 0 &= -3p\pi\alpha e\theta + \tfrac{15}{8}\frac{a}{A}p\pi\mu^2\phi + \cdots, && (a) \\ 0 &= 3p\pi\alpha e + \tfrac{15}{8}\frac{a}{A}p\pi\mu^2 + \cdots, && (b) \\ 0 &= -3p\pi\alpha - 2p\pi\mu - 6p\pi\alpha e + \cdots, && (c) \\ 0 &= 6p\pi\alpha e\theta - \tfrac{15}{4}\frac{a}{A}p\pi\mu^2\phi + \cdots. && (d) \end{aligned}$$

The conditions (24) involve the four quantities α, e, θ, ϕ, and, if independent, would determine them in terms of μ. But the differential equations (19) do not involve τ explicitly and hence admit the integral of JACOBI. This fur-

nishes a relation of the type

$$F(\alpha, e, \theta, \phi, \mu) = \text{constant},$$

and equations (24) are not independent.* It follows that if (a), (b), and (c) are solved for the three quantities α, e and θ in terms of μ and ϕ, and the results substituted in (d), the equation is satisfied identically in ϕ. In this problem the dynamical interpretation is simple. Since the finite bodies move in circles the origin of time is arbitrary. The most convenient choice is $\tau = 0$ when $w = 0$, which is equivalent to choosing $\phi = 0$.

Consider the solution of equations (a), (b), and (c) for α, e, and θ. The equations have the following properties:

i. There are no terms independent of α and μ. This follows from the fact that, in the two-body problem, the period does not depend upon e and θ.

ii. There are no terms involving μ to the first degree except the one term $-2p\pi\mu$ which occurs in (c).

iii. There are no terms in θ independent of e, since θ does not enter the initial conditions independently of e.

It follows from these properties and the particular form of the first terms of the equations that α, e, and θ may be determined uniquely as power series in μ by the following steps:

(1) From (c) we obtain

$$\alpha = \mu\left[-\tfrac{2}{3} + \cdots + \text{func}(\mu, e, \theta)\right].$$

(2) This value of α when substituted in (b) permits a factor μ to be divided out. Then we may solve the result for e as a power series in μ and θ, in which μ is contained as a factor,

$$e = \mu\left[\tfrac{15}{16}\frac{a}{A} + \cdots + \text{func}(\mu, \theta)\right],$$

(3) When the values of α and e are substituted in (a) a factor μ^2 may be divided out and θ obtained as a power series in μ alone, vanishing with μ.

(4) By substitution of the value of θ thus found in the expressions for e and α, we obtain finally

$$\alpha = \mu\mathfrak{P}_1(\mu), \qquad e = \mu\mathfrak{P}_2(\mu), \qquad \theta = \mu\mathfrak{P}_3(\mu).$$

The preceding operations are known to be convergent for all values of α, e, θ, and μ which are sufficiently small. Hence for a given value of μ sufficiently small it is possible to determine the initial conditions (21) as power series in μ such that the solution of the differential equations (19) shall be periodic in τ with the period $2p\pi$.

Since $\mu\nu = N$, μ is determined by choosing the period of the solution in t

* Cf. POINCARÉ, *Les méthodes nouvelles de la mécanique céleste*, vol. 1 (1892), p. 87.

which is $2p\pi/\nu$. All distinct orbits are obtained by taking $p=1$, that is, every orbit is reëntrant after *one* revolution in the rotating plane. For an assigned period there is one and only one direct orbit, and one and only one retrograde orbit.

The analysis and results apply to the case when any number of finite bodies revolve about M in circles with the same angular velocity.

Case III. *When no uniform integral exists.* Suppose that M_1 and M_2 are not, in general, equal; also, that the eccentricity of the orbits of the finite bodies is not zero, and that the w-axis passes through M_1. These conditions are expressed by the relations

$$\epsilon \neq 0, \qquad \varpi_1 = 0, \qquad \varpi_2 = \frac{\pi}{3}. \tag{25}$$

When subject to the conditions (25), let the differential equations of motion be denoted by (7_{III}). In this case it is convenient to consider the equations

$$\begin{aligned} &\frac{d^2\rho}{d\tau^2} - \rho\left(\frac{dw}{d\tau} + \mu\frac{dU}{d\tau}\right)^2 + \frac{1}{\rho^2} = \mu^2 f_{\text{III}}, \\ &\rho\left(\frac{d^2w}{d\tau^2} + \mu\frac{d^2U}{d\tau^2}\right) + 2\frac{d\rho}{d\tau}\left(\frac{dw}{d\tau} + \mu\frac{dU}{d\tau}\right) = \mu^2 g_{\text{III}}, \end{aligned} \tag{26}$$

where $U = \mu/m \cdot V$ and $\mu^2 f_{\text{III}}$ and $\mu^2 g_{\text{III}}$ are obtained from $1/a \cdot \partial\Omega/\partial(a\rho)$ and $1/a^2\rho \cdot \partial\Omega/\partial w$ respectively, by writing μ^2 for the factor m^2 and making the substitution

$$\lambda_2 = \lambda\mu.$$

Equations (20) express the conditions that a solution shall be periodic with the period $2p\pi$. Consider the solution of equations (26) subject to the initial conditions (21), except that the fourth condition contains the term $-\mu \cdot dU/d\tau$ $(\tau = 0)$ instead of $-\mu$. The terms independent of μ are given by equations (22). The only term involving μ to the first degree is $-\mu/m \cdot U$, which occurs in the expression for w.

For the determination of the terms in μ^2 we have the equations

$$\begin{aligned} &\frac{d^2\rho_2}{d\tau^2} - 2\frac{dw_2}{d\tau} - 3\rho_2 = \frac{K\lambda_1}{R^3}\left[\tfrac{1}{2}(1+3\cos 2\tau) + \tfrac{3}{8}\frac{a}{AR}(3\cos\tau + 5\cos 3\tau) + \cdots\right], \\ &\frac{d^2w_2}{d\tau^2} + 2\frac{d\rho_2}{d\tau} = -\frac{K\lambda_1}{R^3}\left[\tfrac{3}{2}\sin 2\tau + \tfrac{3}{8}\frac{a}{AR}(\sin\tau + 5\sin 3\tau) + \cdots\right]. \end{aligned} \tag{27}$$

R is a power series in ϵ whose coefficients involve only cosines of multiples of $m\tau$. It follows that the right hand member of the first and second equations respectively have the forms

$$A_{0,0} + \sum_{g,h} A_{g,h}\cos(g+hm)\tau, \qquad \sum_{g,h} D_{g,h}\sin(g+hm)\tau,$$

where g and h are integers, g taking all positive values and zero, and h taking positive and zero values such that $hm < 1$. Since τ enters the differential equations explicitly, we have $m = q/p$, by equation (4). $A_{g,h}$ and $D_{g,h}$ are constants depending upon ϵ and a/A; they are infinite series which, for small values of the arguments, converge rapidly.

The second equation of the set (27) may be integrated. The result is *

$$\frac{dw_2}{d\tau} + 2\rho_2 = c_1 - \sum_{g,h} \frac{D_{g,h}}{g + hm} \cos(g + hm)\tau, \tag{28}$$

where c_1 is the constant of integration.

On eliminating $dw_2/d\tau$ between the first of equations (27) and equation (28), there results an equation of the form

$$\frac{d^2\rho_2}{d\tau^2} + \rho_2 = \bar{A}_{0,0} + \bar{A}_{1,0}\cos\tau + \cdots + \bar{A}_{g,h}\cos(g + hm)\tau + \cdots;$$

whence †

$$\rho_2 = c_2\cos\tau + c_3\sin\tau + \bar{A}_{0,0} + \tfrac{1}{2}\bar{A}_{1,0}\tau\sin\tau + \cdots + \frac{A_{g,h}}{1-(g+hm)^2}\cos(g+hm)\tau\cdots,$$

where c_2 and c_3 are constants of integration.

Substituting the value of ρ_2 in equation (28) we get

$$\frac{dw_2}{d\tau} = \bar{D}_{0,0} - 2c_3\sin\tau - \bar{A}_{1,0}\tau\sin\tau + \cdots + \bar{D}_{g,h}\cos(g + hm)\tau + \cdots;$$

whence w_2 is obtained by quadrature. The constants of integration are determined by the initial conditions,

$$\rho_2 = \frac{d\rho_2}{d\tau} = w_2 = \frac{dw_2}{d\tau} = 0 \qquad (\tau = 0).$$

It is important to observe that all the terms of the solution of equations (27) have the period $2p\pi$, except the terms τ and $\tau\cos\tau$ which occur in w_2, and $\tau\sin\tau$ which occurs in ρ_2. Consequently when the conditions of periodicity (20) are applied, equations (a) and (d) will contain no term in μ^2 alone; (b) and (c) will each contain a term in μ^2 alone.

For the determination of the terms in $\mu^2\phi$ in the solution we have equations of the form

$$\begin{aligned} \frac{d^2\bar{\rho}}{d\tau^2} - 2\frac{d\bar{w}}{d\tau} - 3\bar{\rho} &= \sum_{g,h} G_{g,h}\sin(g + hm)\tau, \\ \frac{d^2\bar{w}}{d\tau^2} + 2\frac{d\bar{\rho}}{d\tau} &= H_{0,0} + \sum_{g,h} H_{g,h}\cos(g + hm)\tau. \end{aligned} \tag{29}$$

* By the definition of g, h, and m the denominator $g + hm$ cannot vanish.

† By the definition of g, h, and m the denominator $1 - (g + hm)^2$ cannot vanish.

The solution of the set (29) is of the form

$$\bar{\rho} = 2H_{0,0}\tau - \tfrac{1}{2}\bar{G}_{1,0}\tau\cos\tau + \cdots + \frac{\bar{G}_{g,h}}{1-(g+hm)^2}\sin(g+hm)\tau + \cdots,$$

$$\bar{w} = -\tfrac{3}{2}H_{0,0}\tau^2 + \bar{G}_{1,0}\tau\sin\tau + \cdots - \frac{\bar{H}_{g,h}}{g+hm}\cos(g+hm)\tau + \cdots.$$

The conditions that the solution of equations (26) under consideration shall be periodic with the period $2p\pi$ are

$$(30)\qquad \begin{aligned} 0 &= -3p\pi\alpha e\theta + (4H_{0,0}p\pi - \bar{G}_{1,0}p\pi)\mu^2\phi + \cdots, && (a)\\ 0 &= -3p\pi\alpha e + \bar{A}_{1,0}p\pi\mu^2 + \cdots, && (b)\\ 0 &= -3p\pi\alpha + 2p\pi\mu + \cdots, && (c)\\ 0 &= 6p\pi\alpha e\theta - (6H_{0,0}p\pi - 2\bar{G}_{1,0}p\pi)\mu^2\phi + \cdots. && (d) \end{aligned}$$

Equations (30) have the following properties:

i. There are no terms independent of α and μ.

ii. The first power of μ does not occur in (a), (b), and (d).

iii. There are no terms involving ϕ which are independent of μ^2.

iv. There are no terms involving θ which are independent of e.

v. Equations (a), (b), and (d) contain no term in α independent of e.

vi. In equations (a) and (d) there are no terms of degree less than the third.

The knowledge of the preceding properties leads to the conclusion that α, e, θ, and ϕ may be determined uniquely as power series in μ, vanishing with μ and satisfying equations (30). This conclusion is reached by the following steps which are valid for sufficiently small values of α, e, θ, ϕ, and μ.

(1) Let equation (c) be solved for α as a power series in e, θ, ϕ, and μ (which contains μ as a factor by i), giving

$$(31)\qquad \alpha = \tfrac{2}{3}\mu + \mu\mathfrak{P}_c(e, \theta, \phi, \mu).$$

Equation (31) may be used to eliminate α from equations (a), (b), and (d). Let the resulting equations be denoted by (a'), (b'), and (d') respectively.

(2) Then (b') contains a factor μ which may be divided out, leaving

$$0 = -2p\pi e + \bar{A}_{1,0}p\pi\mu + \cdots.$$

This equation contains no term independent of e and μ (by i, ii, and v), and contains a term of the first degree in e. Hence it may be solved for e as a power series in θ, ϕ, and μ, which vanishes with μ,

$$(32)\qquad e = \tfrac{1}{2}\bar{A}_{1,0}\mu + \mu\mathfrak{P}_b(\theta, \phi, \mu).$$

Equation (32) may be used to eliminate e from equations (a') and (d'). Let the resulting equations be denoted by (a'') and (d'') respectively.

(3) Each of equations (a'') and (d'') contains μ^2 as a factor (by i, ii, iii, iv, v, vi), which may be divided out, leaving no term independent of θ, ϕ, and μ:

$$(33)\qquad \begin{aligned} 0 &= -p\pi\bar{A}_{1,0}\theta + (4H_{0,0}p\pi - \bar{G}_{1,0}p\pi)\phi + \cdots, && (a'') \\ 0 &= 2p\pi\bar{A}_{1,0}\theta - (6H_{0,0}p\pi - 2G_{1,0}p\pi)\phi + \cdots. && (b'') \end{aligned}$$

The jacobian with respect to θ and ϕ for $\theta = \phi = \mu = 0$ is $-2p^2\pi^2\bar{A}_{1,0}H_{0,0}$, which does not vanish identically in ϵ and a/A. Hence equations (33) may be solved uniquely for θ and ϕ as power series in μ, and the result used to eliminate θ and ϕ from equations (31) and (32). This yields the final solution of of equations (30),

$$(34)\qquad \alpha = \mu\mathfrak{P}_1(\mu), \qquad e = \mu\mathfrak{P}_2(\mu), \qquad \theta = \mu\mathfrak{P}_3(\mu), \qquad \phi = \mu\mathfrak{P}_4(\mu).$$

When these values are substituted in the solution of equations (26), the expressions for ρ and w become power series in μ alone, and the solution is periodic with the period $2p\pi$, since the conditions (30) are satisfied. The physical interpretation is the same as in the preceding cases.

For the purpose of constructing the solutions in applications, it is convenient to have the existence proof in the form given. It leaves open, however, one question which should be answered. That is, are there any values of ϵ different from zero for which the jacobian of equations (33) vanishes? The jacobian is a power series in ϵ and a/A, vanishing with ϵ, and it is not easy to discuss other values of ϵ for which the series might vanish. Such a discussion is made unnecessary by a slight variation in the details of the existence proof. The proof depends only upon the properties of the solution of the problem of two bodies and the non-periodic terms which enter in ρ_2, w_2, $\bar{\rho}$, and $\bar{w}$, and may be carried through without considering in equations (27) and (29) the infinite series in ϵ which represents $1/R$ and the infinite series in a/A. This is accomplished by picking out from f_{III} in (26) a single term of the type

$$\epsilon^p\left(\frac{a}{A}\right)^{q-1}\cos pm\tau\cos(q-1)w$$

and from g_{III} a single term of the type

$$\epsilon^p\left(\frac{a}{A}\right)^{q-1}\cos pm\tau\sin(q-1)w.$$

These terms are left unchanged, while in all other terms ϵ is replaced by $\sigma\mu$ and a/A by $\eta\mu$. The set of differential equations thus obtained may be treated by steps similar to those employed in the consideration of equations (26), and a set

of equations corresponding to (33) will be obtained. The jacobian of this set is $\epsilon^{2p}(a/A)^{2q-2}$, except for a constant factor, and can vanish only if $\epsilon = 0$.

Construction of the Solutions.

It has been demonstrated for each of the three sets of differential equations (11), (19), and (26) that, for a given value of μ sufficiently small, it is possible to determine the initial conditions uniquely as power series in μ such that the solution as power series in μ shall be periodic in τ with the period $2p\pi$. A method will now be given by which the solution to any desired number of terms may be conveniently constructed. It is not necessary to determine the initial conditions explicitly in advance and the computation involves only algebraic processes. It will be convenient for brevity in the treatment to consider first

Case III. *When no uniform integral exists.* Consider equations (26), and let the periodic solution, which is known to exist, be written

$$\begin{aligned} \rho - 1 &= \rho_1\mu + \rho_2\mu^2 + \rho_3\mu^3 + \cdots + \rho_i\mu^i + \cdots, \\ w - \tau &= w_1\mu + w_2\mu^2 + w_3\mu^3 + \cdots + w_i\mu^i + \cdots. \end{aligned} \tag{35}$$

Since the solution exists and is periodic for all values of μ sufficiently small, each coefficient is periodic with the required period.

Let the solution (35) be substituted into the differential equations (26) and the result arranged as power series in μ. The terms of the left-hand members have the following forms. (The accents indicate derivatives with respect to τ.)

$$U = \tau + \bar{U} = \tau + \frac{2}{m}\epsilon \sin m\tau + \frac{5}{4m}\epsilon^2 \sin 2m\tau + \cdots,$$

$$\frac{dU}{d\tau} = 1 + \bar{U}', \qquad \frac{d^2U}{d\tau^2} = \bar{U}'',$$

$$\frac{d^2\rho}{d\tau^2} = \rho_1''\mu + \rho_2''\mu^2 + \rho_3''\mu^3 + \cdots + \rho_i''\mu^i + \cdots,$$

$$\begin{aligned} \rho\left(\frac{dw}{d\tau} + \mu\frac{dU}{d\tau}\right)^2 = 1 &+ [\rho_1 + 2(w_1' + 1 + \bar{U}')]\mu \\ &+ [\rho_2 + 2w_2' + 2\rho_1(w_1' + 1 + \bar{U}') + (w_1' + 1 + \bar{U}')^2]\mu^2 \\ &+ [\rho_3 + 2w_3' + 2(w_1' + 1 + \bar{U}')w_2' + 2\rho_1 w_2' + 2\rho_2(w_1' + 1 + \bar{U}') \\ &+ \rho_1(w_1 + 1 + \bar{U}')^2]\mu^3 + \cdots + [\rho_i + 2w_i' + 2(w_1' + 1 + \bar{U}')w_{i-1}' \\ &+ 2\rho_{i-1}(w_1' + 1 + \bar{U}') + 2\rho_1 w_{i-1}' + \cdots]\mu^i + \cdots, \end{aligned} \tag{36}$$

$$\begin{aligned} \frac{1}{\rho^2} = 1 - 2\rho_1\mu - (2\rho_2 - 3\rho_1^2)\mu^2 - (2\rho_3 - 6\rho_1\rho_2 + 4\rho_1^3)\mu^3 \\ + \cdots + (2\rho_i - 6\rho_{i-1}\rho_1 + \cdots)\mu^i + \cdots, \end{aligned}$$

$$\rho\left(\frac{d^2 w}{d\tau^2}+\mu\frac{d^2 U}{d\tau^2}\right)=(w_1''+\bar{U}'')\mu+[w_2''+\rho_1(w_1''+\bar{U}'')]\mu^2$$
$$+[w_3''+w_2''\rho_1+\rho_2(w_1''+\bar{U}'')]\mu^3+\cdots$$
$$+[w_i''+w_{i-1}''\rho_1+\cdots+\rho_{i-1}(w_1''+\bar{U}'')]\mu^i+\cdots,$$
$$\frac{d\rho}{d\tau}\left(\frac{dw}{d\tau}+\mu\frac{dU}{d\tau}\right)=\rho_1'\mu+[\rho_2'+\rho_1'(w_1'+1+\bar{U}')]\mu^2$$
$$+[\rho_3'+\rho_2'(w_1'+1+\bar{U}')+\rho_1'w_2']\mu^3+\cdots$$
$$+[\rho_i'+\rho_{i-1}'(w_1'+1+\bar{U}')+\cdots+\rho_1'w_{i-1}']\mu^i+\cdots.$$

The equations hold identically in μ. Hence, equating the coefficients of the first power of μ, we have for the determination of ρ_1 and w_1 the following set of equations:

$$\frac{d^2\rho_1}{d\tau^2}-2\frac{dw_1}{d\tau}-3\rho_1=2(1+\bar{U}'),\qquad \frac{d^2 w_1}{d\tau^2}+2\frac{d\rho_1}{d\tau^2}=-\bar{U}''. \tag{37}$$

It follows that

$$\begin{aligned}\rho_1&=2(1+c_1^{(1)})+c_2^{(1)}\cos\tau+c_3^{(1)}\sin\tau,\\ w_1&=c_4^{(1)}-(4+3c_1^{(1)})\tau-\bar{U}-2c_2^{(1)}\sin\tau+2c_3^{(1)}\cos\tau,\end{aligned} \tag{38}$$

where $c_1^{(1)}$, $c_2^{(1)}$, $c_3^{(1)}$, and $c_4^{(1)}$ are constants of integration. Since ρ_1 and w_1 are periodic, the coefficient of τ in w_1 must vanish. This condition determines the constant $c_1^{(1)}$, namely $c_1^{(1)}=-\frac{4}{3}$. The other constants of integration are so far arbitrary.

On equating the coefficients of the second power of μ, the following set of equations is obtained:

$$\begin{aligned}\frac{d^2\rho_2}{d\tau^2}-2\frac{dw_2}{d\tau}-3\rho_2&=(w_1'+1+\bar{U}')^2+2\rho_1(w_1'+1+\bar{U}')-3\rho_1^2+f_{\mathrm{III},0},\\ \frac{d^2 w_2}{d\tau^2}+2\frac{d\rho_2}{d\tau}&=-\rho_1(w_1''+\bar{U}'')-2\rho_1'(w_1'+1+\bar{U}')+g_{\mathrm{III},0},\end{aligned} \tag{39}$$

where $f_{\mathrm{III},0}$ and $g_{\mathrm{III},0}$ are obtained from f_{III} and g_{III} respectively by writing $\mu=0$, $w=\tau$, $\rho=1$. The right-hand members are known functions of τ and the equations have the form

$$\begin{aligned}\frac{d^2\rho_2}{d\tau^2}-2\frac{dw_2}{d\tau}-3\rho_2&=A_{0,0}^{(2)}+(\tfrac{14}{3}c_2^{(1)}+A_{1,0}^{(2)})\cos\tau+\tfrac{14}{3}c_3^{(1)}\sin\tau\\ &\qquad+\cdots+A_{g,h}^{(2)}\cos(g+hm)\tau+\cdots,\\ \frac{d^2 w_2}{d\tau^2}+2\frac{d\rho_2}{d\tau}&=(\tfrac{10}{3}c_2^{(1)}+D_{1,0}^{(2)})\sin\tau-\tfrac{10}{3}c_3^{(1)}\cos\tau\\ &\qquad+\cdots+D_{g,h}^{(2)}\sin(g+hm)\tau+\cdots.\end{aligned} \tag{40}$$

It is to be noted that $c_4^{(1)}$ does not occur in equations (40). Integrating the second equation we have

$$\frac{dw_2}{d\tau} + 2\rho_2 = c_1^{(2)} - (\tfrac{10}{3}c_2^{(1)} + D_{1,0}^{(2)})\cos\tau - \tfrac{10}{3}c_3^{(1)}\sin\tau + \cdots - \frac{D_{g,h}^{(2)}}{g+hm}\cos(g+hm)\tau + \cdots. \tag{41}$$

On eliminating $dw_2/d\tau$ from the first of equations (40) by means of equation (41), there results

$$\frac{d^2\rho_2}{d\tau^2} + \rho_2 = A_{0,0}^{(2)} + 2c_1^{(2)} + [-2c_2^{(1)} + A_{1,0}^{(2)} - 2D_{1,0}^{(2)}]\cos\tau - 2c_3^{(1)}\sin\tau + \cdots + \left(A_{g,h}^{(2)} - \frac{2D_{g,h}^{(2)}}{g+hm}\right)\cos(g+hm)\tau + \cdots. \tag{42}$$

In order that the solution of equation (42) shall contain no non-periodic term, the coefficients of $\cos\tau$ and $\sin\tau$ must vanish, whence $c_2^{(1)}$ and $c_3^{(1)}$ are determined by the conditions

$$2c_2^{(1)} = A_{1,0}^{(2)} - 2D_{1,0}^{(2)}, \qquad c_3^{(1)} = 0.$$

With these values of $c_2^{(1)}$ and $c_3^{(1)}$, the solution becomes

$$\rho_2 = A_{0,0}^{(2)} + 2c_1^{(2)} + c_2^{(2)}\cos\tau + c_3^{(2)}\sin\tau + \cdots + \alpha_{g,h}^{(2)}\cos(g+hm)\tau + \cdots,$$

where

$$\alpha_{g,h}^{(2)} = \frac{1}{1-(g+hm)^2}\left(A_{g,h}^{(2)} - \frac{2D_{g,h}^{(2)}}{g+hm}\right).$$

Substituting this value of ρ_2 in equation (41) and integrating, we obtain for w_2 a solution of the form

$$w_2 = c_4^{(2)} - (2A_{0,0}^{(2)} + 3c_1^{(2)})\tau - 2c_2^{(2)}\sin\tau + 2c_3^{(2)}\cos\tau + \cdots + \delta_{g,h}^{(2)}\sin(g+hm)\tau + \cdots,$$

where

$$\delta_{g,h}^{(2)} = -\frac{1}{g+hm}\left(\frac{D_{g,h}^{(2)}}{g+hm} + 2\alpha_{g,h}^{(2)}\right).$$

Since w_2 is periodic, $c_1^{(2)}$ is determined by the condition

$$2A_{0,0}^{(2)} + 3c_1^{(2)} = 0.$$

Of the eight constants of integration, which have been introduced in the first two steps, four ($c_1^{(1)}$, $c_2^{(1)}$, $c_3^{(1)}$, $c_1^{(2)}$) have been determined uniquely and four ($c_4^{(1)}$, $c_2^{(2)}$, $c_3^{(2)}$, $c_4^{(2)}$) are still arbitrary.

By equating the coefficients of the third power of μ the following set of equations is obtained:

$$\begin{aligned} \frac{d^2\rho_3}{d\tau^2} - 2\frac{dw_3}{d\tau} - 3\rho_3 &= 2(w_1' + 1 + \bar{U}' + \rho_1)w_2' + 2\rho_2(w_1' + 1 + \bar{U}') \\ &\quad + \rho_1(w_1' + 1 + \bar{U}')^2 - 6\rho_1\rho_2 + 4\rho_1^3 + f_{\mathrm{III},1}, \\ \frac{d^2w_3}{d\tau^2} + 2\frac{d\rho_3}{d\tau} &= -\rho_1 w_2'' - (w_1'' + \bar{U}'')\rho_2 - 2\rho_2'(w_1' + 1 + \bar{U}') - 2\rho_1' w_2' + g_{\mathrm{III},1}, \end{aligned} \tag{43}$$

where $f_{\mathrm{III},1}$ and $g_{\mathrm{III},1}$ denote the coefficients of μ in f_{III} and g_{III} respectively. Equations (43) do not contain $c_4^{(2)}$. The right-hand members contain $c_4^{(1)}$ linearly. The equations have the form

$$
(44)\quad
\begin{aligned}
\frac{d^2\rho_3}{d\tau^2} - 2\frac{dw_3}{d\tau} - 3\rho_3 &= A^{(3)}_{0,0} + (\tfrac{14}{3}c_2^{(2)} + A^{(3)}_{1,0})\cos\tau + (\tfrac{14}{3}c_3^{(2)} + B^{(3)}_{1,0})\sin\tau \\
&\quad + \cdots + A^{(3)}_{g,h}\cos(g+hm)\tau + B^{(3)}_{g,h}\sin(g+hm)\tau + \cdots, \\
\frac{d^2w_3}{d\tau^2} + 2\frac{d\rho_3}{d\tau} &= C^{(3)}_{0,0} + (\tfrac{10}{3}c_2^{(2)} + D^{(3)}_{1,0})\sin\tau + (-\tfrac{10}{3}c_3^{(2)} + C^{(3)}_{1,0})\cos\tau \\
&\quad + \cdots D^{(3)}_{g,h}\sin(g+hm)\tau + C^{(3)}_{g,h}\cos(g+hm)\tau + \cdots.
\end{aligned}
$$

Here $C^{(3)}_{0,0}$ involves $c_4^{(1)}$ linearly. To avoid non-periodic terms in the solution, $C^{(3)}_{0,0}$ must vanish, and this condition determines $c_4^{(1)}$ uniquely, if the coefficient of $c_4^{(1)}$ does not vanish. The coefficient is a power series in ϵ and a/A which vanishes with ϵ but not identically in ϵ. The question concerning special values of ϵ for which it might vanish is the same as the question concerning the vanishing of the jacobian of equations (33) and is answered in the same way.

The treatment of equations (44) proceeds by steps similar to those employed in the solution of equations (40). Four new constants of integration are introduced, namely, $c_1^{(3)}$, $c_2^{(3)}$, $c_3^{(3)}$, $c_4^{(3)}$; while $c_2^{(2)}$, $c_3^{(2)}$, and $c_1^{(3)}$ are uniquely determined by the conditions:

$$
2c_2^{(2)} = A^{(3)}_{1,0} - 2D^{(3)}_{1,0}, \qquad 2c_3^{(2)} = B^{(3)}_{1,0} + 2C^{(3)}_{1,0}, \qquad 3c_1^{(3)} = -2A^{(3)}_{0,0}.
$$

So far $c_4^{(2)}$, $c_2^{(3)}$, $c_3^{(3)}$, and $c_4^{(3)}$ are arbitrary.

It may be established by complete induction that the above process can be carried as far as is desired. Suppose $\rho_1, w_1;\ \rho_2, w_2;\ \cdots \rho_{i-1}, w_{i-1}$ have been determined by this process. The expressions have the following form

$$
\begin{aligned}
\rho_l &= \alpha^{(l)}_{0,0} + \cdots + \alpha^{(l)}_{g,h}\cos(g+hm)\tau + \beta^{(l)}_{g,h}\sin(g+hm)\tau + \cdots, \\
w_l &= \gamma^{(l)}_{0,0} + \cdots + \delta^{(l)}_{g,h}\sin(g+hm)\tau + \gamma^{(l)}_{g,h}\cos(g+hm)\tau + \cdots,
\end{aligned}
\qquad (l = 1, 2, \cdots, i-3).
$$

$$
\begin{aligned}
\rho_{i-2} &= \alpha^{(i-2)}_{0,0} + \cdots + \alpha^{(i-2)}_{g,h}\cos(g+hm)\tau + \beta^{(i-2)}_{g,h}\sin(g+hm)\tau + \cdots, \\
w_{i-2} &= c_4^{(i-2)} + \cdots + \delta^{(i-2)}_{g,h}\sin(g+hm)\tau + \gamma^{(i-2)}_{g,h}\cos(g+hm)\tau + \cdots.
\end{aligned}
$$

$$
\begin{aligned}
\rho_{i-1} &= \alpha^{(i-1)}_{0,0} + c_2^{(i-1)}\cos\tau + c_3^{(i-1)}\sin\tau + \cdots, \\
w_{i-1} &= c_4^{(i-1)} - 2c_2^{(i-1)}\sin\tau + 2c_3^{(i-1)}\cos\tau + \cdots.
\end{aligned}
$$

The equations for the determination of ρ_i and w_i are

$$
(45)\quad
\begin{aligned}
\frac{d^2\rho_i}{d\tau^2} - 2\frac{dw_i}{d\tau} - 3\rho_i &= A^{(i)}_{0,0} + (\tfrac{14}{3}c_2^{(i-1)} + A^{(i)}_{1,0})\cos\tau + (\tfrac{14}{3}c_3^{(i-1)} + B^{(i)}_{1,0})\sin\tau \\
&\quad + \cdots + A^{(i)}_{g,h}\cos(g+hm)\tau + B^{(i)}_{g,h}\sin(g+hm)\tau + \cdots, \\
\frac{d^2w_i}{d\tau^2} + 2\frac{d\rho_i}{d\tau} &= C^{(i)}_{0,0} + (\tfrac{10}{3}c_2^{(i-1)} + D^{(i)}_{1,0})\sin\tau + (-\tfrac{10}{3}c_3^{(i-1)} + C^{(i)}_{1,0})\cos\tau \\
&\quad + \cdots + D^{(i)}_{g,h}\sin(g+hm)\tau + C^{(i)}_{g,h}\cos(g+hm)\sin\tau + \cdots.
\end{aligned}
$$

Equations (45) do not contain $c_4^{(i-1)}$. The coefficient of $c_4^{(i-2)}$ in $C_{0,0}^{(i)}$ is the same as the coefficient of $c_4^{(1)}$ in $C_{0,0}^{(3)}$. Equations (45) are solved by the steps employed for the solution of equations (40). During the process four constants of integration are introduced, namely, $c_1^{(i)}, c_2^{(i)}, c_3^{(i)}, c_4^{(i)}$, and four are uniquely determined by the following conditions, the first of which determines $c_4^{(i-2)}$;

$$
(46)\qquad \begin{aligned} &C_{0,0}^{(i)} = 0, && 2c_2^{(i-1)} = A_{1,0}^{(i)} - 2D_{1,0}^{(i)}, \\ &2c_3^{(i-1)} = B_{1,0}^{(i)} + 2C_{1,0}^{(i)}, && 3c_1^{(i)} = -2A_{0,0}^{(i)}. \end{aligned}
$$

The solution of equations (45) is

$$
(47)\qquad \begin{aligned} \rho_i &= \alpha_{0,0}^{(i)} + c_2^{(i)}\cos\tau + c_3^{(i)}\sin\tau + \cdots \\ &\qquad + \alpha_{g,h}^{(i)}\cos(g+hm)\tau + \beta_{g,h}^{(i)}\sin(g+hm)\tau + \cdots, \\ w_i &= c_4^{(i)} + \delta_{1,0}^{(i)}\sin\tau + \gamma_{1,0}^{(i)}\cos\tau + \cdots \\ &\qquad + \delta_{g,h}^{(i)}\sin(g+hm)\tau + \gamma_{g,h}^{(i)}\cos(g+hm)\tau + \cdots, \end{aligned}
$$

where the coefficients are given by the formulas:

$$
(48)\qquad \begin{aligned} \alpha_{0,0}^{(i)} &= -\tfrac{1}{3}A_{0,0}^{(i)}, \\ \alpha_{g,h}^{(i)} &= \frac{1}{1-(g+hm)^2}\left(A_{g,h}^{(i)} - \frac{2D_{g,h}^{(i)}}{g+hm}\right), \\ \beta_{g,h}^{(i)} &= \frac{1}{1-(g+hm)^2}\left(B_{g,h}^{(i)} + \frac{2C_{g,h}^{(i)}}{g+hm}\right), \\ \delta_{1,0}^{(i)} &= -2c_2^{(i)} - \tfrac{5}{3}A_{1,0}^{(i)} + \tfrac{7}{3}D_{1,0}^{(i)}, \\ \delta_{g,h}^{(i)} &= -\frac{1}{g+hm}\left(\frac{D_{g,h}^{(i)}}{g+hm} + 2\alpha_{g,h}^{(i)}\right), \\ \gamma_{1,0}^{(i)} &= 2c_3^{(i)} + \tfrac{5}{3}B_{1,0}^{(i)} + \tfrac{7}{3}C_{1,0}^{(i)}, \\ \gamma_{g,h}^{(i)} &= -\frac{1}{g+hm}\left(\frac{C_{g,h}^{(i)}}{g+hm} - 2\beta_{g,h}^{(i)}\right). \end{aligned}
$$

This completes the proof of the statement that the solution may be constructed to any desired degree of accuracy. Of the four constants of integration which occur in the calculation of the coefficients of μ^i, one, $c_1^{(i)}$, is determined in the step in which it enters; two more, $c_2^{(i)}$ and $c_3^{(i)}$, are determined in the next following step; while the fourth, $c_4^{(i)}$, is determined in the second following step.

Case II. *When a uniform integral exists.* The method of constructing the solution in this case differs from the preceding in one particular. The general steps are the same and formulas (48) are applicable. The difference occurs in the determination of the constant $c_4^{(i-2)}$. In the proof of the existence of a

periodic solution of equations (19) it was shown that the origin of time is arbitrary, which is equivalent to the statement that at $\tau = \tau_0$ the value $w = w_0$ may be assumed arbitrarily. It follows that $C^{(i)}_{0,0} = 0$ and does not determine $c_4^{(i-2)}$. This constant is determined by assuming an initial condition, for example, $w = 0$ at $\tau = 0$. Then $c_4^{(i-2)}$ is determined in the step in which it enters by the condition $w_{i-2} = 0$ at $\tau = 0$. When the step for computing the coefficients of μ^i is completed, there remain two constants of integration, $c_2^{(i)}$ and $c_3^{(i)}$ which have not been determined. These constants are determined in the next following step.

Case I. *Symmetrical solutions.* In this case there are two arbitrary initial conditions and the method of constructing the solution differs from that of case III in two particulars. The differential equations are (11). Consider the first step in the construction [equations (37) and (38)]. We determine $c_3^{(1)}$ and $c_4^{(1)}$ respectively by the conditions

$$\frac{d\rho_1}{d\tau} = 0, \qquad w_1 = 0 \qquad (\tau = 0).$$

Therefore $c_3^{(1)} = c_4^{(1)} = 0$. And $c_1^{(1)}$ is determined by the condition that w_1 shall be periodic while $c_2^{(1)}$ is not determined until the next step.

This process is applicable to all the following steps. The differential equations (11) have a particular form which admits a symmetrical solution, and it may be established by complete induction that the equations for the determination of ρ_i and w_i have the form

$$\begin{aligned} \frac{d^2\rho_i}{d\tau^2} - 2\frac{dw_i}{d\tau} - 3\rho_i &= A^{(i)}_{0,0} + (\tfrac{14}{3}c_2^{(i-1)} + A^{(i)}_{1,0})\cos\tau + \cdots + A^{(i)}_{g,h}\cos(g+hm)\tau + \cdots, \\ \frac{d^2 w_i}{d\tau^2} + 2\frac{d\rho_i}{d\tau} &= (\tfrac{10}{3}c_2^{(i-1)} + D^{(i)}_{1,0})\sin\tau + \cdots + D^{(i)}_{g,h}\sin(g+hm)\tau + \cdots. \end{aligned} \tag{49}$$

where $c_2^{(i-1)}$ is determined by the condition

$$2c_2^{(i-1)} = A^{(i)}_{1,0} - 2D^{(i)}_{1,0}.$$

The solution of (49) is

$$\rho_i = \alpha^{(i)}_{0,0} + c_1^{(i)}\cos\tau + \cdots + \alpha^{(i)}_{g,h}\cos(g+hm)\tau + \cdots,$$

$$w_i = \delta^{(i)}_{1,0}\sin\tau + \cdots + \delta^{(i)}_{g,h}\sin(g+hm)\tau + \cdots,$$

where

$$\alpha^{(i)}_{0,0} = -\tfrac{1}{3}A^{(i)}_{0,0},$$

$$\alpha^{(i)}_{g,h} = \frac{1}{1-(g+hm)^2}\left(A^{(i)}_{g,h} - \frac{2D^{(i)}_{g,h}}{g+hm}\right),$$

$$\delta^{(i)}_{1,0} = -2c_2^{(i)} - \tfrac{5}{3}A^{(i)}_{1,0} + \tfrac{7}{3}D^{(i)}_{1,0},$$

$$\delta^{(i)}_{g,h} = -\frac{1}{g+hm}\left(\frac{D^{(i)}_{g,h}}{g+hm} + 2\alpha^{(i)}_{g,h}\right).$$

Numerical Examples.

As illustrations of the preceding analysis three numerical examples will be treated briefly. It has not been shown that the processes are valid for the numerical values which are employed and which have been selected for convenience in graphical representation. It is probable that the series are convergent, although it has not been found possible to determine the true radii of convergence.

In each case the mass of M is taken as the unit of mass, and M_1, of mass 10, is supposed to move about M in a circle whose radius is taken as the unit of distance. The unit of time is chosen so that the angular velocity N of M_1 is unity. The orbits considered are those in which the particle revolves about M. The period is chosen so that $\nu = 5$, whence $\mu = 0.2$.

In the first example M_2 of mass 5 is supposed to revolve about M in a circle so that the three finite masses remain always in a straight line in the order $M_2 M M_1$. It follows from the solution of the quintic equation of LAGRANGE that the distance $M_2 M$ is $0.77172\cdots$. The orbit of the particle is symmetrical with respect to the line joining the bodies.

In the second example M_2 of mass 5 is supposed to revolve about M in a circle so that the three bodies remain always at the vertices of an equilateral triangle. The differential equations admit the integral of JACOBI, but there is no symmetry theorem.

In the third case it is supposed that M_2 of unit mass revolves about M_1 in a circle of radius A_2. The orbit of the particle is symmetrical with respect to the radius vector of M_1 and crosses this line when the finite bodies are in conjunction.

The differential equations of motion. With reference to M as origin and an axis passing always through M_1 let the coördinates of M_1, M_2, and P be respectively $(1, 0)$, (R_2, W_2), and (r, w). The equations of motion, see equations (7), of the particle are

$$\frac{d^2\rho}{d\tau^2} - \rho\left(\frac{dw}{d\tau} + \mu\right)^2 + \frac{1}{\rho^2} = k^2 M_1 \mu^2 \rho\,[\tfrac{1}{2}\{1 + 3\cos 2w\} + \tfrac{3}{8}a\rho\{3\cos w + 5\cos 3w\}$$
$$+ \tfrac{1}{16}a^2\rho^2\{9 + 20\cos 2w + 35\cos 4w\} + \cdots]$$
$$+ \frac{k^2 M_2}{R_2^3}\mu^2\rho\Big[\tfrac{1}{2}\{1 + 3\cos 2(w - W_2)\} + \tfrac{3}{8}\frac{a}{R^2}\rho\{3\cos(w - W_2) + 5\cos 3(w - W_2)\}$$
$$+ \tfrac{1}{16}\left(\frac{a}{R_2}\right)^2\rho^2\{9 + 20\cos 2(w - W_2) + 35\cos 4(w - W_2)\} + \cdots\Big],$$

$$\rho\frac{d^2 w}{d\tau^2} + 2\frac{d\rho}{d\tau}\left(\frac{dw}{d\tau} + \mu\right) = -k^2 M_1 \mu^2 \rho\,[\tfrac{3}{2}\sin 2w + \tfrac{3}{8}a\rho\{\sin w + 5\sin 3w\}$$
$$+ \tfrac{5}{16}a^2\rho^2\{2\sin 2w + 7\sin 4w\} + \cdots]$$
$$- \frac{k^2 M_2}{R_2^3}\mu^2\rho\Big[\tfrac{3}{2}\sin 2(w - W_2) + \tfrac{3}{8}\frac{a}{R_2}\rho\{\sin(w - W_2) + 5\sin 3(w - W_2)\}$$
$$+ \tfrac{5}{16}\left(\frac{a}{R_2}\right)^2\rho^2\{2\sin 2(w - W_2) + 7\sin 4(w - W_2)\} + \cdots\Big],$$

where a is given by the relation $\nu^2 a^3 = k^2$. The numerical value of k^2 depends upon the masses and orbits of the finite bodies and must be determined separately for each of the three examples.

Example 1. In this case $R_2 = 0.77172$ and $W_2 = \pi$. Now k^2 is given by the relation

$$N^2 = \frac{M_1 + M + M_2}{M + M_2 + M_2 R_2}\left(M + \frac{M_2}{(1 + R_2^2)}\right)k^2,$$

whence

$$k^2 = 0.23763.$$

It follows that

$$k^2 M_1 = 2.37630, \qquad \frac{k^2 M_2}{R_2^3} = 2.58518,$$

$$a = 1.05914\mu, \qquad \frac{a}{R_2} = 1.37222\mu.$$

The equations of motion become

$$\begin{aligned}
&\frac{d^2\rho}{d\tau^2} - \rho\left(\frac{dw}{d\tau} + \mu\right)^2 + \frac{1}{\rho^2} = (2.48074 + 7.44222\cos 2w)\rho\mu^2 \\
&\qquad - (1.15938\cos w + 1.93230\cos 3w)\rho^2\mu^3 \\
&\qquad + (4.23765 + 9.41700\cos 2w + 16.47975\cos 4w)\rho^3\mu^4 + \cdots, \\
(50)\quad &\rho\frac{d^2 w}{d\tau^2} + 2\frac{d\rho}{d\tau}\left(\frac{dw}{d\tau} + \mu\right) = -(7.44222\sin 2w)\rho\mu^2 \\
&\qquad + (0.38646\sin w + 1.93230\sin 3w)\rho^2\mu^3 \\
&\qquad - (4.70850\sin 2w + 16.47975\sin 4w)\rho^3\mu^4 + \cdots.
\end{aligned}$$

The periodic solution of equations (50) is

$$\begin{aligned}
&\rho = 1 - \tfrac{2}{3}\mu - (0.27136 + 0.96615\cos\tau + 4.96148\cos 2\tau)\mu^2 \\
&\qquad + (0.62584 - 19.13740\cos\tau - 2.47963\cos 2\tau + 0.40256\cos 3\tau)\mu^3 + \cdots, \\
(51)\quad &w = \tau + (1.93230\sin\tau + 6.82204\sin 2\tau)\mu^2 \\
&\qquad + (41.10885\sin\tau + 10.74793\sin 2\tau - 0.48307\sin 3\tau)\mu^3 + \cdots.
\end{aligned}$$

Example 2. In this case $R_2 = 1$ and $W_2 = \pi/3$. Here k^2 is given by the relation

$$N^2 = k^2(M + M_1 + M_2),$$

whence

$$k^2 = 0.06250.$$

It follows that

$$k^2 M_1 = 0.62500, \qquad k^2 M_2 = 1.56250\mu, \qquad a = 0.67860\mu.$$

The equations of motion become

$$
\begin{aligned}
\frac{d^2\rho}{d\tau^2} - \rho\left(\frac{dw}{d\tau} + \mu\right)^2 + \frac{1}{\rho^2} &= (0.31250 + 0.93750 \cos 2w)\rho\mu^2 \\
&\quad + (0.78125 - 1.17188 \cos 2w + 2.02977 \sin 2w)\rho\mu^3 \\
&\quad + (0.47714 \cos w + 0.79523 \cos 3w)\rho^2\mu^3 \\
&\quad + (0.59643 \cos w + 1.03308 \sin w - 1.98808 \cos 3w)\rho^2\mu^4 \\
&\quad + (0.16188 + 0.35974 \cos 2w + 0.62954 \cos 4w)\rho^3\mu^4 + \cdots, \\
\rho\frac{d^2 w}{d\tau^2} + 2\frac{d\rho}{d\tau}\left(\frac{dw}{d\tau} + \mu\right) &= -(0.93750 \sin 2w)\rho\mu^2 \\
&\quad + (1.17188 \sin 2w + 2.02977 \cos 2w)\rho\mu^3 \\
&\quad - (0.15905 \sin w + 0.79523 \sin 3w)\rho^2\mu^3 \\
&\quad + (-0.19881 \sin w + 0.34436 \cos w + 1.98808 \sin 3w)\rho^2\mu^4 \\
&\quad - (0.17987 \sin 2w + 0.62954 \sin 4w)\rho^3\mu^4 + \cdots.
\end{aligned}
\tag{52}
$$

The periodic solution of equations (52) is

$$
\begin{aligned}
\rho &= 1 - \tfrac{2}{3}\mu + (0.45139 + 0.39762 \cos \tau - 0.62500 \cos 2\tau)\mu^2 \\
&\quad + (-0.47647 + 1.04542 \cos \tau + 0.86090 \sin \tau + 0.46875 \cos 2\tau \\
&\quad - 1.35318 \sin 2\tau - 0.16567 \cos 3\tau)\mu^3 + \cdots, \\
w &= \tau + (-0.79524 \sin \tau + 0.85938 \sin 2\tau)\mu^2 \\
&\quad + (0.13882 - 3.25720 \sin \tau + 1.72180 \cos \tau + 0.27995 \sin 2\tau \\
&\quad - 1.86062 \cos 2\tau + 0.19881 \sin 3\tau)\mu^3 + \cdots.
\end{aligned}
\tag{53}
$$

Example 3. In this case M_2 revolves about M_1 in a circle of radius A_2 with an angular velocity N_2. Let N_2 be determined by the relation

$$N_2 - N = 2\nu.$$

A_2, R_2, and W_2 are determined by the relations

$$N_2^2 A_2^3 = k^2(M_1 + M_2),$$

$$R_2 = \sqrt{1 + A_2^2 + 2A_2 \cos 2\nu t} = \sqrt{1 + A_2^2 + 2A_2 \cos 2\tau},$$

$$\sin W_2 = \frac{A_2 \sin 2\tau}{R_2},$$

$$\cos W_2 = \frac{R_2^2 + 1 - A_2^2}{2R_2}.$$

Here k^2 is given by the relation

$$N^2 = k^2(M + M_1),$$

whence

$$k^2 = 0.09091.$$

It follows that

$$k^2 M_1 = 0.90909, \qquad k^2 M_2 = 0.09091,$$

$$A_2 = 1.01200\mu, \qquad a = 0.76910\mu.$$

The differential equations become

$$\begin{aligned}
&\frac{d^2\rho}{d\tau^2} - \rho\left(\frac{dw}{d\tau} + \mu\right)^2 + \frac{1}{\rho^2} = (0.50000 + 1.50000\cos 2w)\rho\mu^2 \\
&\quad + (0.86523\cos w + 1.44205\cos 3w)\rho^2\mu^3 \\
&\quad + (0.33273 + 0.73940\cos 2w + 1.29395\cos 4w)\rho^3\mu^4 \\
&\quad + (0.27600\sin 2\tau\sin 2w - 0.13800\cos 2\tau - 0.41400\cos 2\tau\cos 2w)\rho\mu^3 \\
&\quad + (0.10474 + 0.17456\cos 4\tau + 0.31422\cos 2w + 0.10471\cos 4\tau\cos 2w \\
&\qquad\qquad - 0.55864\sin 4\tau\sin 2w)\rho\mu^4 \\
&\quad + (0.07960\sin 2\tau\sin w - 0.31840\cos 2\tau\cos w - 0.53068\cos 2\tau\cos 3w \\
&\qquad\qquad + 0.39801\sin 2\tau\sin 3w)\rho^2\mu^4 + \cdots, \\
&\rho\frac{d^2w}{d\tau^2} + 2\frac{d\rho}{d\tau}\left(\frac{dw}{d\tau} + \mu\right) = -(1.50000\sin 2w)\rho\mu^2 \\
&\qquad - (0.28841\sin w + 1.44205\sin 3w)\rho^2\mu^3 \\
&\qquad - (0.36970\sin 2w + 1.29395\sin 4w)\rho^3\mu^4 \\
&\quad + (0.27600\sin 2\tau\cos 2w + 0.41400\cos 2\tau\sin 2w)\rho\mu^3 \\
&\quad - (0.31422\sin 2w + 0.10471\cos 4\tau\sin 2w + 0.55864\sin 4\tau\cos 2w)\rho\mu^4 \\
&\quad + (0.02653\sin 2\tau\cos w + 0.10613\cos 2\tau\sin w + 0.39801\sin 2\tau\cos 3w \\
&\qquad\qquad + 0.53068\cos 2\tau\sin 3w)\rho^2\mu^4 + \cdots.
\end{aligned} \tag{54}$$

The periodic solution of equations (54) is

$$\begin{aligned}
\rho &= 1 - \tfrac{2}{3}\mu + (0.38889 + 0.72102\cos\tau - \cos 2\tau)\mu^2 \\
&\quad + (-0.02616 + 2.09168\cos\tau - 0.45400\cos 2\tau - 0.30043\cos 3\tau \\
&\qquad + 0.03450\cos 4\tau)\mu^3 + \cdots, \\
w &= \tau + (-1.44204\sin\tau + 1.37500\sin 2\tau)\mu^2 \\
&\quad + (-6.29838\sin\tau + 2.12066\sin 2\tau + 0.36051\sin 3\tau \\
&\qquad - 0.03881\sin 4\tau)\mu^3 + \cdots.
\end{aligned} \tag{55}$$

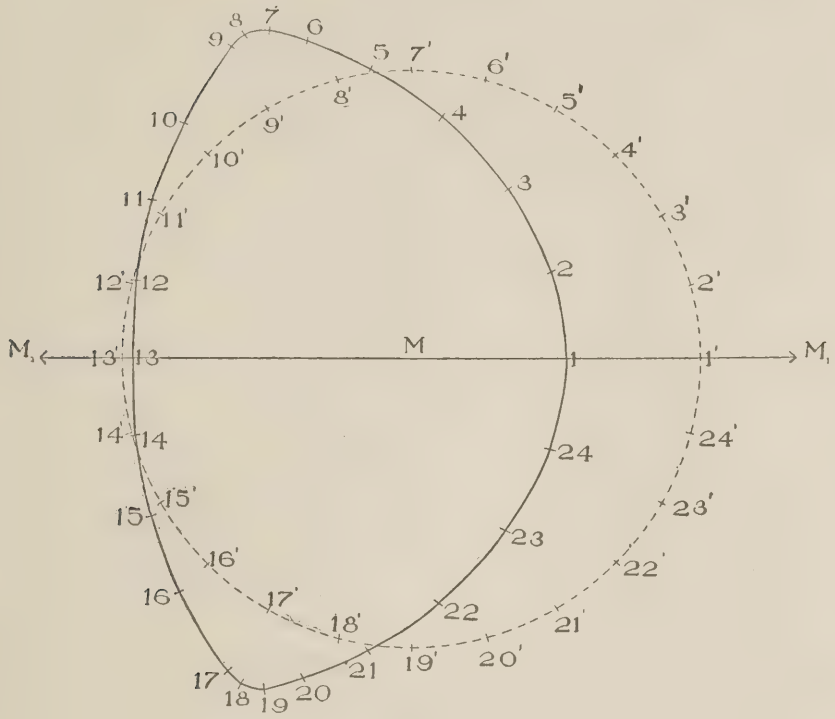

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Fig. 2.

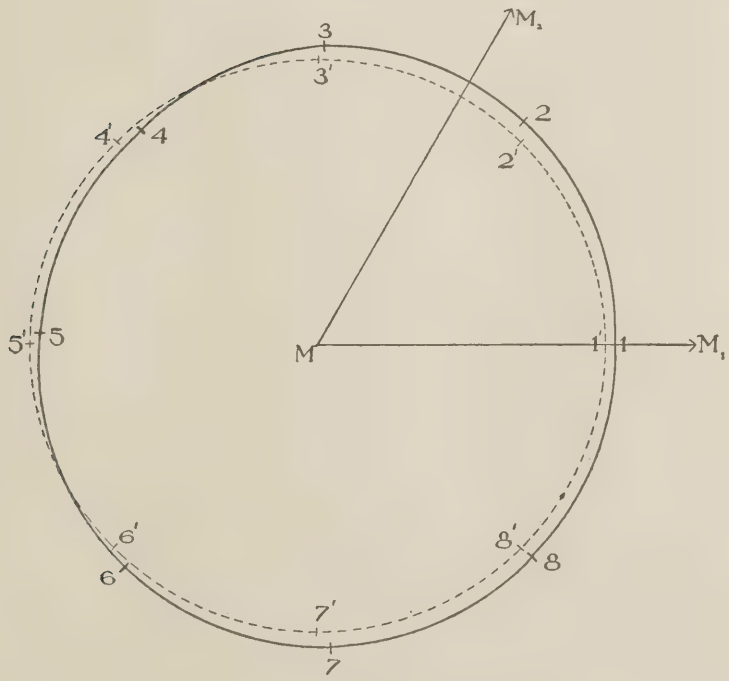
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Fig. 3.

The orbits represented by the solutions (51), (53), and (55) are illustrated by figures 2, 3, and 4 respectively. The comparison circles in the figures are not the circular orbits which have been called the *undisturbed orbits*. The undisturbed orbits are referred to fixed axes while the drawings are made with reference to

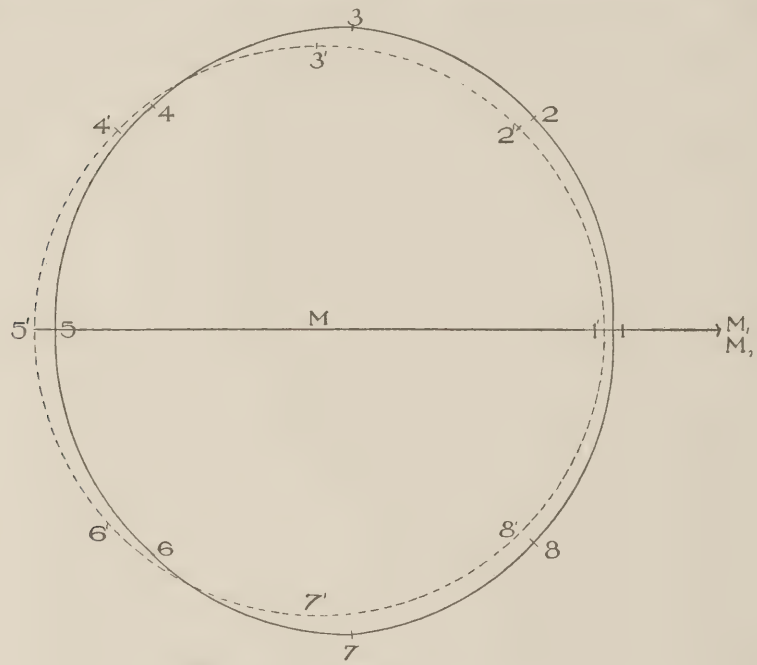

FIG. 4.

rotating axes. The comparison circles represent orbits in which the particle would make a complete revolution with respect to the rotating axes during the period. In figure 2 the points which are numbered 1, 2, $\cdots$ represent the positions of the particle in the periodic orbit at intervals in τ of $\pi/12$. The corresponding positions in the comparison circle are indicated by the numbers $1'$, $2'$, $\cdots$. In figures 3 and 4 the intervals are $\pi/4$.

A PROOF OF SOME THEOREMS ON POINTWISE DISCONTINUOUS FUNCTIONS*

BY

EDWARD B. VAN VLECK

BAIRE in his important thesis † has given a number of interesting theorems concerning pointwise discontinuous functions. For their demonstration he employs the concept of semi- (i. e., upper or lower) continuity. ‡ However necessary the concept may be for subsequent portions of his investigation, its introduction is not needed for the particular theorems referred to. When the unnecessary element is removed, the principles of their demonstration, although very different in form, become equivalent to those used by OSGOOD § in the derivation of his theorem relating to the convergence of series of continuous functions. (Cf. § 5 below.)

The object of the following paper is, in part, to establish the theorems without going so far as to introduce the notion of upper continuity, thus reducing the proof to somewhat lower terms. It is found that BAIRE has restricted $f(x, y)$ more than is necessary in the following theorem:

If $f(x, y)$ is continuous in x and y separately and has a partial derivative $\partial f/\partial x$ at every point of

$$T: \qquad a \leqq x \leqq b, \qquad c \leqq y \leqq d,$$

this derivative, regarded as a function of (x, y), is at most only pointwise discontinuous.

In fact, it suffices here to add to the hypothesis of the existence of $\partial f/\partial x$ the condition that $f(x, y)$ shall be continuous in y upon a set of parallels to the Y-axis which are everywhere dense in T (see theorem II). Other extensions of BAIRE's results will be noted further on (cf. theorems III and V).

It is believed that in the analysis given here the necessary principles have been arranged and combined in so simple a form as to permit quick and easy

* Presented to the Society October 29, 1904, with a supplementary note April 29, 1905. Received for publication January 16, 1907.

† Annali di Matematica, ser. 3, vol. 3 (1899).

‡ Loc. cit., p. 6, or see SCHÖNFLIES' admirable *Bericht über die Mengenlehre*, Jahresbericht der deutschen Mathematiker-Vereinigung, vol. 8 (1900), p. 141. Reference to this report will be indicated hereafter by the word *Bericht*.

§ American Journal of Mathematics, vol. 19 (1897), p. 155.

application to other problems of like character which may arise. As instances of such application I will cite the two following theorems, which I think are new and which constitute a second object of this paper:

THEOREM VI.* *If throughout the field*

$$T: \qquad a \leqq x \leqq b, \qquad c \leqq y \leqq d,$$

the function $f(x, y)$ is continuous in x when y is constant and in y when x is constant, then

$$F(x) \equiv \int_c^d f(x, y)\, dy$$

is at most a pointwise discontinuous function in the interval $a \leqq x \leqq b$.

THEOREM VII. *If at every point of T the integrand possesses a partial derivative $f'_x(x, y)$ which is continuous in the variables x and y considered separately, the integral $F(x)$ will have a derivative in some set of points which is everywhere dense in (a, b), and the value of the derivative at the points here found will be obtained by differentiating with respect to the parameter under the integral sign; i. e.,*

$$F'(x) = \int_c^d f'_x(x, y)\, dy.$$

Furthermore, these points are points of continuity for

$$\int_c^d f'_x(x, y)\, dy.$$

Concerning the former of the two theorems it may be remarked that it has been known previously † that when $|f(x, y)|$ has an upper limit in T the integral $F(x)$ is continuous in (ab). The cases of pointwise discontinuity therefore arise only when there is no upper limit for $|f(x, y)|$.

§ 1. *The preliminary principles.*

We shall first demonstrate the following very simple

Fundamental Principle: ‡ *If to every point p_i of a continuum (or of a perfect set of points)* § *a positive number δ_i is assigned in any manner, then there is some portion of the continuum (some perfect subset) where the points q_i*

* The two theorems are numbered with reference to the place in which they are proved later on.

† ARZELA, Atti della R. Accademia dei Lincei, ser. 4, vol. 1 (1884–5), p. 537. It also can be deduced immediately from OSGOOD's theorem 5, loc. cit., p. 182.

‡ A special case in which this principle is used incidentally will be found in the work of OSGOOD, Mathematische Annalen, vol. 53 (1900), p. 463. Other instances of the use of the principle can doubtless be found.

§ No restriction is here placed upon the perfect point-set. It may consist of a set of points nowhere dense in the space in which it is contained. The continuum may be either closed or open.

which have $\delta_i > D$ are everywhere dense (dense with respect to the subset), provided D is taken sufficiently small.*

Let us suppose, if possible, that this is not true. Denote by $\zeta_1, \zeta_2, \zeta_3, \cdots$ any sequence of decreasing positive numbers having 0 as their limit. Further let Q_i denote the set of points q_i for which $\delta_i > \zeta_i$. Clearly Q_i will be contained in Q_{i+1}. If, now, the principle stated is false, then there is some portion of the continuum (some perfect subset) which contains no points of Q_1. Within this portion (subset) there must be another part (perfect set) which is free from the points of Q_2, and so on. The limit of this series of sub-portions (sub-sets), contained each within the preceding, is either a point † to which belongs a non-negative number smaller than any ζ_i, or a set of such points. But 0 is the only number having this property. This, however, contradicts the hypothesis that to every point there corresponds a *positive* number. The fundamental principle is therefore established.

By placing $\delta_i = 1/\delta_i'$ and $D = 1/D'$ the principle may be put also into the following form: If to every point of the continuum a positive number δ_i' is assigned in any way, the points q_i for which $\delta_i' < D'$ must be everywhere dense in some portion of the continuum, provided D' is sufficiently large.

Consider next the set S of points s_i for which the assigned numbers δ_i fall between D and $1/D$. It will be seen at once that there must be some portion of the continuum (some perfect subset) in which the points s_i are everywhere dense, provided, of course, that D is sufficiently small. Let the interval $(D, 1/D)$ be divided into any number of equal parts,

$$\epsilon = \frac{1/D - D}{n},$$

and denote by $S_1, S_2, \cdots, S_n$ the subsets of S for which the values of δ_i fall successively in the intervals $(D, D+\epsilon), (D+\epsilon, D+2\epsilon), \cdots, (1/D-\epsilon, 1/D)$. Since the points of S are everywhere dense in the part of the continuum (perfect set) now under consideration, there must be some subportion (some perfect subset) in which the points of some one of the sets S_i are everywhere dense. As, moreover, the value of ϵ may be made as small as desired, we obtain the following simple

Extension of the fundamental principle: *If to every point of a continuum (perfect set of points) a positive number δ_i is assigned in any manner, then for a sufficiently small value of D there is some portion of the continuum (some perfect subset) containing a set of points everywhere dense, for which the values of δ_i not only are greater than D but differ from one another by less than an arbitrarily assigned positive quantity ϵ.*

* The points q_i are said to be dense with respect to the subset, if within a sphere of any radius, described about any point of the subset as center, there lies always a point q_i.

† Cf. *Bericht*, p. 58.

In the case of a continuum of two or more dimensions it will be desirable not merely to know the existence of some portion of the continuum in which the points q_i are everywhere dense, but to ascertain also their distribution on a field of curves or surfaces which completely fill that portion. Inasmuch as every surface can be regarded as composed of curves, it will suffice here to consider the distribution of the q_i with respect to a continuous curve-system. For simplicity we shall confine our attention to a system of parallel straight lines, the results obtained being easily capable of extension to other curve-systems.

Let first a portion of space be selected from the continuum in the form of a right cylinder having its axis parallel to the given system of parallel straight lines. For the case of a two dimensional continuum the cylinder shall be understood to be a rectangle. Consider then the segment intercepted upon any one of the parallels by the cylinder. Since each segment is itself a continuum, there must be for a sufficiently small value of D some portion of the segment in which the points q_i having $\delta_i > D$ are everywhere dense. Denote for any segment the length of such a portion by l_i and the corresponding value of D by D_i. We will then assign to the segment (or, if we prefer, to its intersection with the base of the cylinder) the smaller of the two numbers l_i and D_i. Call it s_i. Now the set of segments (intersections) is perfect, and therefore by our "fundamental principle" the segments (intersections) for which $s_i > S$ must be everywhere dense in some portion of the cylinder (base), provided that S is a sufficiently small number. Take any such portion and let this dense set of segments be denoted by G. Divide any element of this new portion into equal parts not exceeding $S/2$ in length, and through the points of division pass planes parallel to the base of the cylinder, thereby dividing the portion under consideration into a finite number of component cylinders. Each segment of G contains at least one interval of length $s_i > S$ along which the points q_i having $\delta_i > s_i > S$ are everywhere dense. Consequently some one of the component cylinders must intercept an interval of length $S/2$, upon which the points q_i are everywhere dense. Since also the lines of G were everywhere dense before the subdivision into component cylinders took place, these intercepted intervals of length $S/2$ must be everywhere dense in some portion of one or more of the component cylinders. We obtain thus the following addendum to our fundamental principle:

ADDENDUM. *When the continuum is a portion of a plane or a space of higher dimensions, it is possible to select the region in which the points q_i are everywhere dense so as to fulfill the following condition: For a sufficiently small but fixed value of D there is throughout the region a dense set of parallels, upon each of which the q_i are everywhere dense; and the direction of the parallels may be assigned arbitrarily before seeking the region.*

§ 2. *Application to derivatives.*

Let $f(x)$ be a function of a real variable having a derivative at every point of an interval (a, b) of the X-axis. If ϵ be a small arbitrarily assigned positive quantity, any point p_i in (a, b) can be made the center of an interval $2\delta_i$ within which

$$\left|\frac{f(p_i+h)-f(p_i)}{h}-f'(p_i)\right|<\epsilon \qquad (|h|<\delta_i).$$

I shall assign to each point p_i such a positive number δ_i. Then, by § 1, for a sufficiently small D the points q_i for which $\delta_i > D$ must be dense in some portion of (a, b). Let (a', b') be such a portion, and denote by Q_i the dense set of points q_i within (a', b'). Then for any two points q_1 and q_2 whose distance $|q_1 - q_2|$ does not exceed D we have

$$\left|\frac{f(q_1)-f(q_2)}{q_1-q_2}-f'(q_1)\right|<\epsilon, \qquad \left|\frac{f(q_1)-f(q_2)}{q_1-q_2}-f'(q_2)\right|<\epsilon,$$

and therefore

$$(1) \qquad |f'(q_1)-f'(q_2)|<2\epsilon.$$

Consequently the variation of $f'(x)$, taken for the set Q_i and for any subinterval of (a', b') not greater than D in length, will not be greater than 4ϵ.

The fundamental principle has been used thus to obtain a portion of the continuum in which there is a dense set of points with a certain specified property. Our second step in this and subsequent demonstrations consists in the extension of the property to all points of the portion. In the special case before us now, every point p of (a', b'), and in particular a point p which does not belong to Q_i, has a small neighborhood $2h$ such that

$$\left|\frac{f(q_i)-f(p)}{q_i-p}-f'(p)\right|<\epsilon' \qquad (|q_i-p|<h),$$

where ϵ' is an arbitrary small prescribed quantity. But also when $|q_i - p| \leqq D$, we have

$$\left|\frac{f(q_i)-f(p)}{q_i-p}-f'(q_i)\right|<\epsilon.$$

Consequently

$$(2) \qquad |f'(p)-f'(q_i)|<\epsilon+\epsilon'.$$

As ϵ' is arbitrarily small, we conclude from (1) and (2) that the variation of $f'(x)$ in any portion of (a', b') not exceeding D in length will not exceed $\zeta_1 = 6\epsilon$.

This brings us to the third and final step of this and similar demonstrations; namely, the establishment of the pointwise discontinuous character of the function under consideration after it has been demonstrated that there is some por-

tion of the original field in which the variation of the function does not exceed an arbitrarily prescribed quantity ζ_1. The reasoning is so familiar that I give it here once for all, remarking merely that it is applicable to fields of two or more dimensions as well as to intervals. Take any series of decreasing positive numbers $\zeta_1, \zeta_2, \cdots$ having 0 as their limit. Inside the part of the field in which the variation does not exceed ζ_1 there must be, by like reasoning, a part in which the variation does not exceed ζ_2; in the interior of this another part in which the variation does not exceed ζ_3, and so on. The limit of this series of intervals or regions, taken each entirely in the interior of the preceding, is a point (or set of points) at which the function is continuous. Furthermore, as the argument applies to any portion of the initial field, the points of continuity must be everywhere dense, and the function is at most only pointwise discontinuous.

Applied to $f'(x)$ in the particular case before us, this reasoning gives at once the following theorem:

THEOREM I. *If a function of a real variable x has a derivative at every point of an interval (a, b), this derivative is at most only pointwise discontinuous in the interval* (BAIRE).*

It is well known that the points of continuity in the case of a pointwise discontinuous function have the power of a continuum. The same remark applies to all other sets of points obtained in this paper (cf. theorem VII) by the above method. In fact, to each value of ζ_i there was found between a and b, or between any two points of (a, b), a certain interval with specified property. Consequently the totality of points which cannot be included in the interior of such intervals is nowhere dense in (a, b). Call this set of points B_i. Corresponding to $\zeta_1, \zeta_2, \cdots$ we have an enumerable set of such sets B_i, each nowhere dense in (a, b), and these together constitute what BAIRE calls a point-set of the *first category*.† The complementary point-set, which consists of such points as we obtain by our method, has the power of a continuum. ‡

Theorem I may be extended to one-side derivatives. The extension is not without some interest, for it is known that the four derivatives of a function, when distinct throughout an interval, must be totally discontinuous. § The possibilities in respect to continuity are accordingly settled for the various species of derivatives. For extension of the above theorem to one-side derivatives BAIRE's proof requires no modification. The proof given here may be modified as follows.

Suppose that only a right-hand derivative $f'_+(x)$ exists in (a, b). Assign

* BAIRE deduces the theorem as a corollary of the fundamental theorem which will be quoted at the beginning of § 3; loc. cit., p. 64 or *Bericht*, p. 224.

† Loc. cit., p. 65.

‡ Cf. *Bericht*, p. 108.

§ Cf. DINI's *Grundlagen für die Theorie der Functionen einer reellen Grösse*, p. 249.

to each point p_i of the interval a positive number δ_i such that

$$(3) \qquad \left| \frac{f(p_i + h) - f(p_i)}{h} - f'_+(p_i) \right| < \frac{\epsilon}{4} \qquad (0 < h < \delta_i).$$

For a sufficiently small D there will be some subinterval (a', b') in which the points q_i for which $\delta_i > D$ are everywhere dense. Take within it two such points, q_1 and $q_2 = q_1 + h$, for which $0 < h \leqq D/2$. We have the three inequalities

$$(4) \qquad \left| \frac{f(q_1 + h) - f(q_1)}{h} - f'_+(q_1) \right| < \frac{\epsilon}{4},$$

$$(5) \qquad \left| \frac{f(q_1 + 2h) - f(q_1 + h)}{h} - f'_+(q_1 + h) \right| < \frac{\epsilon}{4},$$

$$(6) \qquad \left| \frac{f(q_1 + 2h) - f(q_1)}{2h} - f'_+(q_1) \right| < \frac{\epsilon}{4},$$

which combined together give

$$|f'_+(q_1 + h) + f'_+(q_1) - 2f'_+(q_1)| < \epsilon$$

or

$$(7) \qquad |f'_+(q_2) - f'_+(q_1)| < \epsilon.$$

Consequently for Q_i—, i. e., the set of points q_i within (a', b')—the variation of $f'_+(x)$ will not exceed 2ϵ in any segment not exceeding $D/2$ in length.

Take next any point p of (a', b') not belonging to Q_i and let $\delta < D$ be a number assigned to it in accordance with (3). Since p is the limit of a set of points q_i which lie to the right of it, we may choose from Q_i some point $q_i = p + h$ for which $0 < h < \delta/2$. Then if we replace q_1 by p and q_2 by q_i, the inequalities (4)–(7) will hold as before. Consequently

$$|f'_+(q_i) - f'_+(p)| < \epsilon.$$

It follows therefore that the variation of $f'_+(x)$ for any portion of (a', b') which does not exceed $D/2$ in length will not be greater than $\zeta_1 = 4\epsilon$. The conclusion that $f'_+(x)$ is at most only pointwise discontinuous then follows in the manner previously described.

We shall consider next a function $f(x, y)$ which has a partial derivative $\partial f/\partial x$ at every point of a field T defined by

$$a \leqq x \leqq b, \qquad c \leqq y \leqq d.$$

By virtue of this hypothesis $f(x, y)$ must be continuous with respect to x. We suppose also that it is continuous with respect to y upon a set of parallels $x = x_i$ to the Y-axis which are everywhere dense in T, and proceed then to examine the character of $\partial f/\partial x$ in regard to continuity when both x and y are

varied. Any point $p=(x, y)$ may be made the center of an interval parallel to the axis of x and having such a length* 2δ that

$$(8) \qquad \left|\frac{f(x+h, y)-f(x, y)}{h}-\frac{\partial f(x, y)}{\partial x}\right|<\epsilon \qquad (|h|<\delta).$$

By the "addendum" of § 1 it is possible for a sufficiently small value of D to find a part of T

$$T': \qquad a' \leqq x \leqq b', \qquad c' \leqq y \leqq d'$$

where the points q_i having $\delta_i > D$ will be everywhere dense upon each line of some system of parallels, $y=y_j$, which is itself everywhere dense throughout T'. Let $x=x_1$ and $x=x_1+h$ be any two lines within T' along which $f(x, y)$ is continuous with respect to y. Then, *h being fixed in value*, the difference quotient

$$(9) \qquad r(x_1, y)=\frac{f(x_1+h, y)-f(x_1, y)}{h}$$

will be a continuous function of y on the line $x=x_1$. Denote by LL' any segment of the line in which the variation of (9) is less than a prescribed ϵ, and upon LL' as a side construct within T' a rectangle $LL'M'M$ having its second side $L'M'<D$.

Consider the variation of $\partial f(x, y)/\partial x$ in this rectangle, and first of all upon the dense set of parallels $y=y_j$. If the intersection (x_1, y_j) of x_1 with one of these parallels is not itself one of the points q_i, it is the limit of a set of such points upon $y=y_j$, and we may take a point $q_j=(\bar{x}, y_j)$ so close to (x_1, y_j) that the difference of the values of the different quotient $r(x, y)$ at these two points will be numerically less than an arbitrarily assigned quantity ϵ'. At the second point q_j we have

$$\left|r(\bar{x}, y_j)-\frac{\partial f(\bar{x}, y_j)}{\partial x}\right|<\epsilon \qquad (h \leqq D).$$

Since, furthermore, on the one hand, the variation of $r(x, y)$ upon LL' is less than ϵ, and, on the other hand, the variation of $\partial f/\partial x$ upon $y=y_j$ is less than $\zeta=6\epsilon$ by the proof of theorem I, it follows at once that the total variation of $\partial f/\partial x$, taken for the set of parallels $y=y_j$, will be less than $\epsilon+2(6\epsilon)+2(\epsilon+\epsilon')$. Since also ϵ' is arbitrary, it will not exceed 15ϵ.

Bring next into consideration the points of the rectangle which do not lie upon the dense set of parallels $y=y_j$. For any such point (x', y') we can find another point (x'', y') so near to it that

$$(10) \qquad \left|\frac{f(x'', y')-f(x', y')}{x''-x'}-\frac{\partial f(x', y')}{\partial x}\right|<\epsilon'.$$

* In this and other cases BAIRE considers the *maximum* interval 2δ fulfilling the condition imposed and proves that δ as a function of (x, y) is "*semi-continue supérieurement.*" In the method which is here being followed, it is unnecessary to introduce the maximum δ.

Now if neither point lies upon a line of continuity for $f(x, y)$ with respect to y, we can take a pair of such points, $(\bar{x}', y')$ and $(\bar{x}'', y')$, so close to them that

$$(11) \qquad \left| \frac{f(\bar{x}'', y') - f(\bar{x}', y')}{\bar{x}'' - \bar{x}'} - \frac{f(x'', y') - f(x', y')}{x'' - x'} \right| < \epsilon',$$

while if either point lies upon such a line of continuity we have merely to identify it in (11) with the corresponding point of the second pair. Having now two points upon lines of y-continuity, by varying y we can find one of the dense set of parallels $y = y_j$ so near to $y = y'$ that

$$(12) \qquad \left| \frac{f(\bar{x}'', y_j) - f(\bar{x}', y_j)}{\bar{x}'' - \bar{x}'} - \frac{f(\bar{x}'', y') - f(\bar{x}', y')}{\bar{x}'' - \bar{x}'} \right| < \epsilon'.$$

But upon this parallel it is possible to select from the dense set of points q_i such a point $q_i = (x_i, y_j)$ that

$$(13) \qquad \left| \frac{f(x_i, y_j) - f(\bar{x}', y_j)}{x_i - \bar{x}'} - \frac{f(\bar{x}'', y_j) - f(\bar{x}', y_j)}{\bar{x}'' - \bar{x}'} \right| < \epsilon' \quad (|x_i - \bar{x}'| < D).$$

Since, lastly, for q_i

$$(14) \qquad \left| \frac{\partial f(x_i, y_j)}{\partial x_i} - \frac{f(x_i, y_j) - f(\bar{x}', y_j)}{x_i - \bar{x}'} \right| < \epsilon,$$

it follows immediately from (10)–(14) that

$$\left| \frac{\partial f(x_i, y_j)}{\partial x} - \frac{\partial f(x', y')}{\partial x} \right| < \epsilon + 4\epsilon',$$

where ϵ' is arbitrarily small. As the variation of $\partial f/\partial x$ upon the dense set of parallels $y = y_j$ does not exceed 15ϵ, we conclude from the last inequality that the variation of $\partial f/\partial x$ in the rectangle $LL'M'N$ will not exceed $15\epsilon + 2\epsilon = 17\epsilon$.

This rectangle can now be substituted in place of the original field. By indefinitely decreasing $\zeta_1 = 17\epsilon$ we obtain as the limit of a series of rectangles, lying each within the preceding, a point of continuity of $\partial f/\partial x$ considered as a function of (x, y). This proves the following theorem;

THEOREM II. *If $f(x, y)$ at every point of T possesses a partial derivative $\partial f(x, y)/\partial x$ and if, furthermore, there is in T an everywhere dense set of parallels $x = x_i$ upon which $f(x, y)$ is continuous in y, then $\partial f(x, y)/\partial x$ as a function of (x, y) is at most only pointwise discontinuous.*

For the sake of completeness I call attention to BAIRE's important remark* that when the derivative $\partial f/\partial y$ exists as well as $\partial f/\partial x$, there must be at least a set of points everywhere dense throughout T at which both derivatives are continuous, and at which, therefore, $f(x, y)$ has a differential. The proof follows immediately by making the values of $\zeta_1, \zeta_2, \zeta_3, \cdots$ refer alternately to the variation of the two derivatives.

*Loc. cit., pp. 108–110.

§ 3. *Theorems on the continuity of* $f(x, y)$.

BAIRE in his thesis has given the following important theorem:

If $f(x, y)$ *is continuous with respect to* y *at every point of a rectangle*

$$T: \qquad a \leqq x \leqq b, \qquad c \leqq y \leqq d,$$

and with respect to x *on a set of parallels to the* X*-axis everywhere dense in* T, *then* $f(x, y)$ *as a function of the two variables together is at most only point-wise discontinuous; and the points of continuity in* T *will be everywhere dense upon any curve having an equation of the form* $y = \phi(x)$, *wherein* $\phi(x)$ *is a continuous function of* x.*

SCHÖNFLIES,† commenting on this theorem, has remarked that although BAIRE's hypothesis of continuity with respect to x is stated for a dense set of parallel straight lines, this is clearly an unessential specialization, for in place of this set of parallels "*eine überall dichte Curvenschar*" could be substituted. There is, however, nothing in BAIRE's work to indicate the mode of distribution of the points of continuity along such a system of curves $y = \phi_i(x)$ as SCHÖNFLIES suggests. I shall complete the result by establishing the following theorem:

THEOREM III. *If* $f(x, y)$ *is continuous with respect to* y *at every point of* T *and with respect to* x *upon a set of curves everywhere dense and having equations of the form* $y = \phi_i(x)$, *in which* $\phi_i(x)$ *is continuous in* x, *then* $f(x, y)$ *as a function of* (x, y) *is continuous at all the intersections of these curves with some set of parallels to the* Y*-axis which are likewise everywhere dense in* T.

Proof. To every line $x = x_i$ in the field T let a positive number δ_i be so assigned that the variation of $f(x, y)$ in any segment not exceeding δ_i in length will not exceed a fixed but arbitrarily prescribed ϵ. According to our fundamental principle, for a sufficiently small value of D the set of lines $x = \xi_i$ for which $\delta_i > D$ must be dense in some portion of T. Suppose such a portion T' to be included between $x = \xi_1$ and $x = \xi_2$, and denote the dense set of lines between these limits by Ξ. We shall establish first that the characteristic property of the lines of Ξ holds within T' for every parallel to the Y-axis.

Consider for this purpose any line $x = x'$ which is not known to belong to Ξ. It is cut by the curves $y = \phi_i(x)$ in a dense set of points (x', y_i). Take now any two points whatsoever of the line, (x', y') and (x', y''), for which $|y' - y''| \leqq D$. If these points are not themselves included in the dense set of intersections (x', y_i), we can take two of the intersections (x', y_i), (x', y_j) which lie so near to these two points that simultaneously

$$|y_i - y_j| \leqq D,$$

$$|[f(x', y') - f(x', y'')] - [f(x', y_i) - f(x', y_j)]| < \epsilon',$$

* Loc. cit., p. 27.

† *Bericht*, p. 140. The specification as to their distribution upon $y = \phi(x)$ is omitted in SCHÖNFLIES' summary of the theorem (p. 142).

in which ϵ' is arbitrarily small. Then since (x', y_i), (x', y_j) are points of continuity of $f(x, y)$ as a function of x, they may be made the centers of intervals parallel to the X-axis and so small that in either interval the variation of $f(x, y)$ will be less than ϵ'. Furthermore, since the lines of Ξ are everywhere dense in T', the two intervals will be cut by certain lines of the set. Let (ξ', y_i), (ξ', y_j) be the intersections of the two intervals with some one of these lines. We now have the inequalities

$$|[(f(x', y_i) - f(x', y_j)] - [f(\xi', y_i) - f(\xi', y_j)]| < 2\epsilon',$$

$$|f(\xi', y_i) - f(\xi', y_j)| \leqq \epsilon.$$

It follows then by combination of these with the inequality above that

$$|f(x', y') - f(x', y'')| < \epsilon + 3\epsilon',$$

and hence, since ϵ' is arbitrary,

$$|f(x', y') - f(x', y'')| \leqq \epsilon.$$

We have thus shown that the variation of $f(x, y)$ in T' upon any parallel to the Y-axis will not exceed ϵ for any segment of the parallel not greater than D in length.

Any point (x_1, y_1) within T' at which $f(x, y)$ is continuous with respect to x can now be made the center of a rectangle, with sides parallel to the axes of x and of y, in which the variation of $f(x, y)$ does not exceed 3ϵ. For suppose it first to be made the center of an interval $(x_1 - \delta', x_1 + \delta')$ parallel to the X-axis in which the variation does not exceed ϵ. Then the rectangle bounded by

$$x = x_1 - \delta', \qquad x = x_1 + \delta', \qquad y = y_1 - D, \qquad y = y_1 + D,$$

or so much of it as lies in T', fulfills the conditions stated.

Let the value of ϵ be now diminished. Corresponding to an indefinitely diminishing series $\epsilon', \epsilon'', \epsilon''', \cdots$ we will have a series of rectangles $T', T'', T''', \cdots$ having the same altitude as T and lying each within the preceding. The limit of this series of rectangles is a line $y = \eta$ (or a continuous set of such lines) which has the following property: Every point of the line at which $f(x, y)$ is continuous with respect to x is also a point of continuity of $f(x, y)$ with respect to (x, y), inasmuch as it can be enclosed in a series of rectangles in which the variation of $f(x, y)$ is successively not greater than $3\epsilon', 3\epsilon'', 3\epsilon''', \cdots$. In particular, every point in which the line is cut by the curves $y = \phi_i(x)$ is a point of continuity of $f(x, y)$.

If the original field T is divided into two parts by a parallel to the Y-axis, our reasoning applies to each part. The lines $y = \eta$ are therefore everywhere dense in T. This proves the theorem.

One particular case of theorem III is worthy of special note, that in which $f(x, y)$ is continuous in x and y separately at every point of T. Applied to this case the theorem becomes the following corollary:

THEOREM IV. *If $f(x, y)$ is continuous in x and y separately, there exists in T an everywhere dense set of parallels to the Y-axis (also to the X-axis), at every point of which $f(x, y)$ is continuous in (x, y).*

This is merely another statement of the following conclusion of BAIRE:*

THEOREM IV′. *If $f(x, y)$ is continuous in x and y separately within T, the projections upon the axis of x of all points of discontinuity whose measure is greater than an arbitrarily prescribed positive quantity σ will be nowhere dense.*

To establish the equivalence of the two statements, it suffices to note that, by the proof of IV, between any two parallels to the Y-axis containing points of discontinuity which have a measure greater than σ, there may be inserted one of our strips T' for which $\sigma_1 \equiv 3\epsilon$ may be taken less than σ. Because 3ϵ is an upper limit for the measure of discontinuity at all interior points of T', it follows that the projections of the discontinuities having a measure $> \sigma$ can be nowhere dense. Conversely, if IV′ be assumed, theorem IV at once results by application of familiar principles.

In the method which I have given here, attention has been concentrated upon the parallels of continuity for $f(x, y)$ rather than upon those containing points of discontinuity. This makes it easy to make the following advance beyond BAIRE's result:

THEOREM V. *Upon the hypothesis enunciated in theorem* IV *there exists also a set of lines everywhere dense in T and parallel to the Y-axis, upon each of which $f(x, y)$ as a function of (x, y) is uniformly continuous.*†

To prove this theorem consider again the strip T'. Divide any interior line $x = x'$ into segments equal to or less than $D/2$ in length by means of the points

$$(x', y_1),\ (x', y_2),\ \cdots,\ (x', y_n).$$

Each point (x', y_i) can be made the center of an interval $2\sigma_i$ parallel to the axis, in which the variation of $f(x, y)$ does not exceed a prescribed ϵ. Let σ be the smallest of the numbers σ_i and mark out a strip of T' bounded by the parallels

$$x = x' - \sigma, \qquad x = x' + \sigma.$$

Take now any two points (x, y), (x'', y'') in the interior of the strip such that $|y - y''| \leqq D/2$, and make each point the center of an interval of length D

* The proof given above is much simpler than that of BAIRE (loc. cit., pp. 88–94), who does not make use of the theorem of which III was an extension.

† The term "uniformly continuous" here signifies that if any ε is arbitrarily prescribed, a value δ can be found such that for the vicinity of all points (x', y') of the line considered the variation of $f(x, y)$ will be less than ε when $|x - x'| \leqq \delta$, $|y - y'| \leqq \delta$, provided, of course, (x, y) lies within the field T.

parallel to the axis of y. It has been seen that the variation of $f(x, y)$ in either of these intervals does not exceed ϵ. Since also they will cut in common some one of the n intervals σ_i, we have

$$|f(x, y) - f(x'', y'')| \leqq 3\epsilon.$$

If now we confine ourselves to the strip S bounded by

$$x = x' - \frac{\sigma}{2}, \qquad x = x' + \frac{\sigma}{2},$$

every point in S may be made the center of a square, the sides of which are parallel to the axes and have a length equal to the smaller of the two numbers σ and D. Within the square (or such part of the square as is contained in T) the variation of $f(x, y)$ will not exceed 6ϵ.

The strip S may now be substituted in place of T, and the reasoning be repeated. To a sequence of indefinitely diminishing values $\epsilon_1, \epsilon_2, \epsilon_3, \cdots$ there corresponds a series of strips of decreasing breadth, contained each in the preceding, and the limit is a line $x = \bar{x}$ on which $f(x, y)$ is uniformly continuous. Obviously, also, such lines will be found between any two parallels to the axis of y, and they are accordingly everywhere dense in T.

§ 4. *Theorems concerning integrals.*

Suppose that

$$F(x) \equiv \int_c^d f(x, y)\,dy \tag{15}$$

is the integral of any function $f(x, y)$ which in the rectangle T is continuous in x and y separately. We have just found in T a strip S for which the following property holds: If $|x' - x''|$ does not exceed σ, then

$$|f(x', y) - f(x'', y)| \leqq 3\epsilon.$$

Consequently we have for the difference of the values of (15) along the parallel paths $x = x'$, $x = x''$

$$|F(x') - F(x'')| = \left| \int_c^d f(x', y)\,dy - \int_c^d f(x'', y)\,dy \right| \leqq 3\epsilon(d - c).$$

In other words, in the interval (a, b) may be found a subinterval in which the variation of $F(x)$ does not exceed $\zeta_1 = 6\epsilon(d - c)$. The conclusion concerning the pointwise discontinuous character of $F(x)$ which has been stated previously in theorem VI of the introduction, follows at once in the manner explained in § 1.

Consider next the difference quotient

$$\frac{F(x + \Delta x) - F(x)}{\Delta x} = \int_c^d \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}\,dy.$$

If $f(x, y)$ has a derivative with respect to x, the application of the mean-value theorem gives for the value of the right-hand member

$$\int_c^d f'_x(x + \theta \cdot \Delta x, y)\, dy,$$

where θ is a number between 0 and 1 which varies with y. If, now, $f'_x(x, y)$ is continuous in x and y separately, there is a strip of T between two appropriately chosen parallels $x = x' \pm \delta$, within which the variation of $f'_x(x, y)$ for constant y will not exceed 3ϵ. Hence when x and $x + \Delta x$ both lie between these limits for x, we have

$$\left| \int_c^d f'_x(x + \theta \cdot \Delta x, y)\, dy - \int_c^d f'_x(x, y)\, dy \right| \leqq 3\epsilon(d - c).$$

Consequently

$$\frac{F(x + \Delta x) - F(x)}{\Delta x} = \int_c^d f'_x(x, y)\, dy + \epsilon_1,$$

where

$$|\epsilon_1| \leqq 3\epsilon(d - c).$$

The interval $(x' - \delta, x' + \delta)$ and the corresponding strip of T may now be substituted in place of (a, b) and the field T, and the argument may be repeated with a smaller value of ϵ. Continuing in this way we obtain with diminishing ϵ a series of intervals contained each in the interior of the preceding and having for their limit a point x at which

$$\lim_{\Delta x = 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} = F'(x) = \int_c^d f'_x(x, y)\, dy.$$

As the argument applies to any portion of (a, b), the first part of theorem VII is established.

The second part of the theorem follows from the remark that in the succession of strips diminishing with ϵ the variation of $f'_x(x, y)$ for constant y does not exceed 3ϵ, so that the variation of

$$F_1(x) = \int_c^d f'_x(x, y)\, dx$$

does not exceed $3\epsilon(d - c)$. The limiting point x is therefore a point of continuity of $F_1(x)$.

§ 5. *On series.*

Let

$$S(x) = u_1(x) + u_2(x) + \cdots$$

be a series, whose terms are continuous in (a, b) and which converges at every point of this interval. Denote by $S_n(x)$ the sum of the first n terms of the

series. To each point x_i of the interval a number n_i can be so assigned that

$$|S_{n_i+m}(x_i) - S(x_i)| < \epsilon \qquad (m = 1, 2, 3, \cdots).$$

If ϵ is fixed and a sufficiently large N is selected, the set of points $x = q_i$ for which $n_i < N$ will be everywhere dense in some subinterval, (a', b'). Let Q denote this set of points. Contract this interval so much as may be necessary to make the variation of the continuous function $S_N(x)$ within (a', b') less than ϵ. Then for q_1 and q_2, any two points of Q, we have

$$|S_N(q_1) - S(q_1)| < \epsilon,$$

$$|S_N(q_2) - S(q_2)| < \epsilon,$$

and

$$|S_N(q_1) - S_N(q_2)| < \epsilon.$$

Hence

$$|S(q_1) - S(q_2)| < 3\epsilon; \tag{16}$$

that is to say, the variation of $S(x)$ in (a', b') over the set Q does not exceed 6ϵ.

Consider next a point p in (a', b') which does not belong to Q. A fixed value $N' \geqq N$ can be found such that

$$|S_{N'}(p) - S(p)| < \epsilon'.$$

As the points q_i are everywhere dense in $(a'b')$, one of these points be found so near to p that

$$|S_{N'}(p) - S_{N'}(q_i)| < \epsilon'.$$

Since also

$$|S_{N'}(q_i) - S(q_i)| < \epsilon,$$

it follows from the last three inequalities that

$$|S(p) - S(q_i)| < \epsilon + 2\epsilon'. \tag{17}$$

But ϵ' is arbitrary. We conclude therefore from (16) and (17) that the variation of $S(x)$ within (a', b') can not exceed $\zeta_1 = 8\epsilon$. By § 2 this gives at once the following theorem:

THEOREM VIII. *If the series*

$$S(x) = u_1(x) + u_2(x) + u_3(x) + \cdots$$

converges at every point of an interval (a, b) and each term of the series is a continuous function of x, the sum of the series can be at most only pointwise discontinuous in (a, b) (BAIRE).*

This theorem was first proved by OSGOOD† for the case in which the limit is a

* Loc. cit., p. 62, or *Bericht*, p. 224.

† American Journal of Mathematics, vol. 19 (1897).

continuous function, and he showed, moreover, that the points of uniform convergence* are everywhere dense. Recently HOBSON† has proved that the points of uniform convergence are also dense in the more comprehensive case considered in theorem VIII. This conclusion results also from the proof given above, for it can be shown at once that all the points of continuity for $S(x)$ obtained through the demonstration are points of uniform convergence. Take, in fact, any one of these points. It can be made first the center of an interval (α, β) in which the variation of $S(x)$ is less than ϵ. But it also lies in an interval (a', b') in which the points q_i are everywhere dense. For any of these points we had

$$|S_n(q_i) - S(q_i)| < \epsilon \qquad (n \geqq N). \tag{18}$$

In the vicinity of any point p of (a', b') we can find a point q_i so near that, for given n and ϵ',

$$|S_n(p) - S_n(q_i)| < \epsilon'.$$

But for two points q_i and p interior to (α, β),

$$|S(q_i) - S(p)| < \epsilon.$$

Hence from the last three inequalities

$$|S_n(p) - S(p)| \leqq 2\epsilon \qquad (n > N). \tag{19}$$

The two inequalities (18), (19) hold simultaneously for the common portion of (a', b') and (α, β), and consequently the center of (α, β) is, as stated, a point of uniform convergence.

* By a point of uniform convergence is to be understood one which, for any arbitrarily assigned ε, can be enclosed in an interval in which

$$|S(x) - S_n(x)| < \varepsilon \qquad (n > N)$$

for all values of x and for a sufficiently large N.

† Proceedings of the London Mathematical Society, vol. 34 (1902), p. 245, or Acta Mathematica, vol. 27 (1903), p. 209. An earlier proof was given in TOWNSEND's dissertation (Göttingen, 1900), p. 68.

INVARIANTS OF BINARY FORMS UNDER MODULAR TRANSFORMATIONS*

BY

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For a non-modular field F and a binary form f with coefficients in F, the problem of the determination of functions of the coefficients and variables of f, invariant under all binary linear transformations in F, is formally identical with the corresponding problem of the ordinary algebraic invariant theory. But for a finite modular field the problem is essentially different; the terms of an invariant need not be of the same degree nor of constant weight; the annihilators are quite complicated, involving higher partial derivatives. Fortunately, the difficulty in the direct computation of the invariants is in marked contrast with the regularity observed in the actual form of the invariants and with the simplicity of the relations between the invariants, common to the algebraic and modular theories, and the additional invariants peculiar to the modular theory.

In the study of the invariants of a given quantic in the Galois field of order p^n, we have a doubly infinite system of problems, corresponding to a single problem in the algebraic theory. Interest naturally centers in a comparative study, rather than in the individual problems. The aim of the present paper is to give in correlation the results of a rather extensive comparative study. Formal proofs of the laws observed are given only in a few instances. It is hoped that proofs of the remaining properties observed will become more practicable when a satisfactory symbolic treatment is constructed.

Since there is evidently only a finite number of linearly independent invariants I of a given binary form of degree m in the $GF[p^n]$, each set of numbers m, n, p uniquely defines a commutative linear associative algebra, whose units are these invariants I.

In the final sections of the paper, application is made to the invariantive reduction of binary quantics to canonical forms.†

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† The invariants of k-ary quadratic forms are treated in papers to appear in the Proceedings of the London Mathematical Society and the American Journal of Mathematics.

1. THEOREM. *The binary form* with coefficients in the* $GF[p^n]$,

$$f = \sum_{i=0}^{m} a_i x^{m-i} y^i, \tag{1}$$

has the absolute invariant

$$I = \prod_{i=0}^{m} (a_i^\mu - 1), \qquad \mu \equiv p^n - 1. \tag{2}$$

Here and below we employ the generators (t, λ any marks $\neq 0$)

$$x = x' + ty', \qquad y = y'; \tag{3}$$

$$x = x', \qquad y = \lambda y'; \tag{4}$$

$$x = y', \qquad y = -x'; \tag{5}$$

of the group of all binary linear transformations with coefficients in the $GF[p^n]$. Under (3) f becomes a form f' with the coefficients

$$a_0' = a_0, \quad a_1' = a_1 + mta_0, \cdots, \quad a_i' = a_i + \sum_{j=0}^{i-1} C_{m-j,\, i-j} t^{i-j} a_j, \cdots. \tag{6}$$

Under transformations (4) and (5) we have, respectively,

$$a_i' = \lambda^i a_i \qquad (i = 0, 1, \cdots, m), \tag{7}$$

$$a_i' = (-1)^{m-i} a_{m-i} \qquad (i = 0, 1, \cdots, m). \tag{8}$$

Under the replacements (7) and (8) the function (2) is unaltered, since $\lambda^\mu = 1$ for every mark $\lambda \neq 0$. Consider next the replacement (6). The first factor $a_0^\mu - 1$ is unaltered. Then, since $(a_0^\mu - 1)a_0 = 0$,

$$(a_0'^\mu - 1)(a_1'^\mu - 1) = (a_0^\mu - 1)(a_1^\mu - 1 + Aa_0) = (a_0^\mu - 1)(a_1^\mu - 1).$$

The proof may be completed by induction from $i-1$ to i. Let

$$P_i = \prod_{j=0}^{i} (a_j^\mu - 1).$$

Then

$$P_i' = P_{i-1}(a_i'^\mu - 1) = P_{i-1}(a_i^\mu - 1 + B_0 a_0 + B_1 a_1 + \cdots + B_{i-1} a_{i-1})$$
$$= P_{i-1}(a_i^\mu - 1) = P_i.$$

2. The absolute invariant (2) has a simple interpretation. According as $a \neq 0$ or $a = 0$, we have $a^\mu - 1 = 0$ or -1. *Hence the form f vanishes identically if and only if $I \neq 0$.*

* We do not at present introduce binomial coefficients C_{mi}, some of which are divisible by p for certain values of m and p.

Frequently a property expressible by a single invariant in the modular theory requires in the algebraic theory the use of a covariant or else an invariantive simultaneous system of equations. A less obvious instance relates to the minors of the determinant of a k-ary quadratic form.

3. THEOREM. *For m a power of p, the binary form* (1) *of degree m with coefficients in the $GF[p^n]$ has the absolute invariant*

$$P = \prod_{i=1}^{m-1} (a_i^\mu - 1), \qquad \mu \equiv p^n - 1. \tag{9}$$

The proof is similar to that in § 1. We now have

$$C_{mi} \equiv 0 \pmod{p} \qquad (i = 1, \cdots, m-1),$$

so that there is no term a_0 in a_i' $(i = 1, \cdots, m-1)$ in formulæ (6).

4. An obvious invariant of f is the eliminant E of

$$f = 0, \qquad x^{p^n} = x, \qquad y^{p^n} = y.$$

For example, let $m = 3$, $p^n = 3$. Multiply f by xy, x^2y^2 and x^2 in turn and reduce by means of $x^3 = x$, $y^3 = y$. The corresponding equations

$$a_0x + a_1x^2y + a_2xy^2 + a_3y = 0, \qquad (a_0 + a_2)x^2y + (a_1 + a_3)xy^2 = 0,$$

$$(a_1 + a_3)x^2y + (a_0 + a_2)xy^2 = 0, \qquad a_0x + (a_1 + a_3)x^2y + a_2xy^2 = 0,$$

have the eliminant, identical with (17) of § 9,

$$a_0a_3\{(a_1 + a_3)^2 - (a_0 + a_2)^2\}.$$

Evidently, $E = a_m R$, where R is the resultant of

$$f_1 \equiv a_0 + a_1y + \cdots + a_my^m = 0, \qquad y^{p^n} = y.$$

By the elimination of $1, y, y^2, \cdots, y^{p^n-1}$ by SYLVESTER's dialytic method, we obtain for R a determinant of order p^n in which each element in the main diagonal is a_0. The numerical factor is thereby determined so that $R = a_0^{p^n} + \cdots$.

Except in the simplest cases, it is more convenient to have E expressed as the product of the linear functions obtained from f by giving to x, y the $p^n + 1$ sets of values $0, 1$; and $1, \rho$, where ρ ranges over the marks of the field:

$$E = a_m \Pi(a_0 + a_1\rho + \cdots + a_m\rho^m), \quad \rho \text{ ranging over the } GF[p^n].$$

For the case of quadratic forms $(m = 2)$ in the $GF[p^n]$, simple expressions may be given to E. For $p = 2$, $E = a_1^2\chi$, where χ is the absolute

invariant given by (20). For $p > 2$,

$$E = \tfrac{1}{2}(\Delta^{\frac{1}{2}(p^n+1)} - \Delta), \qquad \Delta = a_1^2 - 4a_0a_2.$$

Note, in verification of this result, that $f_1 = 0$ has a root in the field if, and only if, the discriminant Δ is zero or a square, so that $E = 0$ if, and only if,

$$\Delta(\Delta^{\frac{1}{2}(p^n-1)} - 1) = 0.$$

For cubic forms, consider first the case $p^n = 3^2$.

We define the $GF[9]$ by the primitive irreducible congruence $i^2 \equiv i + 1 \pmod 3$; then the marks $\neq 0$ are $1, i, i^2, i^3$, and their negatives. Hence $E = a_0a_3\pi$, where

$$\pi = \Pi\{(a_0 + a_2\rho^2)^2 - (a_1\rho + a_3\rho^3)^2\} \qquad (\rho = 1, i, i^2, i^3)$$
$$= \Pi(a_0^2 - \alpha\rho^2 + \beta\rho^4 - a_3^2\rho^6) \quad (\alpha = a_1^2 + a_0a_2,\ \beta = a_2^2 + a_1a_3).$$

Forming the product of the factors given by $\rho = 1, i^2$, and those given by $\rho = i, i^3$, we get, since $i^4 = -1$,

$$\pi = \{(a_0^2 + \beta)^2 - (a_3^2 + \alpha)^2\}\{(a_0^2 - \beta)^2 + (a_3^2 - \alpha)^2\}$$
$$= (a_0^4 + a_3^2\alpha + \beta^2)^2 - (a_3^4 + a_0^2\beta + \alpha^2)^2.$$

Since $a_0a_3(a_0^8 - a_3^8) = 0$ in the field, are readily obtain the result:

$$\begin{aligned} E = {} & a_0^3a_1a_3^6 - a_0a_1^4a_3^5 + (a_0^2a_1a_2^3 - a_0a_1^3a_2^2)a_3^4 - (a_0^2a_2^5 + a_0a_1^2a_2^4 + a_0^6a_2)a_3^3 \\ & + (a_0a_1a_2^6 - a_0^4a_1^3a_2 + a_0^3a_1^5)a_3^2 + (a_0a_2^8 + a_0^5a_2^4 + a_0^4a_1^2a_2^3 + a_0^3a_1^4a_2^2 - a_0^2a_1^6a_2 - a_0a_1^8)a_3. \end{aligned} \tag{10}$$

For $p \neq 3$, we shall employ the cubic form (40) with binomial coefficients. Then π is the product of the functions $a_0 + 3a_1\rho + 3a_2\rho^2 + a_3\rho^3$, ρ ranging over the marks $\neq 0$. For $p^n = 2, 2^2, 2^3$, the expressions for π and E are given in § 23. For $p^n = 5$,

$$\pi = (a_0^2 - a_2^2 - a_1a_3)^2 - (a_1^2 - a_3^2 + a_0a_2)^2$$
$$= a_0^4 - a_1^4 + a_2^4 - a_3^4 + 2a_0^2a_2^2 - 2a_1^2a_3^2 + 2a_0a_2a_3^2 - 2a_0^2a_1a_3 + 2a_1a_2^2a_3 - 2a_0a_1^2a_2,$$

so that E is given by (90). For $p^n = 7$, it is convenient to set

$$\alpha = a_0^2 - a_3^2, \qquad \beta = a_1a_3 + 2a_2^2, \qquad \gamma = -a_0a_2 - 2a_1^2.$$

Forming the product of the factors by twos, viz., those with $\rho = \pm 1$, those with $\rho = \pm 2$, and those with $\rho = \pm 3$, we obtain the interesting result:

$$\pi \equiv (\alpha + \beta + \gamma)(\alpha + 2\beta + 4\gamma)(\alpha + 4\beta + 2\gamma) \equiv \alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma \pmod 7.$$

The resulting expression for E is given by (93).

5. THEOREM. *The weights* $\sigma = e_1 + 2e_2 + \cdots + me_m$ *of the various terms*

$$a_0^{e_0} a_1^{e_1} \cdots a_m^{e_m}$$

of an invariant ϕ *differ by multiples of* $\mu = p^n - 1$.

Here and in § 6, let ϕ become $\phi' = D^d \phi$ under every binary transformation of determinant D. Consider the special transformation (4) in which λ is a primitive root of the $GF[p^n]$. In view of (7), the above general term of ϕ is multiplied by λ^σ. Hence

$$\lambda^\sigma = \lambda^d, \qquad \sigma \equiv d \pmod{\mu}.$$

6. THEOREM. *The degrees* ρ *of the various terms of an invariant differ by multiples of* μ/δ, *where* δ *is the greatest common divisor of* m *and* $\mu = p^n - 1$. *Further,* $m\rho - 2\sigma$ *is a multiple of* μ.

For the transformation $x = \lambda x'$, $y = \lambda y'$, we have $a_i' = \lambda^m a_i$. Hence a term of degree ρ of ϕ is multiplied by $\lambda^{m\rho}$, so that $m\rho \equiv 2d \pmod{\mu}$.

7. The differential operators which annihilate an invariant are here more complicated than in the algebraic theory. This is due primarily to the fact that, in a series of powers of an arbitrary mark t of the $GF[p^n]$, certain terms now combine, viz., t^i, $t^{i+\mu}$, $t^{i+2\mu}$, $\cdots$, where $\mu = p^n - 1$. Since $a^{p^n} = a$ for every mark a, we may assume that the exponent of each a_i in an invariant does not exceed μ. Then, for the case $n = 1$, we can employ TAYLOR's theorem for the expansion of a polynomial of degree $\mu = p - 1$,

$$\psi(a + t) = \psi(a) + t\psi'(a) + \cdots + \frac{1}{i!} t^i \psi^{(i)}(a) + \cdots + \frac{1}{\mu!} t^\mu \psi^{(\mu)}(a),$$

since the denominator $i!$ is prime to the modulus p. The case $n > 1$ apparently (cf. § 10) offers a theoretical difficulty; it actually presents a difficulty from another source (§ 12). We therefore begin with examples illustrating the simpler case $n = 1$.

8. For the first illustrative example, consider the form

$$a_0 x^2 + a_1 xy + a_2 y^2, \tag{11}$$

in which the coefficients are integers taken modulo 3. Under the transformation (3), a_1 and a_2 receive, in view of (6), the respective increments

$$\alpha_1 = 2ta_0, \qquad \alpha_2 = ta_1 + t^2 a_0.$$

Let ϕ be a polynomial in a_0, a_1, a_2, with exponents $\leqq 2$. By TAYLOR's theorem,

$$\phi' - \phi = \alpha_1 \phi_{a_1} + \alpha_2 \phi_{a_2} + \tfrac{1}{2}\alpha_1^2 \phi_{a_1^2} + \alpha_1 \alpha_2 \phi_{a_1 a_2} + \tfrac{1}{2}\alpha_2^2 \phi_{a_2 a_2}$$
$$+ \tfrac{1}{2}\alpha_1^2 \alpha_2 \phi_{a_1^2 a_2} + \tfrac{1}{2}\alpha_1 \alpha_2^2 \phi_{a_1 a_2^2} + \tfrac{1}{4}\alpha_1^2 \alpha_2^2 \phi_{a_1^2 a_2^2},$$

where $\phi_{a_1^2 a_2}$ denotes $\partial^3 \phi / \partial a_1^2 \partial a_2$, etc. Then $\phi' - \phi = t\delta\phi + t^2 \delta_1 \phi$, where

$$\delta\phi \equiv 2a_0 \phi_{a_1} + a_1 \phi_{a_2} + 2a_0^2 \phi_{a_1 a_2} + a_0 a_1 \phi_{a_2^2} + 2a_0^2 a_1 \phi_{a_1^2 a_2} + (a_0 + a_0 a_1^2)\phi_{a_1 a_2^2} + 2a_0 a_1 \phi_{a_1^2 a_2^2},$$
$$\delta_1 \phi \equiv a_0 \phi_{a_2} + 2a_0^2 \phi_{a_1^2} + 2a_0 a_1 \phi_{a_1 a_2} - (a_0^2 + a_1^2)\phi_{a_2^2} + 2a_0 \phi_{a_1^2 a_2} + 2a_0^2 a_1 \phi_{a_1 a_2^2}$$
$$+ (a_0^2 + a_0^2 a_1^2)\phi_{a_1^2 a_2^2}.$$

A necessary condition for $\phi' = \phi$ is $\delta\phi = 0$. This condition is also sufficient. Indeed, if $[\delta\phi]$ denotes $\delta\phi$ in reduced form, i. e., with every exponent $\leqq 2$ (in virtue of $a^3 \equiv a$), then $\delta[\delta\phi] \equiv \delta_1 \phi$. To verify the latter, note that if ψ is a reduced function of a, viz., $\psi = ra^2 + sa + t$, then $[a\psi] \equiv ra + sa^2 + ta$, and $[a\psi]_a \equiv a\psi_a + \psi - \psi_{aa} \pmod 3$. Again, if $\chi = sa + t$, $[a^2\chi] \equiv sa + ta^2$, and

$$[a^2\chi]_a \equiv a^2 \chi_a + (a^2)_a \chi + \chi_a \pmod 3.$$

Proceeding to the computation of the invariants, we may set

$$\phi = \sum_{i,j}^{0,1,2} A_{ij} a_1^i a_2^j \qquad (A_{ij} \text{ quadratic functions of } a_0).$$

Employing the annihilator δ and giving to $\delta\phi$ its reduced form $[\delta\phi]$ with exponents $\leqq 2$, we require that $[\delta\phi] \equiv 0$, identically in a_1, a_2. Among the resulting conditions occur

$$A_{11} \equiv 0, \qquad A_{12} \equiv 0, \qquad A_{10} a_0 \equiv 0, \qquad A_{22} a_0 \equiv 0 \pmod 3.$$

In view of these, the remaining conditions reduce to

$$A_{21} a_0 - A_{22} - A_{02} \equiv 0, \qquad A_{20} a_0 + A_{21} + A_{01} - A_{21} a_0^2 - A_{02} a_0 \equiv 0.$$

From $A_{22} a_0 \equiv 0$ we get $A_{22} \equiv r(a_0^2 - 1)$, where r is a constant. The term $ra_0^2 a_1^2 a_2^2$ is unaltered by transformation (4), so that $r = 0$ unless ϕ is an absolute invariant. In the latter case we replace ϕ by $\phi - rI$, where I is the invariant given by (2). In either case, it remains to consider ϕ with $A_{22} \equiv 0$. Under transformation (5),

$$(8') \qquad a_0' = a_2, \qquad a_1' = -a_1, \qquad a_2' = a_0.$$

This replacement must leave ϕ unaltered. But by $A_{22} \equiv 0$, $a_1^2 a_2^2$ is a factor of no term of ϕ. Hence $a_1^2 a_0^2$ is a factor of no term of ϕ. We may therefore set

$$A_{21} = \alpha + \beta a_0, \qquad A_{20} = \lambda + \mu a_0 \qquad (\alpha, \beta, \lambda, \mu \text{ constants}).$$

Similarly, by $A_{12} \equiv 0$, $a_1 a_0^2$ is a factor of no term, so that $A_{10} \equiv 0$. The above long conditions now give

$$A_{02} = \alpha a_0 + \beta a_0^2, \qquad A_{01} = -\alpha + (\beta - \lambda) a_0 - (\alpha + \mu) a_0^2.$$

The conditions that ϕ shall be unaltered by (8′) are $\mu = \alpha$, $A_{00} = k - \alpha a_0$, k a constant. Hence

$$\phi = \beta a_0^2 a_2^2 + (\beta - \lambda) a_0 a_2 + \lambda a_1^2 + \beta a_0 a_2 a_1^2 + \alpha Q + k,$$

where

$$Q = a_0 a_2^2 + a_0^2 a_2 + a_1^2 a_2 + a_1^2 a_0 - a_2 - a_0. \tag{12}$$

The coefficient of λ is the discriminant $\Delta = a_1^2 - a_0 a_2$, that of β is congruent to $\Delta^2 - \Delta$. *As the linearly independent invariants of the quadratic form* (11) *modulo* 3, *we may take* I, Q, Δ, Δ^2. *Now* $I \equiv Q^2 + \Delta^2 - 1$. *As the independent invariants we may take* Q *and* Δ.

9. Consider the binary cubic $(1)_{m=3}$ with integral coefficients modulo 3. The discriminant Δ and (9) give the absolute invariants

$$\Delta = a_1^2 a_2^2 - a_0 a_2 - a_1 a_3, \qquad P = (a_1^2 - 1)(a_2^2 - 1). \tag{13}$$

Hence there is an absolute invariant of the second degree:

$$K \equiv \Delta - P + 1 = a_1^2 + a_2^2 - a_0 a_2 - a_1 a_3. \tag{14}$$

In view of (6), a_0 and a_1 are unaltered by transformation (3), while a_2 and a_3 have the respective increments $2ta_1$ and $t(a_0 + a_2) + t^2 a_1$. The coefficient of t in $\phi' - \phi$ gives the annihilator

$$\begin{aligned} &2a_1\phi_{a_2} + (a_0 + a_2)\phi_{a_3} - a_1^2\phi_{a_2 a_3} + a_1(a_0 + a_2)\phi_{a_3^2} - a_1^2(a_0 + a_2)\phi_{a_2^2 a_3} \\ &\quad + \{a_1 + a_1(a_0 + a_2)^2\}\phi_{a_2 a_3^2} - a_1(a_0 + a_2)\phi_{a_2^2 a_3^2}. \end{aligned} \tag{15}$$

By §§ 5, 6, the terms of an invariant ϕ are all of even degrees, while all are of even or all of odd weights. Further, by (5) and (8), ϕ must be unaltered by the substitution

$$a_0' = -a_3, \qquad a_3' = a_0, \qquad a_1' = a_2, \qquad a_2' = -a_1. \tag{16}$$

Let first the weights be even, so that by § 5, $e_1 \equiv e_3 \pmod 2$ in every term of ϕ. Then, in view of (16), ϕ must have the form

$$\begin{aligned} &b(a_0^2 + a_3^2) + c(a_1^2 + a_2^2) + d(a_0 a_2 + a_1 a_3) + e a_0^2 a_3^2 + f a_1^2 a_2^2 + g(a_0^2 a_1^2 + a_2^2 a_3^2) \\ &\quad + h(a_0^2 a_2^2 + a_1^2 a_3^2) + j(a_0^2 a_1 a_3 + a_3^2 a_0 a_2) + k(a_1^2 a_0 a_2 + a_2^2 a_1 a_3) + l a_0 a_1 a_2 a_3 \\ &\quad + m(a_0^2 a_1^2 a_2^2 + a_1^2 a_2^2 a_3^2) + n(a_0^2 a_1^2 a_3^2 + a_0^2 a_2^2 a_3^2) + q(a_0^2 a_2^2 a_1 a_3 + a_1^2 a_3^2 a_0 a_2) \\ &\quad + r a_0^2 a_1^2 a_2^2 a_3^2. \end{aligned}$$

By subtracting from ϕ suitable multiples of P, K, Δ^2, and I, given by (2), we may set $c = d = k = r = 0$. Then (15) vanishes if and only if

$$n = f = 0, \qquad g = q = e = h = -m, \qquad b = j = l = m.$$

The coefficient of m is the absolute invariant

$$T = a_0^2 + a_3^2 - a_0^2a_3^2 - a_0^2a_1^2 - a_2^2a_3^2 - a_0^2a_2^2 - a_1^2a_3^2 + a_0^2a_1a_3 + a_0a_2a_3^2$$
$$+ a_0a_1a_2a_3 + a_0^2a_1^2a_2^2 + a_1^2a_2^2a_3^2 - a_0^2a_1a_2^2a_3 - a_0a_1^2a_2a_3^2.$$

Finally, when the weights are odd, the possible terms are

$$B(a_0a_1 - a_2a_3) + G(a_0^2a_1a_2 - a_3^2a_1a_2) + H(a_0^2a_2a_3 - a_3^2a_0a_1)$$
$$+ J(a_1^2a_0a_3 - a_2^2a_0a_3) + K(a_1^2a_2a_3 - a_2^2a_0a_1) + L(a_0^2a_1^2a_2a_3 - a_3^2a_2^2a_0a_1).$$

The conditions that (15) shall vanish are $B = G = K = L = 0$, $J = H$. Hence

$$E = a_1^2a_0a_3 - a_2^2a_0a_3 + a_0^2a_2a_3 - a_3^2a_0a_1 \tag{17}$$

is the only invariant of odd weights. By §§ 4, 27, the cubic form is irreducible modulo 3 if, and only if, $E \neq 0$.

The relations between T and the remaining invariants are

$$IT = 0, \qquad PT = P - I, \qquad \Delta T = I - P + T, \qquad ET = E, \qquad T^2 = T.$$

We obtain the simpler relations * (19) by introducing $W \equiv I - P + T$, viz.,

$$W = a_3^2(a_0^2a_1^2a_2^2 - a_0^2a_2^2 - a_0a_1^2a_2 + a_0a_2 - a_0^2a_1^2) + a_3(-a_0^2a_1a_2^2 + a_0a_1a_2 + a_0^2a_1). \tag{18}$$

THEOREM. *As a complete set of linearly independent invariants of the cubic form modulo* 3, *we may take* I, P, Δ, Δ^2, E, W. *The product of any two invariants can be reduced to a linear function of these six by means of*

$$\begin{cases} I^2 = I, \quad IP = I, \quad I\Delta = IE = IW = 0, \quad P^2 = P, \quad P\Delta = PE = PW = 0, \\ \Delta^3 = \Delta, \quad \Delta E = E, \quad \Delta W = W, \quad E^2 = W, \quad EW = E, \quad W^2 = W. \end{cases} \tag{19}$$

10. The next illustrative examples relate to the invariants of a quantic in the $GF[p^n]$, $n > 1$. We require the expansion of $\psi(a + t)$ in powers of t, where $\psi(a)$ is a polynomial of degree at most $\mu = p^n - 1$. This can be done by TAYLOR's theorem, the coefficient of t^i being the quotient obtained algebraically by dividing $\psi^{(i)}(a)$, all of whose coefficients are exact multiples of $i!$, by the number $i!$. Similar remarks apply to TAYLOR's theorem for two or more variables.

* Conforming with the relations (§ 21) between the invariants of the cubic form in the $GF[9]$. Note that $I - P + T$ and $-\Delta - \Delta^2$ are the only invariants W satisfying the relations (19) involving W. Hence there is a single invariant W which can replace T.

11. Consider the quadratic form (11) with coefficients in the $GF[2^n]$. Under the transformation (3), a_0 and a_1 are unaltered while the increment to a_2 is $ta_1 + t^2 a_0$. Let ϕ be a polynomial in a_0, a_1, a_2, with exponents $\leqq 2^n - 1$.

Let first $n = 2$. Then the coefficient of t in $\phi' - \phi$ equals

$$a_1 \phi_{a_2} + a_0^2 [\tfrac{1}{2}\phi_{a_2^2}] + a_0 a_1^2 [\tfrac{1}{6}\phi_{a_2^3}],$$

where the indicated divisions of the derivatives by 2 and 6 are to be performed algebraically and the quotients only are interpreted in the $GF[4]$. If we set

$$\phi = \sum_{i=0}^{3} B_i a_2^i \qquad (B\text{'s functions of } a_0, a_1),$$

the above expression for the coefficient of t becomes, in the $GF[4]$,

$$a_1(B_1 + B_3 a_2^2) + a_0^2(B_2 + B_3 a_2) + a_0 a_1^2 B_3.$$

This must vanish identically in a_2. From $a_1 B_3 = a_0 B_3 = 0$ we get

$$B_3 = r(a_0^3 - 1)(a_1^3 - 1) \qquad (r = \text{constant}).$$

The remaining condition* becomes $a_1 B_1 + a_0^2 B_2 = 0$. Replacing ϕ by $\phi - rI$, where I is the absolute invariant (2), we may set $B_3 = 0$. By (8), ϕ must be symmetrical in a_0 and a_2. Hence no term of ϕ contains the factor a_2^3 or the factor a_0^3. By §§ 5, 6, the degrees ρ (and likewise the weights σ) of the various terms of ϕ differ by multiples of 3, and $\rho \equiv \sigma \pmod 3$.

Let first $\rho \equiv \sigma \equiv 0$. Then ϕ involves only the terms a_1^3, $a_0 a_1 a_2$, $a_0^2 a_1^2 a_2^2$, so that

$$B_0 = l a_1^3, \qquad B_1 = m a_0 a_1, \qquad B_2 = k a_0^2 a_1^2.$$

Then $m = k$ by $a_1 B_1 + a_0^2 B_2 = 0$. The linearly independent invariants† are

$$I, \qquad a_1^3, \qquad J = a_0 a_1 a_2 + a_0^2 a_1^2 a_2^2.$$

For $\rho \equiv \sigma \equiv 1$, ϕ involves only the terms a_1, $a_0 a_1^2 a_2$, $a_0^2 a_2^2$, $a_0^2 a_1^3 a_2^2$, so that

$$B_0 = b a_1, \qquad B_1 = c a_0 a_1^2, \qquad B_2 = d a_0^2 + e a_0^2 a_1^3.$$

By $a_1 B_1 + a_0^2 B_2 = 0$, we get $d = 0$, $e = c$. The invariants† are a_1 and $a_1 J$.

The case $\rho \equiv \sigma \equiv 2$ may be reduced to the preceding by squaring, an operation here reversible. Hence the invariants are a_1^2 and $a_1^2 J$.

---

* The corresponding conditions from the coefficient of t^2 are $a_1 B_3 = a_0 B_3 = 0$, $a_0 B_1 + a_1^2 B_2 = 0$. Thus the vanishing of the coefficient of t does not imply the vanishing of that of t^2 (but does that of t^3, viz., $a_0^3 B_3 + a_1^3 B_3$). In this respect the case of invariants in the $GF[p^n]$, $n > 1$, appears to be in contrast to the case $n = 1$.

† The remaining conditions from t^2 (preceding foot-note) are satisfied.

For general n, we can prove (cf. §§ 4, 26) the following

THEOREM. *The independent invariants of the quadratic form* (11) *in the* $GF[2^n]$ *may be taken to be* a_1 (*which is multiplied by the determinant of the transformation*), I *and the absolute invariant*

$$(20) \qquad \chi(a_0 a_2 a_1^{2^n-3}), \qquad \textit{where} \qquad \chi(c) \equiv c + c^2 + c^4 + \cdots + c^{2^{n-1}},$$

the last invariant being replaced by $a_0 a_2 a_1$ *in the case* $n = 1$.

12. The illustrative examples employed thus far were chosen on account of their comparative simplicity. As the order of the $GF[p^n]$ increases, the complexity of the computation increases very rapidly. Certain remarks will be found to be very useful. While, for $n > 1$, certain divisions are called for in the definition of an annihilator δ (see §§ 10, 11), the actual performance of the divisions can be dispensed with in the computation of $\delta\phi$. For example, in the $GF[3^2]$ the result of operating on a term $a_0^e a_1^i a_2^j$ by $1/3!\ 1/2!\ \partial^5/\partial a_1^3 \partial a_2^2$ is zero unless $i \geqq 3$, $j \geqq 2$, while in the latter case the result is $C_{i3} C_{j2} a_0^e a_1^{i-3} a_2^{j-2}$. It thus suffices to have a table* of the residues modulo 3 of the binomial coefficients C_{rs} $(r, s \leqq 8)$. In view of this remark, the following permanent notations will be used for the terms of $\delta\phi$, the notations not exhibiting the numerical divisors. We shall set

$$(21) \qquad (1^i) = \frac{1}{i!}\frac{\partial^i \phi}{\partial a_1^i}, \qquad (1^i 2^j) = \frac{1}{i!\ j!}\frac{\partial^{i+j}\phi}{\partial a_1^i \partial a_2^j}, \quad \text{etc.}$$

In the algebraic theory ϕ is invariant under the special transformation (3) if, and only if, $\delta\phi = 0$, where $\delta\phi$ denotes the coefficient of t in $\phi' - \phi$. The same theorem appears to hold in the modular theory when $n = 1$. In a few cases (cf. § 8), I have made a direct proof of this statement; in very many cases I have secured indirect verification in showing that the computed functions are actually invariants. But for $n > 1$ such a theorem does not hold (cf. foot-notes to § 11). However, it appears to be true that the vanishing of the coefficients of $t, t^p, t^{p^2}, \cdots, t^{p^{n-1}}$ implies the vanishing of the coefficients of the remaining powers (the last one being $p^n - 1$). The importance of this conjecture as a guide in actual computations is obvious; its effect on the explicit form of the invariant is noted in § 17.

13. Consider the quadratic form (11) in the $GF[3^2]$. Under transformation (3), a_1 and a_2 receive the respective increments $\delta_1 = -ta_0$, $\delta_2 = ta_1 + t^2 a_0$. Expanding the powers $\delta_2^2, \cdots, \delta_2^8$, reducing the coefficients modulo 3 and the exponents by means of $x^9 = x$, we write down by inspection (in view of the

* The residues of multinomial coefficients modulo p, a prime, are obtained rapidly by means of two general theorems given in the writer's dissertation, Annals of Mathematics, ser. 1, vol. 11 (1896–7), pp. 75, 76.

simple form of δ_1) the coefficient (22) of t and the coefficient (23) of t^3 in $\phi' - \phi$, employing the notations (21).

$$
(22)\quad
\begin{aligned}
&- a_0(1) + a_1(2) - a_0^8(1^7 2) + a_0^8 a_1(1^8 2) - a_0^7(1^5 2^2) - a_0^7 a_1(1^6 2^2)\\
&- a_0^7 a_1^2(1^7 2^2) - a_0^6(1^3 2^3) + a_0^6 a_1^3(1^6 2^3) - a_0^5(12^4) + a_0^5 a_1(1^2 2^4)\\
&+ a_0^5 a_1^3(1^4 2^4) - a_0^5 a_1^4(1^5 2^4) - a_0^4 a_1(2^5) - a_0^4 a_1^2(12^5) + a_0^4 a_1^3(1^2 2^5)\\
&+ a_0^4 a_1^4(1^3 2^5) + a_0^4 a_1^5(1^4 2^5) - a_0^4(1^7 2^5) - a_0^4 a_1(1^8 2^5) - a_0^3 a_1^3(2^6)\\
&- a_0^3 a_1^6(1^3 2^6) - a_0^3(1^5 2^6) - a_0^3 a_1^3(1^8 2^6) - a_0^2 a_1^6(12^7) + a_0^2 a_1^7(1^2 2^7)\\
&- a_0^2(1^3 2^7) + a_0^2 a_1(1^4 2^7) - a_0^2 a_1^3(1^6 2^7) + a_0^2 a_1^4(1^7 2^7) - a_0 a_1^7(2^8)\\
&- (a_0 + a_0 a_1^8)(12^8) - a_0 a_1(1^2 2^8) - a_0 a_1^2(1^3 2^8) - a_0 a_1^3(1^4 2^8)\\
&- a_0 a_1^4(1^5 2^8) - a_0 a_1^5(1^6 2^8) - a_0 a_1^6(1^7 2^8) - a_0 a_1^7(1^8 2^8).
\end{aligned}
$$

$$
(23)\quad
\begin{aligned}
&- a_0^3(1^3) - a_0^2(12) + a_0^2 a_1(1^2 2) - a_0 a_1(2^2) - a_0 a_1^2(12^2) - a_0(1^7 2^2)\\
&- a_0 a_1(1^8 2^2) + a_1^3(2^3) - a_0^8(1^5 2^3) + a_0^8 a_1^3(1^8 2^3) - a_0^7(1^3 2^4)\\
&+ a_0^7 a_1(1^4 2^4) + a_0^7 a_1^3(1^6 2^4) - a_0^7 a_1^4(1^7 2^4) - a_0^6(12^5) - a_0^6 a_1(1^2 2^5)\\
&- a_0^6 a_1^2(1^3 2^5) + a_0^6 a_1^3(1^4 2^5) + a_0^6 a_1^4(1^5 2^5) + a_0^6 a_1^5(1^6 2^5) - a_0^5 a_1^3(1^2 2^6)\\
&- a_0^5 a_1^6(1^5 2^6) - a_0^5(1^7 2^6) - a_0^4 a_1^3(2^7) + a_0^4 a_1^4(12^7) - a_0^4 a_1^6(1^3 2^7)\\
&+ a_0^4 a_1^7(1^4 2^7) - a_0^4(1^5 2^7) + a_0^4 a_1(1^6 2^7) - a_0^4 a_1^3(1^8 2^7) - a_0^3 a_1^5(2^8)\\
&- a_0^3 a_1^6(12^8) - a_0^3 a_1^7(1^2 2^8) - (a_0^3 + a_0^3 a_1^8)(1^3 2^8) - a_0^3 a_1(1^4 2^8)\\
&a_0^3 a_1^2(1^5 2^8) - a_0^3 a_1^3(1^6 2^8) - a_0^3 a_1^4(1^7 2^8) - a_0^3 a_1^5(1^8 2^8).
\end{aligned}
$$

We may set

$$
(24)\qquad \phi = \sum_{i,j}^{0,\cdots,8} A_{ij} a_1^i a_2^j \qquad (A\text{'s functions of } a_0).
$$

Since a_2^8 occurs only in the first term of (22) and (23), we get

$$
A_{i8} a_0 = 0 \qquad (i = 1, \cdots, 8).
$$

Employing these to simplify the coefficient of a_2^7 in the first four terms of (22) and that in the first three terms of (23), we get

$$
- a_0 \sum i A_{i7} a_1^{i-1} - a_1 \sum A_{i8} a_1^i, \qquad - a_0^3 \sum C_{i3} A_{i7} a_1^{i-3}.
$$

These must vanish identically in $a_1 = a_1^0$. Hence

$$
A_{i7} a_0 = 0 \ (i = 1, 3, \cdots, 8), \quad A_{i8} = 0 \ (i = 1, \cdots, 7), \quad A_{27} a_0 - A_{08} - A_{88} = 0.
$$

Before examining the further conditions we shall introduce a decided simplification. From $A_{88} a_0 = 0$, $A_{88} = k(a_0^8 - 1)$, k a constant. But the term

$ka_0^8 a_1^8 a_2^8$ of ϕ is unaltered by transformation (4). Hence k is zero unless ϕ is an absolute invariant. In the latter case, we replace ϕ by $\phi - kI$, where I is the absolute invariant (2). In either case we have $A_{88} = 0$. *Then* $A_{ij} a_0 = 0$ $(i \neq 0)$ *implies* $A_{ij} = 0$. For, then $A_{ij} = c(a_0^8 - 1)$, while $a_0^8 a_1^i a_2^j$ is not a term of ϕ. If it were, then would $a_0^j a_1^i a_2^8$ occur in ϕ since ϕ must be unaltered by (5) and hence by (8′) of § 8, whereas $A_{i8} = 0$ $(i = 1, \cdots, 8)$.

We next examine the coefficients of a_2^6 in (22) and (23), then those of a_2^5, etc. At each step we utilize the conditions previously found. In view of the above result and the fact that the binomial coefficients C_{i2} $(i = 3, 4, 6, 7)$, C_{64}, C_{65}, C_{75} are multiples of the modulus 3, the determination of the coefficient of a_2^k may be done readily by inspection. The resulting conditions,* including the earlier ones, are:

$$
\begin{aligned}
&A_{i8} = 0\,(i = 1, \cdots, 8),\ A_{i7} = 0(i = 1, 3, \cdots, 8),\ A_{i6} = 0(i = 1, 3, 5, 6, 7, 8),\\
&A_{i5} = 0\,(i = 1, 2, 3, 4, 5, 7, 8),\ A_{i4} = 0\,(i = 1, 3, 4, 5, 7),\\
&A_{i3} = 0\,(i = 1, 3, 5, 7),\ A_{i2} = 0\,(i = 1, 2, 3, 5, 7, 8),\\
&A_{i1} = 0\,(i = 1, 3, 5, 7),\ A_{i0} = 0\,(i = 1, 3, 5, 7);
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
&A_{08} = A_{27} a_0,\ A_{27} = A_{46} a_0,\ A_{07} = -A_{26} a_0,\ A_{65} = A_{84} a_0,\ A_{27} = A_{84} a_0^3,\\
&A_{05} = A_{24} a_0,\ A_{07} = A_{64} a_0^3,\ A_{64} = -A_{83} a_0,\ A_{24} = A_{43} a_0,\ A_{26} = A_{83} a_0^3,\\
&A_{06} = A_{63} a_0^3,\ A_{46} = A_{84} a_0^2,\ A_{05} = -A_{62} a_0^3,\ A_{24} = -A_{81} a_0^3,\ A_{64} = A_{41} a_0^3,\\
&A_{42} = A_{84} a_0^6,\ A_{62} = A_{81} a_0,\ A_{02} = A_{21} a_0,\ A_{23} a_0 + A_{04} + A_{84} + A_{84} a_0^8 = 0,\\
&A_{61} a_0^3 = A_{23} a_0,\ A_{41} = -A_{83} a_0^6,\ A_{21} = A_{40} a_0 - A_{63} a_0^6,\ A_{61} + A_{80} a_0 + A_{27} a_0^2 = 0,\\
&A_{20} a_0 + A_{01} + A_{81} + A_{81} a_0^8 = 0,\ A_{63} = A_{40} a_0^3 - A_{21} a_0^2,\ A_{43} = -A_{81} a_0^2,\\
&A_{23} + A_{80} a_0^3 + A_{84} a_0^7,\ A_{03} + A_{83} + A_{60} a_0^3 - A_{07} a_0^4 = 0.
\end{aligned}
\tag{26}
$$

We note that each A_{ij} not in the set (25) has a non-vanishing value in at least one invariant given below.

In the notations of § 5, we have here

$$
e_1 + 2e_2 \equiv 2e_0 + e_1 \equiv d \pmod 8. \tag{27}
$$

Consider first the absolute invariants, so that $d \equiv 0$. For $e_1 = 8$, $e_2 = 0$ or 4, we have $e_0 = 0$, 4 or 8. The value $e_0 = 8$ is excluded, since (as shown above) $a_0^8 a_1^8 a_2^{e_2}$ is not a term of ϕ. Hence we may set

$$
A_{84} = e + f a_0^4, \qquad A_{80} = l + m a_0^4, \qquad A_{00} = \lambda + \mu a_0^4 + \nu a_0^8.
$$

* Those in set (25) and likewise the remaining ones occur approximately in the order of their determination. Obvious simplifications have been made in the latter conditions. Certain conditions have been omitted as being simple consequences of those retained.

By (27) the only further non-vanishing A_{ij} are the following, whose values are derived at once from the non-identical conditions (26):

$$A_{65} = ea_0 + fa_0^5, \quad A_{27} = ea_0^3 + fa_0^7, \quad A_{08} = ea_0^4 + fa_0^8, \quad A_{46} = ea_0^2 + fa_0^6,$$

$$A_{42} = ea_0^6 + fa_0^2, \quad A_{23} = -(f+l)a_0^3 - (e+m)a_0^7, \quad A_{61} = -(f+l)a_0 - (e+m)a_0^5,$$

$$A_{04} = -e + (l-f)a_0^4 + ma_0^8.$$

The necessary and sufficient conditions that the resulting function (24) shall be unaltered by (5) and hence by (8′) are $m = e$, $\mu = -e$, $\nu = 0$. The independent parameters are thus e, f, l. The coefficients of l and f are Δ^4 and $\Delta^8 - \Delta^4$, respectively, where

$$\Delta = a_1^2 - a_0 a_2 \tag{28}$$

is the discriminant of the quadratic form. The coefficient of e is

$$\begin{aligned} Q = {} & a_1^8 a_2^4 + a_0^4 a_1^8 - a_0^4 + a_0 a_1^6 a_2^5 + a_0^3 a_1^2 a_2^7 + a_0^4 a_2^8 \\ & + a_0^2 a_1^4 a_2^6 + a_0^6 a_1^4 a_2^2 + a_0^7 a_1^2 a_2^3 + a_0^5 a_1^6 a_2 - a_2^4 + a_0^8 a_2^4. \end{aligned} \tag{29}$$

By way of check we note here the important relations

$$Q^2 + \Delta^8 - 1 = I, \qquad Q\Delta = 0. \tag{30}$$

Next, there is no invariant with $d \equiv 1 \pmod 8$. Indeed, each of the nine $A_{e_1 e_2}$ for which $e_1 + 2e_2 \equiv 1 \pmod 8$ is zero by (25). There is no invariant with $d \equiv 3$; for, if so, its cube would be an invariant with $d \equiv 1$. Similarly, the cases $d \equiv 7$, $d \equiv 5$ are excluded.

In an invariant with $d \equiv 2 \pmod 8$, the possible non-vanishing A_{ij} are $A_{20}, A_{01}, A_{81}, A_{62}, A_{43}, A_{24}, A_{05}$. Now A_{81} is a linear function of a_0 and a_0^5 by (27). But $a_0^5 a_1^8 a_2$ is not a term of ϕ in view of (8′). Hence $A_{81} = la_0$. Similarly, $A_{20} = r + sa_0^4$. Then the seven non-vanishing conditions (26) give

$$A_{62} = la_0^2, \quad A_{05} = -la_0^5, \quad A_{24} = -la_0^4, \quad A_{43} = -la_0^3, \quad A_{01} = (l-r)a_0 - sa_0^5.$$

The resulting function ϕ must be unaltered by (8′); hence $s = 0$. The coefficients of r and l are Δ and $\Delta^5 - \Delta$, respectively. By cubing, we conclude that the only invariants with $d \equiv 6 \pmod 8$ are the linear functions of Δ^3 and Δ^7.

Finally, for $d \equiv 4$, every A_{ij} vanishes except $A_{40}, A_{21}, A_{02}, A_{63}, A_{06}$, while $A_{40} = c$, $A_{21} = ka_0$. The four non-vanishing conditions (26) give

$$A_{02} = ka_0^2, \qquad A_{63} = (c-k)a_0^3, \qquad A_{06} = (c-k)a_0^6.$$

For $c = k = 1$, $\phi = \Delta^2$; for $c = 1$, $k = 0$, $\phi = \Delta^6$.

THEOREM. *Every invariant of a quadratic form in the* $GF[9]$ *is a linear function of* I, Q, $\Delta^i (i = 1, \cdots, 8)$. *As independent invariants we may take* Q *and* Δ.

14. The results of §§8, 13 on the invariants of a quadratic form in the $GF[3^n]$, $n = 1, 2$, may be given a simpler form, better adapted to the extension to the case of general n. We set

$$(31) \qquad J = Q + \Delta^2 - 1 \text{ (for } n = 1), \qquad J = Q + \Delta^8 - 1 \text{ (for } n = 2).$$

Then in each case

$$(32) \qquad J\Delta = 0, \qquad J^2 - J = I,$$

as follows from (30) and the similar relations for $n = 1$. The importance of the introduction of the invariant J lies in the simplicity of relations (32) and in the factorizations

$$(33) \qquad J = (a_0 + 1)(a_2 + 1)(a_1^2 + a_0 a_2 - 1) \qquad \text{(for } n = 1),$$

$$(34) \qquad J = (a_0^4 + 1)(a_2^4 + 1)(a_1^8 + a_0^4 a_2^4 + a_0 a_2 a_1^6 + a_0^3 a_2^3 a_1^2 + a_0^2 a_2^2 a_1^4 - 1) \qquad \text{(for } n = 2).$$

For general n, the corresponding function is

$$(35) \qquad J = (a_0^\nu + 1)(a_2^\nu + 1)\left\{ \sum_{i=0}^{\nu} a_0^i a_2^i a_1^{2\nu - 2i} - 1 \right\},$$

where $\nu = \frac{1}{2}(3^n - 1)$. The discussion in § 15 will apply here since (28) and (35) are unaltered modulo 3 when a_1 is multiplied by 2.

15. Consider the quadratic form* in the $GF[p^n]$, $p > 2$,

$$(36) \qquad a_0 x^2 + 2a_1 xy + a_2 y^2.$$

We investigate the function J given by (35) for $\nu = \frac{1}{2}(p^n - 1)$. It is obviously unaltered by the substitutions (7) and (8), so that the proof of its absolute invariance† depends only upon the verification that it is unaltered by the transformation (3). Let J' denote the function (35) of the transformed coefficients a_0', a_1', a_2'. Here $a_0' = a_0$, $\Delta' = \Delta$. As shown below, $J\Delta = 0$, $J'\Delta' = 0$. Hence if $\Delta \neq 0$, $J = J' = 0$. Let next $\Delta = 0$, so that $a_1^2 = a_0 a_2$, $a_1'^2 = a_0' a_2'$. Then the sum in (35) equals $(\nu + 1) a_0^\nu a_2^\nu$. Now $\nu + 1 = \frac{1}{2}(p^n + 1)$ is the mark $\frac{1}{2}$ in the field. Hence the final factor of J equals $\frac{1}{2} a_0^\nu a_2^\nu - 1$. If a_0 is a not-square, $a_0^\nu = -1$, so that $J = 0$, $J' = 0$. If $a_0 = 0$, then $a_1 = 0$, while

* If we employ (11) instead of (36), we must introduce the factor 4^i under the summation sign in (35). For the essentially distinct case $p = 2$, here excluded, see § 11.

† The existence of an invariant independent of I and Δ follows from the canonical form theory (§ 26).

the form $a_2 y^2$ is unaltered by transformation (3). Finally, let a_0 be a square $\neq 0$, so that $a_0^\nu = 1$. Then

$$J = (a_2^\nu + 1)(a_2^\nu - 2), \qquad J' = (a_2'^\nu + 1)(a_2'^\nu - 2).$$

Since $a_0 a_2$ is a square or zero, a_2 is a square or zero, so that $a_2^\nu = 1$ or 0; in either case $J = -2$. Likewise $J' = -2$. We have now proved the following

THEOREM. *The quadratic form* (36) *in the* $GF[p^n]$, $p < 2$, *has the absolute invariant* J *defined by* (35) *for* $\nu = \frac{1}{2}(p^n - 1)$.

We next prove the following relations, which reduce to (32) for $p = 3$:

$$J\Delta = 0, \qquad (J+1)^2 = I + 1, \tag{37}$$

so that, *for the functions obtained from* I *and* J *by deleting their constant terms* -1, *the one is the square of the other.* We here have

$$\Delta = a_1^2 - a_0 a_2, \qquad I = (a_0^{2\nu} - 1)(a_1^{2\nu} - 1)(a_2^{2\nu} - 1). \tag{38}$$

Since $a^{2\nu+1} = a$, a simple change of summation indices gives

$$J\Delta = P(a_0 a_2 - a_0^{\nu+1} a_2^{\nu+1}), \qquad P \equiv (a_0^\nu + 1)(a_2^\nu + 1).$$

Then $J\Delta = 0$ follows from

$$a_0^{\nu+k} a_2^{\nu+k} P = (a_0^{2\nu+k} + a_0^{\nu+k})(a_2^{2\nu+k} + a_2^{\nu+k}) = (a_0^k + a_0^{\nu+k})(a_2^k + a_2^{\nu+k}) = a_0^k a_2^k P \quad (k > 0). \tag{39}$$

To determine J^2, call σ the last factor, $\sum - 1$, in (35). Then

$$(a_2^\nu + 1)^2 \sigma^2 = \{a_2^{2\nu} - 1 + 2(a_2^\nu + 1)\}\sigma^2 = -(a_2^{2\nu} - 1)(a_1^{2\nu} - 1) + 2(a_2^\nu + 1)\sigma^2,$$

since $(a^{2\nu} - 1)a = 0$. Modifying $(a_0^\nu + 1)^2$ similarly, we get

$$\begin{aligned} J^2 &= -I - 2(a_0^\nu + 1)(a_2^{2\nu} - 1)(a_1^{2\nu} - 1) + 2(a_2^\nu + 1)\{-(a_0^{2\nu} - 1)(a_1^{2\nu} - 1) + 2(a_0^\nu + 1)\sigma^2\} \\ &= -I + 2P\{(a_1^{2\nu} - 1)(2 - a_0^\nu - a_2^\nu) + 2\sigma^2\}. \end{aligned}$$

Now

$$\sigma = a_1^{2\nu} - 1 + a_0^\nu a_2^\nu + \Sigma, \qquad \Sigma \equiv \sum_{i=1}^{\nu-1} a_0^i a_2^i a_1^{2\nu-2i},$$

$$\sigma^2 = -(a_1^{2\nu} - 1) + 2(a_1^{2\nu} - 1)a_0^\nu a_2^\nu + a_0^{2\nu} a_2^{2\nu} + 2a_0^\nu a_2^\nu \Sigma + \Sigma^2.$$

Since $2J = 2P\sigma$, we have

$$J^2 + 2J = -I + 2P\{(a_1^{2\nu} - 1)(1 - a_0^\nu - a_2^\nu + a_0^\nu a_2^\nu) + L\} = I + 2PL,$$

$$L \equiv a_0^\nu a_2^\nu + 2a_0^{2\nu} a_2^{2\nu} + 3a_0^\nu a_2^\nu (a_1^{2\nu} - 1) + 4a_0^\nu a_2^\nu \Sigma + \Sigma + 2\Sigma^2.$$

Hence (37_2) will follow if we show that $PL = 0$. In view of (39),

$$PL = P(3a_0^\nu a_2^\nu a_1^{2\nu} + 5\Sigma + 2\Sigma^2).$$

The second member may be shown, by means of (39), to equal

$$P(2\nu+1)(a_0^\nu a_2^\nu a_1^{2\nu}+\Sigma),$$

and hence vanishes in the $GF[p^n]$, since $2\nu+1=p^n$.

16. This section and the four succeeding sections will be devoted primarily to the exhibition of a remarkable absolute invariant K of degree $\mu=p^n-1$ of the binary cubic form in the $GF[p^n]$, and to the simple relation between K and the discriminant Δ of the form. Except for the case $p=3$ (cf. § 9), we shall write the form with binomial coefficients

$$a_0x^3+3a_1x^2y+3a_2xy^2+a_3y^3. \tag{40}$$

Under transformation (3), a_1, a_2, a_3 take the respective increments

$$\delta_1=ta_0,\qquad \delta_2=2ta_1+t^2a_0,\qquad \delta_3=3ta_2+3t^2a_1+t^3a_0. \tag{41}$$

For each term (§ 5) of an absolute invariant ϕ we have

$$3e_0+2e_1+e_2=r\mu,\qquad e_1+2e_2+3e_3=s\mu \qquad (r,\ s \text{ integers}). \tag{42}$$

The condition that the term shall be of degree μ is $r+s=3$. For $r=3$ and $r=0$, we obtain the respective terms

$$a_0^\mu,\qquad a_3^\mu. \tag{43}$$

The general term with $r=1$ is

$$a_0^{e_0}a_1^{e_1}a_2^{e_2}a_3^{e_3},\qquad 3e_0+2e_1+e_2=\mu,\qquad e_1+2e_2+3e_3=2\mu. \tag{44}$$

In view of (5), an invariant ϕ must be unaltered by the substitution

$$a_0'=-a_3,\qquad a_3'=a_0,\qquad a_1'=a_2,\qquad a_2'=-a_1. \tag{45}$$

For $p=2$, the signs may be taken positive. For $p>2$, μ is even, and hence e_0, e_2 are both even or both odd. Further, we cannot have $e_0=e_3$ and $e_1=e_2$ in a term (44). Hence each term (44) of ϕ is accompanied by another term, not of type (44), having the same coefficient as the former. These new terms give all the terms of ϕ for which $r=2$, $s=1$ in (42).

In all the cases, viz., for $p^n=2^n(n=1,\cdots,5)$, 3^n (n general), 5, 7, 11, 13, in which I have actually constructed an absolute *invariant* K of degree $\mu=p^n-1$, I find that the two terms (43) do not occur. Excluding those terms, we have the simpler* problem: *to construct an absolute invariant K whose terms are those given by* (44) *with suitable coefficients, and those derived by applying the substitution* $(a_0a_3)(a_1a_2)$.

* Since the exponent of a_3 is at most $\frac{2}{3}\mu$. By (41), the powers of δ_3 are complicated.

The results for the case $n = 1$ are very simple; all the terms just mentioned actually occur in K. *The cubic form with coefficients modulo p, a prime, has an absolute invariant K of degree $p-1$ containing $2k$ terms, where k is the number of partitions of $p-1$ into $3e_0 + 2e_1 + e_2$.* For $p = 3$, K is given by (14). For $p \neq 3$, the results relate to the notation (40) for the cubic form. Then, for $p = 2$, $K = a_1 + a_2$. For $p = 5, 7, 11, 13$, the expressions for K follow in that order. The first terms in the parentheses are the terms (44), arranged so that if two terms have different exponents to a_0, that with the lesser precedes, while, if the exponents of a_0 are equal the term with the lesser exponent to a_1 precedes. The second term in any parenthesis is derived from the first by the substitution $(a_0 a_3)(a_1 a_2)$.

$$(46)\quad (a_2^4+a_1^4)-(a_1a_2^2a_3+a_0a_1^2a_2)-(a_1^2a_3^2+a_0^2a_2^2)+(a_0a_2a_3^2+a_0^2a_1a_3),\quad GF[5].$$

$$(47)\quad \begin{aligned}&(a_2^6+a_1^6)+3(a_1a_2^4a_3+a_0a_1^4a_2)+3(a_1^2a_2^2a_3^2+a_0^2a_1^2a_2^2)+(a_1^3a_3^3+a_0^3a_2^3)\\&+3(a_0a_2^3a_3^2+a_0^2a_1^3a_3)+2(a_0a_1a_2a_3^3+a_0^3a_1a_2a_3)+(a_0^2a_3^4+a_0^4a_2^3),\quad GF[7].\end{aligned}$$

$$(48)\quad \begin{aligned}&(a_2^{10}+a_1^{10})-3(a_1a_2^8a_3+a_0a_1^8a_2)-3(a_1^2a_2^6a_3^2+a_0^2a_1^6a_2^2)+4(a_1^3a_2^4a_3^2+a_0^3a_1^4a_2^3)\\&+(a_1^4a_2^2a_3^4+a_0^4a_1^2a_2^4)-(a_1^5a_3^5+a_0^5a_2^5)+6(a_0a_2^7a_3^2+a_0^2a_1^7a_3)\\&+6(a_0a_1a_2^5a_3^3+a_0^3a_1^5a_2a_3)+5(a_0a_1^2a_2^3a_3^4+a_0^4a_1^3a_2^2a_3)-3(a_0a_1^3a_2a_3^5+a_0^5a_1a_2^3a_3)\\&+5(a_0^2a_2^4a_3^4+a_0^4a_1^4a_3^2)+4(a_0^2a_1a_2^2a_3^5+a_0^5a_1^2a_2a_3^2)-(a_0^2a_1^2a_3^6+a_0^6a_2^2a_3^2)\\&+(a_0^3a_2a_3^6+a_0^6a_1a_3^3),\qquad GF[11].\end{aligned}$$

$$(49)\quad \begin{aligned}&(a_2^{12}+a_1^{12})+5(a_1a_2^{10}a_3+a_0a_1^{10}a_2)+5(a_1^2a_2^8a_3^2+a_0^2a_1^8a_2^2)-6(a_1^3a_2^6a_3^3+a_0^3a_1^6a_2^3)\\&+6(a_1^4a_2^4a_3^4+a_0^4a_1^4a_2^4)+2(a_0^2a_2^6a_3^4+a_0^4a_1^6a_3^2)+2(a_1^5a_2^2a_3^5+a_0^5a_1^2a_2^5)+(a_1^6a_3^6+a_0^6a_2^6)\\&-3(a_0a_2^9a_3^2+a_0^2a_1^9a_3)+2(a_0a_1a_2^7a_3^3+a_0^3a_1^7a_2a_3)-3(a_0a_1^2a_2^5a_3^4+a_0^4a_1^5a_2^2a_3)\\&+3(a_0a_1^3a_2^3a_3^5+a_0^5a_1^3a_2^3a_3)-3(a_0a_1^4a_2a_3^6+a_0^6a_1a_2^4a_3)+4(a_0^2a_1a_2^4a_3^5+a_0^5a_1^4a_2a_3^2)\\&-3(a_0^2a_1^2a_2^2a_3^6+a_0^6a_1^2a_2^2a_3^2)+3(a_0^2a_1^3a_3^7+a_0^7a_2^3a_3^2)-6(a_0^3a_2^3a_3^6+a_0^6a_1^3a_3^3)\\&+2(a_0^3a_1a_2a_3^7+a_0^7a_1a_2a_3^3)+(a_0^4a_3^8+a_0^8a_3^4),\qquad GF[13].\end{aligned}$$

17. For a cubic form in the $GF[p^n]$, $n > 1$, the invariant K of degree $\mu = p^n - 1$ does not involve all the terms (44). Indeed, when the multiplicative constant is suitably chosen, K is identical* with K^p, so that a term (44) occurs in K only when

$$(50)\qquad [p^i e_0] + [p^i e_1] + [p^i e_2] + [p^i e_3] = \mu \quad (i = 1, \cdots, n-1),$$

where $[x]$ denotes the least positive (or zero) residue of x modulo μ. Inversely, *K contains every term* (44) *satisfying conditions* (50). In exhibiting an invariant, we enclose within a parenthesis terms derived as follows: first a term (44); then in turn its pth, (p^2)th, $\cdots$, (p^{n-1})th powers, when they are distinct

* The explanation of this property appears to lie in the fact noted at the end of § 12.

from the first term; finally, the terms derived from the preceding by the substitution $(a_0 a_3)(a_1 a_2)$, when they are distinct from the preceding.

18. For a cubic form in the $GF[2^n]$, the absolute invariant K of degree $2^n - 1$ contains exactly 2^n terms. We exhibit K for $n \leqq 5$, arranging the terms as explained in §§ 16, 17.

$$(a_2 + a_1), \qquad GF[2]. \tag{51}$$

$$(a_2^3 + a_1^3) + (a_0 a_3^2 + a_0^2 a_3), \qquad GF[4]. \tag{52}$$

$$(a_2^7 + a_1^7) + (a_0 a_2^4 a_3^2 + a_0^2 a_2 a_3^4 + a_0^4 a_2^2 a_3 + a_0^2 a_1^4 a_3 + a_0^4 a_1 a_3^2 + a_0 a_1^2 a_3^4), \quad GF[8]. \tag{53}$$

$$\begin{aligned}&(a_2^{15} + a_1^{15}) + (a_0 a_2^{12} a_3^2 + a_0^2 a_2^9 a_3^4 + a_0^4 a_2^3 a_3^8 + a_0^8 a_2^6 a_3 + a_0^2 a_1^{12} a_3 + a_0^4 a_1^9 a_3^2 + a_0^8 a_1^3 a_3^4 \\ &+ a_0 a_1^6 a_3^8) + (a_0 a_1^2 a_2^8 a_3^4 + a_0^2 a_1^4 a_2 a_3^8 + a_0^4 a_1^8 a_2^2 a_3 + a_0^8 a_1 a_2^4 a_3^2), + (a_0^5 a_3^{10} + a_0^{10} a_3^5), \\ &GF[16].\end{aligned} \tag{54}$$

$$\begin{aligned}&(a_2^{31} + a_1^{31}) + (a_0 a_2^{28} a_3^2 + a_0^2 a_2^{25} a_3^4 + a_0^4 a_2^{19} a_3^8 + a_0^8 a_2^7 a_3^{16} + a_0^{16} a_2^{14} a_3 + a_0^2 a_1^{28} a_3 \\ &+ a_0^4 a_1^{25} a_3^2 + a_0^8 a_1^{19} a_3^4 + a_0^{16} a_1^7 a_3^8 + a_0 a_1^{14} a_3^{16}) + (a_0 a_1^2 a_2^{24} a_3^4 + a_0^2 a_1^4 a_2^{17} a_3^8 + a_0^4 a_1^8 a_2^3 a_3^{16} \\ &+ a_0^8 a_1^{16} a_2^6 a_3 + a_0^{16} a_1 a_2^{12} a_3^2 + a_0^4 a_1^{24} a_2^2 a_3 + a_0^8 a_1^{17} a_2^4 a_3^2 + a_0^{16} a_1^3 a_2^8 a_3^4 + a_0 a_1^6 a_2^{16} a_3^8 \\ &+ a_0^2 a_1^{12} a_2 a_3^{16}) + (a_0^5 a_2^{16} a_3^{10} + a_0^{10} a_2 a_3^{20} + a_0^{20} a_2^2 a_3^9 + a_0^9 a_2^4 a_3^{18} + a_0^{18} a_2^8 a_3^5 + a_0^{10} a_1^{16} a_3^5 \\ &+ a_0^{20} a_1 a_3^{10} + a_0^9 a_1^2 a_3^{20} + a_0^{18} a_1^4 a_3^9 + a_0^5 a_1^8 a_3^{18}), \qquad GF[32].\end{aligned} \tag{55}$$

19. For the cubic form, with coefficients in the $GF[3^n]$,

$$a_0 x^3 + a_1 x^2 y + a_2 x y^2 + a_3 y^3, \tag{56}$$

we can determine immediately the absolute invariant K of degree $3^n - 1$ in terms of the absolute invariant P, given by (9), and the discriminant Δ of (56), viz.,

$$P = (a_1^{3^n-1} - 1)(a_2^{3^n-1} - 1), \qquad \Delta = a_1^2 a_2^2 - a_0 a_2^3 - a_1^3 a_3. \tag{57}$$

In fact, the invariant K is simply the reduced form of

$$K = \Delta^{(3^n-1)/2} - P + 1. \tag{58}$$

The reduced form of K depends upon the character of the product

$$\pi \cdot \Delta^{3^{n-1}}, \qquad \pi \equiv \prod_{i=0}^{n-2} \Delta^{3^i}. \tag{59}$$

But in the $GF[3^n]$, we have

$$\Delta^{3^i} = (a_1 a_2)^{2 \cdot 3^i} - a_0^{3^i} a_2^{3^{i+1}} - a_3^{3^i} a_1^{3^{i+1}} (i < n-1), \quad \Delta^{3^{n-1}} = (a_1 a_2)^{2 \cdot 3^{n-1}} - a_0^{3^{n-1}} a_2 - a_3^{3^{n-1}} a_1.$$

Now every term of the expanded product π is of degree

$$4 \sum_{i=0}^{n-2} 3^i = 2d, \qquad d \equiv 3^{n-1} - 1.$$

Further, $\pi = (a_1 a_2)^d + \sigma$, where each term of σ is of degree higher than d in one of the letters a_1, a_2. Indeed, if we employ either of the last two terms of $\Delta^{3^{n-2}}$ the resulting term of the product π is of degree $\geqq 3^{n-1}$ in a_1 or a_2. Hence the statement follows by induction from $n-1$ to n.

It now follows that, in the complete product (59), a term is exactly of degree $3^n - 1$ if one of its factors is chosen from the last two terms of $\Delta^{3^{n-1}}$. Finally, in

$$\pi \cdot (a_1 a_2)^{2 \cdot 3^{n-1}} = (a_1 a_2)^{3^n - 1} + \sigma (a_1 a_2)^{2 \cdot 3^{n-1}}$$

every term, except the first, is of degree higher than $d + 2 \cdot 3^{n-1} \equiv 3^n - 1$ in one of the letters a_1, a_2, and hence admits of a reduction in total degree (by means of the equation $a^{3^n} = a$, satisfied by every mark of the field) from $2d + 4 \cdot 3^{n-1} = 2(3^n - 1)$ to $3^n - 1$. The exceptional first term cancels a term of $-P$. Hence *every term in the reduced form of the absolute* invariant* (58) *is of degree* $3^n - 1$.

In the expansion of (59) no two terms have the same set of exponents, so that there result 3^n distinct terms, no one identical with $a_j^{3^n-1}$ ($j = 1$ or 2). Hence *the reduced form of* (58) *contains exactly* $3^n + 1$ *terms.* For $n = 1$, they are given by (14); for $n = 2$ by (75). For $n \leqq 3$, I have verified that the terms of K obey the laws stated in § 17. As these (empirical) laws were formulated from another standpoint (and in fact prior to the investigation of the present case $p = 3$), we have independent evidence of the validity of the conjectures.

From (57) and (58) we deduce immediately the important relation

$$(\Delta^{(3^n+1)/2} - \Delta) K = 0. \tag{60}$$

Indeed, $\Delta P = 0$, so that

$$\Delta K = \Delta^{(3^n+1)/2} + \Delta, \qquad \Delta^{(3^n+1)/2} K = \Delta^{3^n} + \Delta^{(3^n+1)/2}, \qquad \Delta^{3^n} = \Delta.$$

20. The discriminant of the cubic form (40) in the $GF[p^n]$, $p \neq 3$, is

$$\Delta = -3a_1^2 a_2^2 + 4a_0 a_2^3 + 4a_1^3 a_3 - 6a_0 a_1 a_2 a_3 + a_0^2 a_3^2. \tag{61}$$

Now the invariants K, given by (46)-(49), satisfy the relation

$$(\Delta^{(p^n+1)/2} - \epsilon\Delta) K = 0, \qquad \epsilon = (-3)^{(p^n-1)/2}, \tag{62}$$

ϵ being $+1$ or -1 according as -3 is a square or a not-square in the $GF[p^n]$, $p > 3$. If we define Δ to be the function (61) multiplied by -3 (or by $-3\lambda^2$, λ any mark $\neq 0$), relation (62) is replaced by the simpler relation

$$(\Delta^{(p^n+1)/2} - \Delta) K = 0. \tag{63}$$

* Under a transformation of determinant D, the discriminant Δ is multiplied by D^6, so that $\Delta^{(3^n-1)/2}$ is an absolute invariant in the $GF[3^n]$.

THEOREM.* *If the indeterminate constant factor in the discriminant* Δ *of a cubic form in the* $GF[p^n]$, $p>2$, *be chosen so that the coefficient of* $a_1^2a_2^2$ *is a square in the field, then* Δ *and the absolute invariant* K *of degree* p^n-1 *satisfy the relation* (63).

21. Consider next invariants of the cubic form (56) in the $GF[3^n]$. Under transformation (3), the increments to a_2 and a_3 are $-ta_1$ and $ta_2+t^2a_1+t^3a_0$, respectively, while a_0 and a_1 are unaltered. The case $n=1$ was treated in § 9. Let here $n=2$. Then the coefficients of t and t^3 in $\phi'-\phi$ give the respective annihilators: †

$$
\begin{aligned}
&-a_1(2)+a_2(3)+a_0a_1^6(2^63)-a_1^8(2^73)+a_1^8a_2(2^83)-a_0^2a_1^3(2^33^2)\\
&-a_0a_1^5(2^43^2)+(a_0a_1^5a_2-a_1^7)(2^53^2)-a_1^7a_2(2^63^2)-a_1^7a_2^2(2^73^2)-a_0^3(3^3)\\
&-a_1^6(2^33^3)+a_1^6a_2^3(2^63^3)+a_0^3a_1^8(2^83^3)+a_0a_1^3(3^4)-a_1^5(23^4)+a_1^5a_2(2^23^4)\\
&-a_0a_1^3a_2^3(2^33^4)+a_1^5a_2^3(2^43^4)-a_1^5(a_0^4+a_2^4)(2^53^4)+a_0^3a_1^7(2^63^4)-a_0^3a_1^7a_2(2^73^4)\\
(64)\quad &+a_0a_1^3(2^83^4)+(a_0^2a_2^3-a_1^4a_2)(3^5)+(a_0a_1^2a_2^3-a_1^4a_2^2)(23^5)+(a_0^4a_1^4+a_1^4a_2^4)(2^33^5)\\
&+(a_0^3a_1^6a_2^2-a_0a_1^2)(2^63^5)-a_1^3a_2^3(3^6)-a_0^3a_1^5(2^23^6)-a_1^3a_2^6(2^33^6)\\
&+(a_0^3a_1^5a_2^3-a_1^3)(2^53^6)-a_0^6a_1^7(2^73^6)-a_1^3a_2^3(2^83^6)+(a_0a_2^6-a_0^3a_1^4)(3^7)\\
&+(a_0^3a_1^4a_2-a_1^2a_2^6)(23^7)+(a_1^2a_2^7-a_0^4a_1^2a_2^3+a_0a_1^8)(2^23^7)+(a_0^3a_1^4a_2^3-a_1^2)(2^33^7)\\
&+(a_0^4a_1a_2^3-a_0^3a_1^3a_2^2-a_0a_1^7-a_1a_2^7)(3^8);
\end{aligned}
$$

$$
\begin{aligned}
&-a_1^3(2^3)+a_0(3)-a_1^2(23)+a_1^2a_2(2^23)+a_0a_1^8(2^83)-a_1a_2(3^2)-a_1a_2^2(23^2)\\
&-a_0^2a_1^5(2^53^2)-a_0a_1^7(2^63^2)+(a_0a_1^7a_2-a_1)(2^73^2)-a_1a_2(2^83^2)+a_2^3(3^3)\\
(65)\quad &+a_0^3a_1^2(2^23^3)-a_1^8(2^53^3)+a_1^8a_2^3(2^83^3)+a_0^3a_1(3^4)-a_0^3a_1a_2(23^4)+a_0a_1^5(2^23^4)\\
&-a_1^7(2^33^4)+a_1^7a_2(2^43^4)-a_0a_1^5a_2^3(2^53^4)+a_1^7a_2^3(2^63^4)-(a_0^4a_1^7+a_1^7a_2^4)(2^73^4)\\
&+a_0^3a_1(2^83^4)+(a_0^3a_2^2-a_0a_1^4)(3^5)+(a_0a_1^4a_2-a_1^6)(23^5)+(a_0a_1^4a_2^3-a_1^6a_2^2)(2^33^5)
\end{aligned}
$$

* If, instead of the mere verification of this theorem for the computed invariants K, we had a direct proof, it would be of decided aid to the computation. We may compute K by means of (63) alone. For example, let $p^n=3^2$, and give K its necessary form (75) with the four parentheses multiplied by $1, m, s, r$, respectively. In $(\Delta^5-\Delta)K$, we readily find that the factor of a_3^8 is $(s-r)a_0a_1^8a_2^3-(s+m)a_1^2a_2^2$, while the terms independent of a_1 and a_3 are $(s-1)(a_0^5a_2^7-a_0a_2^3)$. Hence $-m=r=s=1$.

In general, the labor of determining the high power of Δ would be great. We may, however, employ only the first two terms of (61) and so readily compute the terms of (63) independent of a_3. We thereby get the coefficients of the terms of ϕ independent of a_3. The coefficients of the terms involving a_3 to the first power are then either known or else occur in the terms involving a_3^e $(e\geqq 2)$, in view of (45). The work with the annihilators is now greatly simplified. It suffices to find the coefficients of $a_3^r, \cdots, a_3^3, a_3^2$, in turn, where r is the greatest integer in $2/3(p^n-1)$. We need only a part of the annihilators; for instance, the derivative with respect to a_3 need not be of order higher than $r-2$.

† We have omitted the terms (2^i3^8), $i\geqq 1$, (2^i3^7), $i\geqq 4$, (2^i3^5), $i=2, 4, 5, 7, 8$ (which have long coefficients), since they have no effect upon a function ϕ satisfying (67), the binomial coefficients C_{6i} $(i=2, 4, 5)$ being multiples of 3.

$$+(a_1^6a_2^5+a_0^3a_1^8-a_0^4a_1^6a_2)(2^63^5)-a_0^6a_1(23^6)-a_1^5a_2^3(2^23^6)-a_0^3a_1^7(2^43^6)$$
$$-a_1^5a_2^6(2^53^6)+(a_0^3a_1^7a_2^3-a_1^5)(2^73^6)+(a_0^6a_2-a_1^4a_2^3)(3^7)+(a_0^4a_1^4+a_1^4a_2^4)(23^7)$$
$$+(a_0a_1^2a_2^6-a_0^3a_1^6)(2^23^7)+(a_0^3a_1^6a_2-a_1^4a_2^6)(2^33^7)$$
$$+(a_0^4a_1^3a_2-a^3a_1^5-a_0a_1a_2^6-a_1^3a_2^5)(3^8).$$

We may set

$$\phi=\sum_{i,j}^{0,\cdots,8}B_{ij}a_2^ia_3^j \qquad (B\text{'s functions of } a_0,\ a_1). \tag{66}$$

Since a_3^8 occurs only the first term of (64) and of (65), we get

$$B_{i8}a_1=0 \qquad (i=1,\cdots,8).$$

In view of these relations, a_3^7 occurs only in the first two terms of (64) and the first two terms of (65). The resulting sets of conditions react on each other and give the first two conditions (69) and

$$B_{i8}=0\ (i=1,\cdots,7),\qquad B_{i7}a_1=0\ (i=1,4,5,6,7,8),$$

together with $B_{88}a_0=0$. From the latter and $B_{88}a_1=0$ follows

$$B_{88}=k\pi,\qquad \pi\equiv(a_0^8-1)(a_1^8-1).$$

Hence on replacing ϕ by $\phi-kI$, when I is the absolute invariant (2), we have $B_{88}=0$. If both $B_{ij}a_0$ and $B_{ij}a_1$ vanish, then $B_{ij}=c\pi$, and hence $B_{ij}=0$ in view of (16).

We consider in turn the coefficients of $a_3^6,\cdots,a_3$ in (64) and (65) and finally the terms* independent of a_3 in (65). The process of writing down the two coefficients of a_3^7 is simplified by holding in mind those of the relations (67) and (68) which have been found in the earlier steps. The two resulting sets of conditions react on each other very considerably, and some further simplifications result from the earlier conditions. The final results are as follows:

$$B_{i8}=0\,(i=1,\cdots,8),\quad B_{i7}=0\,(i=1,4,\cdots,8),\quad B_{i5}=0\,(i=2,3,4,5,7,8); \tag{67}$$

$$B_{i6}a_1=0\,(i=1,7,8),\quad B_{i4}a_1=0\,(i=5,7),\quad B_{i2}a_1=0\,(i=3,5,8); \tag{68}$$

together with the set of conditions, referred to as (69):

$$B_{08}=P_{27}a_1,\ B_{08}a_0=-B_{37}a_1^3,\ B_{07}=-B_{26}a_1,\ B_{37}=-B_{56}a_1,\ B_{27}=B_{46}a_1,$$
$$B_{37}a_0=-B_{66}a_1^3,\ B_{27}a_0=B_{56}a_1^3,\ B_{07}a_0=B_{36}a_1^3,\ B_{15}a_1=-B_{08}a_0^3,\ B_{65}a_1^3=B_{08},$$

* Those in (64) are not required in computing the invariants; they were determined directly for each invariant and found to vanish, thus giving a check.

$B_{65} = B_{84} a_1$, $B_{15} = - B_{27} a_0^3$, $B_{05} = B_{24} a_1$, $B_{44} a_1 = - B_{37} a_0^3$, $B_{65} a_0 = - B_{37}$,
$B_{84} a_1^3 = B_{27}$, $B_{14} a_1 + B_{65} a_0 a_1^6 + B_{07} a_0^3 = 0$, $B_{44} a_1^3 = - B_{15} a_0$, $B_{07} = B_{64} a_1^3$,
$B_{05} a_0 = B_{15} a_1^2 - B_{34} a_1^3$, $B_{74} = B_{86} a_0^3$, $B_{64} = B_{76} a_0^3 - B_{83} a_1$, $B_{54} = B_{66} a_0^3 + B_{73} a_1$,
$B_{44} = B_{56} a_0^3$, $B_{34} = B_{46} a_0^3 - B_{53} a_1$, $B_{24} = B_{36} a_0^3 + B_{43} a_1 - B_{66} a_1^6$,
$B_{14} = B_{26} a_0^3 - B_{84} a_0 a_1^6$, $B_{04} + B_{84} + B_{23} a_1 - B_{16} a_0^3 + B_{46} a_1^6 = 0$,
$B_{64} a_0 a_1^6 = B_{13} a_1 + B_{06} a_0^3$, $B_{56} = B_{84} a_0$, $B_{46} = B_{74} a_0 + B_{84} a_1^2$, $B_{36} = B_{64} a_0$,
$B_{26} = B_{54} a_0 + B_{83} a_1^3$, $B_{16} = B_{44} a_0 + B_{73} a_1^3$, $B_{06} + B_{86} = B_{34} a_0 - B_{44} a_1^2 + B_{63} a_1^3$,
$B_{76} = B_{24} a_0 - B_{53} a_1^3 - B_{08} a_0^3$, $B_{66} = B_{14} a_0 - B_{24} a_1^2 - B_{43} a_1^3 - B_{05} a_1$,
$B_{04} a_0 = B_{33} a_1^3 + B_{26} a_0^3 a_1^2 - B_{56} a_1^8$, $B_{12} a_1 = B_{05} a_0^3$, $B_{22} a_1 = - B_{15} a_0^3$,
$B_{42} a_1 = - B_{65} a_1^6 - B_{08} a_1^3$, $B_{15} = - B_{72} a_1^3$, $B_{05} = - B_{62} a_1^3$, $B_{65} = B_{42} a_1^3$,
$B_{72} = B_{84} a_0^3$, $B_{62} = B_{81} a_1 + B_{74} a_0^3$, $B_{52} = B_{64} a_0^3 + B_{37} a_1^3 - B_{71} a_1$, $B_{42} = B_{54} a_0^3 + B_{27} a_3^1$,
$B_{32} = B_{51} a_1 + B_{44} a_0^3$, $B_{22} = B_{34} a_0^3 + B_{07} a_1^3 - B_{41} a_1$, $B_{12} = B_{24} a_0^3$,
$B_{02} + B_{82} = B_{21} a_1 - B_{72} a_0 a_1^6 + B_{14} a_0^3$, $B_{62} a_0 a_1^6 = B_{04} a_0^3 + B_{84} a_0^3 a_1^8 - B_{11} a_1$,
$B_{54} = B_{82} a_0$, $B_{44} = B_{72} a_0$, $B_{34} = B_{62} a_0 - B_{84} a_0^3 a_1^2$, $B_{24} = B_{52} a_0 - B_{81} a_1^3$,
$B_{14} = B_{42} a_0 - B_{71} a_1^3$, $B_{04} + B_{84} = B_{32} a_0 - B_{61} a_1^3 - B_{27} a_1^5$, $B_{74} = B_{22} a_0 + B_{51} a_1^3$,
$B_{64} + B_{22} a_1^2 = B_{12} a_0 + B_{41} a_1^3$, $B_{12} a_1^2 = B_{02} a_0 + B_{31} a_1^3$, $B_{53} + B_{81} a_0 + B_{72} a_1 = 0$,
$B_{43} + B_{71} a_0 + B_{62} a_1 = 0$, $B_{33} + B_{61} a_0 - B_{71} a_1^2 = 0$, $B_{23} + B_{51} a_0 + B_{80} a_1^3 + B_{84} a_1^7 = 0$,
$B_{13} + B_{41} a_0 - B_{51} a_1^2 + B_{70} a_1^3 = 0$, $B_{03} + B_{83} + B_{31} a_0 - B_{41} a_1^2 + B_{60} a_1^3 = 0$,
$B_{73} + B_{21} a_0 + B_{12} a_1 - B_{50} a_1^3 = 0$, $B_{63} + B_{11} a_0 - B_{21} a_1^2 - B_{02} a_1 - B_{40} a_1^3 = 0$,
$B_{01} a_0 - B_{11} a_1^2 - B_{30} a_1^3 + B_{23} a_0^3 a_1^2 - B_{53} a_1^8 + B_{04} a_0^3 a_1 + B_{24} a_0 a_1^5 = 0$.

The general term of ϕ, in the notations of § 5, is

$$(70) \quad a_0^{e_0} a_1^{e_1} a_2^{e_2} a_3^{e_3}, \qquad 3e_0 + 2e_1 + e_2 \equiv d, \qquad e_1 + 2e_2 + 3e_3 \equiv d \pmod 8.$$

For use in the later cases, we first determine the absolute invariants, viz., those with $d \equiv 0$. If $B_{e_2 e_3} a_1 = 0$, then $e_1 \equiv 0$, and hence $B = 0$ unless $e_3 \equiv 2e_2 \pmod 8$. Hence by (68),

$$(71) \quad B_{16} = B_{86} = B_{54} = B_{74} = B_{32} = B_{82} = 0, \quad B_{76} = c a_0^3 (a_1^8 - 1), \quad B_{52} = k a_0 (a_1^8 - 1).$$

For B_{27}, $e_1 = 7$, $e_0 = 0$ or 8; but $a_0^8 a_1^7$ does not occur in view of (16) since $B_{78} = 0$. Hence $B_{27} = r a_1^7$, r a constant. Then $B_{08} = r a_1^8$, $B_{37} = - r a_0 a_1^5$,

etc., by (69). But Δ^4 and Δ^8 are absolute invariants, Δ being the discriminant (57), and

$$\Delta^8 = a_3^8 a_1^8 + a_3^7(a_1^7 a_2^2 - a_0 a_1^5 a_2^3) + \cdots. \tag{72}$$

Hence if we replace ϕ by $\phi - r\Delta^8$, we may set $r = 0$. Hence, by (69)

$$B_{08},\ B_{27},\ B_{37},\ B_{56},\ B_{46},\ B_{66},\ B_{15},\ B_{65},\quad B_{84},\ B_{44},\ B_{73},\ B_{22},\ B_{42},\ B_{72},\ B_{51} \text{ all vanish.} \tag{73}$$

Now ϕ must be unaltered by substitution (16) and hence by $(a_0 a_3)(a_1 a_2)$, since e_0 and e_2 are both even or both odd, by (70). Thus, from (71), $B_{83} = ca_0^6 a_1^7$. Note that from $B_{07} = -ca_0^6 a_1^3$, we can conclude initially only that $B_{36} = -ca_0^7 + \rho a_0^7 a_1^8$, but subsequently that $\rho = 0$ since $B_{87} = 0$. In this way we readily find that conditions (69) are all satisfied if and only if, in addition to (71) and (73), the following relations hold:

$$B_{83} = ca_0^6 a_1^7,\ B_{03} = -ca_0^6 a_1^7,\ B_{81} = ka_0^2 a_1^5,\ B_{01} = -ka_0^2 a_1^5,\ B_{64} = -ca_0^6,$$
$$B_{06} = -ca_0^4 a_1^6,\ B_{07} = -ca_0^6 a_1^3,\ B_{36} = -ca_0^7,\ B_{14} = ca_0 a_1^2,\ B_{21} = ca_0^4 a_1,$$
$$B_{26} = ca_0^6 a_1^2,\ B_{24} = -ka_0^2,\ B_{02} = -ka_0^4 a_1^2,\ B_{05} = -ka_0^2 a_1,\ B_{12} = -ka_0^5,$$
$$B_{62} = ka_0^2 a_1^6,\ B_{71} = -ka_0 a_1^7,\ k = c,\ B_{41} = B_{31} = B_{43} = 0,\ B_{34} = ca_0^3 a_1^6,$$
$$B_{63} = ca_0^4 a_1^3,\ B_{53} = -ca_0^3 a_1^5,\ B_{23} = wa_1^3\ (w \text{ new parameter}),\ B_{30} = wa_0^3 a_1^2,$$
$$B_{04} = -wa_1^4,\ B_{40} = -wa_0^4,\ B_{11} = -(k+w)a_0^3 a_1^3,\ B_{33} = -(k+w)a_0 a_1,$$
$$B_{61} = wa_1,\ B_{10} = wa_0 a_1^6,\ B_{20} = B_{50} = B_{60} = B_{70} = B_{13} = 0,$$
$$B_{80} = l(a_1^8 - 1) - w\ (l \text{ new parameter}),\ B_{00} = -(w+l)a_1^8 + \text{constant}.$$

For convenience, we take the constant in B_{00} (the absolute term of ϕ) to be l. Then there are three independent parameters w, l and $c = k$. The coefficient of the latter is the negative of the absolute invariant (of degrees 24, 16, 8):

$$\begin{aligned} W = {} & a_3^7 a_0^6 a_1^3 + a_3^6\{-a_0^3(a_1^8 - 1)a_2^7 + a_0^7 a_2^3 - a_0^6 a_1^2 a_2^2 + a_0^4 a_1^6\} + a_3^5 a_0^2 a_1 \\ & + a_3^4(a_0^6 a_2^6 - a_0^3 a_1^6 a_2^3 + a_0^2 a_2^2 - a_0 a_1^2 a_2) + a_3^3\{-a_0^6 a_1^7(a_2^8 - 1) - a_0^4 a_1^3 a_2^6 + a_0^3 a_1^5 a_2^5 + a_0 a_1 a_2^3\} \\ & + a_3^2\{-a_0^2 a_1^6 a_2^6 - a_0(a_1^8 - 1)a_2^5 + a_0^5 a_2 + a_0^4 a_1^2\} \\ & + a_3\{-a_0^2 a_1^5(a_2^8 - 1) + a_0 a_1^7 a_2^7 - a_0^4 a_1 a_2^2 + a_0^3 a_1^3 a_2\}. \end{aligned} \tag{74}$$

The coefficient of l is the invariant P, given by (57). The negative of the coefficient of w is the absolute invariant of degree 8 (see § 19):

$$K = a_3^4 a_1^4 + a_3^3(a_0 a_1 a_2^3 - a_1^3 a_2^2) + a_3(a_0^3 a_1^3 a_2 - a_1 a_2^6) + a_2^8 + a_0^4 a_2^4 - a_0^3 a_1^2 a_2^3 - a_0 a_1^6 a_2 + a_1^8. \tag{78}$$

Now $K = \Delta^4 - P + 1$, in agreement with (58). *Hence the absolute invariants are the linear functions of* $I, P, W, \Delta^4, \Delta^8$. The system should be closed also under multiplication. In verification, we note the relations

$$(76)\qquad \begin{gathered} I^2 = I,\ IP = I,\ IW = 0,\ I\Delta = 0,\ P^2 = P,\ PW = 0, \\ P\Delta = 0,\ W^2 = W,\ W\Delta^4 = W. \end{gathered}$$

Let next $d \equiv 1 \pmod 8$. Then $B_{e_2 e_3} a_1 = 0$ implies $B = 0$ unless $e_3 \equiv 2e_2 + 3 \pmod 8$. Hence the B_{ij} in (68) all vanish. For B_{27}, $e_1 = 0$ or 8, $e_0 = 5$; but $a_0^5 a_1^8$ does not occur in view of (16) since $B_{85} = 0$. Thus $B_{27} = ca_0^5$. Also $B_{84} = ra_0^5 a_1^5$. But one of relations (69) is $B_{84} a_1^3 = B_{27}$. Hence $B_{27} = B_{84} = 0$. In view of relations (69) and the invariance* of ϕ under (16), we readily find that the only non-vanishing B_{ij} are the sixteen for which i, j range over the exponents of a_2, a_3 in the following invariant, the value of B_{ij} being the product of a fixed parameter l by the coefficient of $a_2^i a_3^j$:

$$(77)\qquad \begin{aligned} & a_3^7 a_0^3 a_1^4 + a_3^6(a_0^4 a_1 a_2^3 - a_0^3 a_1^3 a_2^2 - a_0 a_1^7) + a_3^4(a_0^3 a_1 a_2^6 - a_0^6 a_1^3 a_2) \\ & \quad + a_3^3(-a_0^3 a_2^8 - a_0 a_1^4 a_2^6 - a_0^7 a_2^4 + a_0^6 a_1^2 a_2^3 - a_0^4 a_1^6 a_2 + a_0^3 a_1^8) + a_2^3(-a_0 a_1^3) \\ & \quad + a_3(a_0^6 a_2^7 + a_0^3 a_1^6 a_2^4 + a_0^2 a_2^3), \end{aligned}$$

which equals $-E^3$, E being given by (10). By (82), $E^3 = \Delta E$.

For $d \equiv 3 \pmod 8$, we conclude that E is the only invariant. Indeed, the operation of cubing is uniquely reversible in the $GF[9]$. Similarly, the case $d \equiv 2$ reduces to the case $d \equiv 8$, and $d \equiv 7$ to $d \equiv 5$.

For $d \equiv 6 \pmod 8$, we may set $B_{08} = ka_0^6 a_1^6$. Then $B_{27} = ka_0^6 a_1^5$, etc. If we replace ϕ by $\phi + k\Delta W$, where

$$(78)\qquad \Delta W = a_3^8(-a_0^6 a_1^6) + a_3^7(a_0^7 a_1^3 a_2^3 - a_0^6 a_1^5 a_2^2 - a_0^4 a_1) + \cdots,$$

we may set $k = 0$. The resulting invariant is easily found to be a linear function of Δ and Δ^5. Hence for $d \equiv 2$, the invariants are the linear functions of Δ^3, Δ^7 and $\Delta^3 W^3 \equiv \Delta^3 W$.

For $d \equiv 4$, we have $B_{27} = da_0^4 a_1^3$, $B_{08} = da_0^4 a_1^4$, $B_{37} = -da_0^5 a_1$, $B_{07} = da_0^2 a_1^7$, etc. We replace ϕ by $\phi - d\Delta^2 W$, where

$$(79)\qquad \Delta^2 W = a_3^8 a_0^4 a_1^4 + a_3^7(-a_0^5 a_1 a_2^3 + a_0^4 a_1^3 a_2^2 + a_0^2 a_1^7) + \cdots,$$

and hence may set $d = 0$. The resulting invariant is readily found to be a linear function of Δ^2 and Δ^6.

---

* Here a more exacting condition than for $d \equiv 0$. Thus $B_{21} = ma_0 a_1^2$ yields $m = 0$.

For $d \equiv 5$, we have $B_{27} = ra_0 a_1^4$, $B_{08} = ra_0 a_1^5$, $B_{37} = -ra_0^2 a_1^2$, etc. We replace ϕ by $\phi - r\Delta^3 E$, where

$$\Delta^3 E = a_3^8 a_0 a_1^5 + a_3^7 (a_0 a_1^4 a_2^2 - a_0^2 a_1^2 a_2^2) + \cdots, \tag{80}$$

and easily find that the resulting invariant vanishes identically. Then for $d \equiv 7$, the only invariant is $(\Delta^3 E)^3 \equiv \Delta^2 E$, by (82).

THEOREM. *The cubic form* (57) *in the* $GF[9]$ *has exactly* 18 *linearly independent invariants. These may be taken to be*

$$I,\ P,\ \Delta^i\ (i = 1, \cdots, 8), \qquad \Delta^j W,\ \Delta^j E\ (j = 0, 1, 2, 3). \tag{81}$$

The product of any two invariants can be reduced to a linear function of these 18 *invariants by the application of relations* (76) *and*

$$IE = PE = 0, \qquad WE = E, \qquad E^2 = \Delta W, \qquad \Delta^4 E = E, \qquad \Delta^9 = \Delta. \tag{82}$$

22. The invariants I and P of the cubic form in the $GF[3^n]$ can be expressed in terms of a single invariant Γ. Thus

$$\Gamma = I + P, \qquad I = \Gamma - \Gamma^2, \qquad P = \Gamma^2. \tag{83}$$

For $n = 1$, $W = E^2$ by (19). For $n = 2$, we have, by (76), (82),

$$W = W\Delta^4 = E^2 \Delta^3.$$

Hence we have proved, for $n = 1$ and $n = 2$, the following

THEOREM. *The invariants of the cubic form in the* $GF[3^n]$ *are all rational integral functions of the three fundamental invariants* $\Gamma = I + P$, E, *and the discriminant* Δ, *where* I, P, E, *are defined in* §§ 1, 3, 4.

23. For the cubic form in the $GF[2^n]$, the power 2^{n-1} of the discriminant $a_0^2 a_3^2 + a_1^2 a_2^2$ gives the invariant

$$D = a_0 a_3 + a_1 a_2. \tag{84}$$

The eliminant E (§ 4) is $a_0 a_3 \pi$, where for $n = 1, 2, 3$, respectively, π equals

$$\sum a_i, \qquad \sum a_i^3 + D(a_0 + a_3), \qquad \sum a_i^7 + \lambda + \mu,$$

λ being the expression in the second parenthesis of (53), and

$$\mu = a_1 a_2^5 a_3 + a_0 a_1 a_2^2 a_3^3 + a_0^3 a_1 a_2^3 + a_0 a_1^5 a_2 + a_0^3 a_1^2 a_2 a_3 + a_1^3 a_2 a_3^3.$$

Thus $a_0 a_3 \mu = a_1 a_2 \lambda$. Hence, in each case, $E = DK$, K being given in § 18.

For $n = 1$ *or* 2, *every invariant is an absolute invariant.* This is evident for $n = 1$. For $n = 2$, we add the congruences (70), the modulus now being

3; there results $0 \equiv d \pmod 3$. By use of the annihilators, I have found the invariants

$$R_{n=1} = a_0 + a_3 + a_0 a_1 + a_0 a_2 + a_1 a_2 + a_1 a_3 + a_2 a_3, \tag{85}$$

$$R_{n=2} = a_0 + a_3 + a_0 a_1^3 + a_0 a_2^3 + a_0 a_3^3 + a_0^3 a_3 + a_1^3 a_3 + a_2^3 a_3 + a_0^2 a_1 a_2 + a_1 a_2 a_3^2 + a_0 a_1 a_2 a_3, \tag{86}$$

and have proved that, for $n = 1, 2$, every invariant is a rational integral function of D, K, R. In particular, an expression for (2) is

$$I = (D^\mu - 1)(K^\mu - 1)(R^\mu - 1), \qquad \mu = 2^n - 1. \tag{87}$$

For $n = 1$, every invariant is a linear function of D, K, R, DK, DR. Any relation between these follows from $KR = 0$, $D^2 = D$, $K^2 = K$, $R^2 = R$.

For $n = 2$, the 13 linearly independent invariants may be taken to be

$$K, \qquad D^i, \qquad R^i, \qquad D^i R, \qquad D^i K \qquad (i = 1, 2, 3). \tag{88}$$

Any relation between these (absolute) invariants follows from

$$KR = 0, \qquad K^2 = K, \qquad (DR)^2 = DR, \qquad D^4 = D, \qquad R^4 = R. \tag{89}$$

For a proof that D, R, K are independent, see §§ 27, 28.

24. For the cubic form (40) with the integral coefficients modulo 5, the only invariant with $d \equiv 3 \pmod 4$ is the eliminant, given by § 4,

$$E = a_3^3(2a_0^2 a_2 - 2a_0 a_1^2) + a_3^2(2a_0 a_1 a_2^2 - 2a_0^3 a_1) + a_3(a_0 a_2^4 + 2a_0^3 a_2^2 - 2a_0^2 a_1^2 a_2 - a_0 a_1^4); \tag{90}$$

while ΔE is the only invariant with $d \equiv 1$. The only (absolute) invariants with $d \equiv 0$ are the linear functions of $I, \Delta^2, \Delta^4, K, \Delta^2 K$, where K is given by (46). The only invariants with $d \equiv 2$ are the linear functions of $\Delta, \Delta^3, \Delta K$.

The ten linearly independent invariants of the cubic form (40) in the $GF[5]$ may be taken to be

$$I, \ E, \ \Delta E, \ K, \ \Delta K, \ \Delta^2 K, \ \Delta, \ \Delta^2, \ \Delta^3, \ \Delta^4. \tag{91}$$

Any relation between these follows from

$$\begin{gathered} I^2 = I, \quad IE = I\Delta = IK = 0, \quad E^2 = \Delta - \Delta^3 - \Delta K, \quad \Delta^2 E = -E, \\ KE = -E, \quad K^2 = K - \Delta^2 + \Delta^4, \quad \Delta^3 K = -\Delta K, \quad \Delta^5 = \Delta. \end{gathered} \tag{92}$$

25. For the cubic form (40) with integral coefficients modulo 7, the non-absolute invariants have $d \equiv 3 \pmod 6$. Indeed, if we add the congruences $\pmod 6$, analogous to (70), we find that $0 \equiv d \pmod 3$.

The only invariants with $d \equiv 3 \pmod 6$ are the linear functions of E, ΔE, and $\Delta^2 E$, where the eliminant E is (end of § 4)

$$(93)\quad \begin{aligned} E = {} & a_3^5(3a_0^3) + a_3^4(2a_0a_1^3 - 3a_0^2a_1a_2) + a_3^3(a_0^2a_2^3 + a_0a_1^2a_2^2 - 3a_0^5) \\ & + a_3^2(5a_0a_1a_2^4 + 3a_0^4a_1a_2 - a_0^3a_1^3) + a_3(a_0a_2^6 - 2a_0^4a_2^3 - a_0^3a_1^2a_2^2 + 2a_0^2a_1^4a_2 - a_0a_1^6). \end{aligned}$$

The absolute invariants are functions of Δ, K, given by (47), and Q:

$$(94)\quad \begin{aligned} Q = {} & a_3^6(2a_0^2) + a_3^5(6a_0a_1a_2 + 3a_1^3) + a_3^4(a_0a_2^3 + a_1^2a_2^2) + a_3^3(2a_1a_2^4 + 3a_0^3a_1a_2 + 4a_0^2a_1^3) \\ & + a_3^2(2a_2^6 + 4a_0^3a_2^3 + 3a_0^2a_1^2a_2^2 + a_0a_1^4a_2 - a_1^6 + 2a_0^6 - 2) \\ & + a_3(a_0^2a_1a_2^4 - a_0a_1^3a_2^3 + 2a_1^5a_2^2 - a_0^5a_1a_2 + a_0^4a_1^3) \\ & - a_0^2a_2^6 + 2a_0a_1^2a_2^5 + 3a_0^5a_2^3 + a_0^4a_1^2a_2^2 + 2a_0^3a_1^4a_2 + 2a_0^2a_1^6 - 2a_0^2. \end{aligned}$$

The sixteen linearly independent invariants of the cubic form (40) *in the* $GF[7]$ *may be taken to be*

$$(95)\quad E,\ \Delta E,\ \Delta^2 E,\ Q,\ Q^2,\ Q^3,\ \Delta^i\ (i = 1, \cdots, 6),\ \Delta^j K\ (j = 0, 1, 2, 3).$$

Any relation between these follows from

$$(96)\quad \begin{aligned} & \Delta^3 E = E,\ EQ = 0,\ EK = -E,\ Q^4 = 3\Delta^5 - 3\Delta^2 - Q,\ QK = 0,\ \Delta^7 = \Delta, \\ & \Delta^4 K = \Delta K,\ \Delta Q = 2\Delta^3 - 2\Delta^6,\ K^2 = K + \Delta^6 + \Delta^3,\ E^2 = 2\Delta K - 2\Delta^4 - 2\Delta. \end{aligned}$$

The invariant (2) has the following expression:

$$(97)\quad I = Q^3 + \Delta^3 K - K + 2\Delta^6 - 3\Delta^3 + 1.$$

26. The following tables give a complete set of (non-equivalent) canonical forms of binary quadratic forms in the $GF[p^n]$ and the values of the independent invariants for each canonical form:

$p > 2$ (invariants, § 15).

Canonical.	Δ	J
$x^2 - \nu y^2$	ν	0
xy	1	0
x^2	0	-2
νx^2	0	0
Vanishing	0	-1

$p = 2$ (invariants, § 11).

Canonical.	a_1	I	K
$x^2 + xy + cy^2$	1	0	1
xy	1	0	0
x^2	0	0	0
Vanishing	0	1	0

Here ν denotes a particular not-square in the $GF[p^n]$, $p > 2$; while c is a particular solution of $c + c^2 + \cdots + c^{2^{n-1}} = 1$ in the $GF[2^n]$.

27. As a complete set of (non-equivalent) canonical cubic forms in the $GF[p^n]$, we may take

$$\beta C,\ \beta x Q,\ \beta xy(x+y),\ x^2y,\ \beta x^3,\ \text{vanishing form},$$

where C and Q are any particular irreducible cubic and quadratic forms, e. g., $C = x^3 - xy^2 + \tau y^3$, for a suitable value of τ, and (§ 26)

$$Q = x^2 - \nu y^2 \quad (p > 2), \qquad Q = x^2 + xy + cy^2 \quad (p = 2);$$

while $\beta = 1$ if $p^n = 3^n$ or $3l + 2$; $\beta = 1, \epsilon, \epsilon^2$, if $p^n = 3l + 1$, ϵ being a primitive root of the field.

$GF[2]$ (invariants, § 23).

Canonical.	D	R	K
$x^3 - xy^2 + y^3$	1	0	1
$x(x^2 + xy + y^2)$	1	0	0
$xy(x+y)$	1	1	0
x^2y	0	0	1
x^3	0	1	0
Vanishing	0	0	0

$GF[2^2]$, $i^2 \equiv i + 1 \pmod 2$ (§ 23).

Canonical.	D	R	K
$\beta(x^3 - xy^2 + y^3)$	β^2	0	β
$\beta x(x^2 + xy + iy^2)$	$i\beta^2$	$i^2\beta$	$i^2\beta$
$\beta xy(x+y)$	β^2	0	0
x^2y	0	0	1
βx^3	0	β	0
Vanishing	0	0	0

$GF[3^n]$, $n = 1, 2$ (§ 22).

Canonical.	Δ	E	Γ
$x^3 - xy^2 + y^3$	1	1	0
$x(x^2 - \nu y^2)$	ν^3	0	0
$xy(x+y)$	1	0	0
x^2y	0	0	0
x^3	0	0	1
Vanishing	0	0	-1

$GF[5]$ (§ 24).

Canonical.	Δ	K	I	E
$x^3 - xy^2 + 2y^3$	2	-1	0	1
$x(x^2 - 2y^2)$	-1	0	0	0
$xy(x+y)$	2	2	0	0
x^2y	0	1	0	0
x^3	0	0	0	0
Vanishing	0	0	1	0

28. An inspection of the tables §§ 26, 27 shows that all the invariants given are necessary to characterize the canonical forms (since if one invariant is omitted, the others have equal values for some two canonical forms), and that no invariant is a rational function of the others (since any one takes different values when the others are equal). The only exception to these statements is the case of invariant E of the cubic form in the $GF[5]$; the characterization is complete without E; but the independence of E follows from the weights d (§ 24).

THE UNIVERSITY OF CHICAGO,
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PROJECTIVE DIFFERENTIAL GEOMETRY OF CURVED SURFACES*

(FIRST MEMOIR)

BY

E. J. WILCZYNSKI

§ 1. *The simultaneous solutions of two linear homogeneous partial differential equations of the second order with two independent variables.*

Consider a system of partial differential equations of the form

$$(1)\qquad \begin{aligned} Ay_{uu} + By_{uv} + Cy_{vv} + Dy_u + Ey_v + Fy &= 0, \\ A'y_{uu} + B'y_{uv} + C'y_{vv} + D'y_u + E'y_v + F'y &= 0, \end{aligned}$$

where

$$(2)\qquad y_u = \frac{\partial y}{\partial u}, \qquad y_v = \frac{\partial y}{\partial v}, \qquad y_{uu} = \frac{\partial^2 y}{\partial u^2}, \quad \text{etc.},$$

and where $A, B, \cdots, F'$ are analytic functions of u and v.

If the determinant $AC' - A'C$ does not vanish identically, (1) may be written in the form

$$(3)\qquad \begin{aligned} y_{uu} &= ay_{uv} + by_u + cy_v + dy, \\ y_{vv} &= a'y_{uv} + b'y_u + c'y_v + d'y. \end{aligned}$$

If the determinant $AC' - A'C$ is equal to zero, while $BC' - B'C$ does not vanish, we may write (1) in the form

$$(4)\qquad \begin{aligned} y_{uu} &= \alpha y_{vv} + \beta y_u + \gamma y_v + \delta y, \\ y_{uv} &= \beta' y_u + \gamma' y_v + \delta' y, \end{aligned}$$

provided that A and A' do not vanish simultaneously. If they do, i. e., if $A = A' = 0$, while $BC' - B'C \neq 0$, we obtain from (1) a system of the same form as (4) except for an interchange of u and v, with the further specialization that $\alpha = 0$. Finally, if

$$AC' - A'C = BC' - B'C = 0,$$

one of the equations deducible from (1) reduces to the first order.

* Presented to the Society (San Francisco) February 24, 1906, as a preliminary report under the title "Outline of a projective differential geometry of curved surfaces." Received for publication December 3, 1906.

Transform (4) by putting

$$u = r, \qquad v = r + s,$$

and let

$$y_r = \frac{\partial y}{\partial r}, \qquad y_s = \frac{\partial y}{\partial s}, \quad \text{etc.}$$

The system (4) becomes

$$y_{rr} - 2y_{rs} + y_{ss} = \alpha y_{ss} + \beta(y_r - y_s) + \gamma y_r + \delta y,$$
$$y_{rs} - y_{ss} = \beta'(y_r - y_s) + \gamma' y_s + \delta' y,$$

which equations may be solved for y_{rr} and y_{ss}. By such a transformation of the independent variables (4) may therefore be reduced to the form (3).

Unless, therefore, there exists a linear combination of the two equations (1) *which is of the first order only, system* (1) *may be reduced to the form* (3), *a reduction which may involve a linear transformation of the independent variables.*

Consider a system of form (3); y_{uu} and y_{vv} are expressed in terms of y, y_u, y_v, y_{uv}. Differentiation of the first of the two equations with respect to u will express y_{uuu} in terms of the same four quantities and of y_{uuv}. Similarly y_{vvv} will depend upon y_{uvv} as well as upon y, y_u, y_v, y_{uv}. In general, however, y_{uuv} and y_{uvv} may themselves be expressed in terms of y, y_u, y_v, y_{uv}. In fact differentiation of the first equation of (3) with respect to v, and of the second with respect to u will give

$$\begin{aligned} y_{uuv} - ay_{uvv} &= (a_v + b)y_{uv} + cy_{vv} + b_v y_u + (c_v + d)y_v + d_v y, \\ -a'y_{uuv} + y_{uvv} &= (a'_u + c')y_{uv} + b'y_{uu} + (b'_u + d')y_u + c'_u y_v + d'_u y, \end{aligned} \tag{5}$$

whence follows the truth of the above statement provided that

$$aa' - 1 \neq 0 \tag{6}$$

In this case, clearly all higher derivatives of y may be expressed in the form

$$\alpha y + \beta y_u + \gamma y_v + \delta y_{uv}, \tag{7}$$

where α, β, γ, δ are, for every such derivative, perfectly definite functions of u and v. It is to be noticed, however, that of the five derivatives of the fourth order, three may be computed in two different ways; for example

$$y_{uuuv} = \frac{\partial}{\partial u}(y_{uuv}) = \frac{\partial}{\partial v}(y_{uuu}.)$$

Therefore, in order that the eight expressions of form (7) for the five fourth derivatives may be consistent, three conditions must be satisfied. It is not difficult to see that, if these three so-called *integrability conditions* are satisfied, the expression of form (7) for a derivative of any higher order will be unique.

Let y be a solution of system (3), developable in powers of $u - u_0$ and $v - v_0$, and denote by $c_1, \cdots, c_4$ the values of y, y_u, y_v, y_{uv} for $u = u_0, v = v_0$. Then y may be expressed in the form

$$y = c_1 + c_2(u - u_0) + c_3(v - v_0) + c_4(u - u_0)(v - v_0)$$
$$+ \tfrac{1}{2}(d_0 c_1 + c_0 c_2 + b_0 c_3 + a_0 c_4)(u - u_0)^2 + \tfrac{1}{2}(d_0' c_1 + c_0' c_2 + b_0' c_3 + a_0' c_4)(v - v_0)^2 + \cdots,$$

where a_0, b_0, etc. are the values of $a, b, c, \cdots$ for $u = u_0, v = v_0$. All of the coefficients of this development will be linear combinations of $c_1, \cdots, c_4$. We may write, therefore,

$$y = c_1 y' + c_2 y'' + c_3 y^{(3)} + c_4 y^{(4)},$$

where $y', \cdots, y^{(4)}$ are holomorphic functions of $u - u_0$ and $v - v_0$. The most general analytic solution of (3) can, therefore, contain only four arbitrary constants which enter linearly.

The integrability conditions are of the form

$$\alpha y_{uv} + \beta y_u + \gamma y_v + \delta y = 0. \tag{8}$$

They may be satisfied *identically*, so that

$$\alpha = \beta = \gamma = \delta = 0.$$

In that case the solution just indicated in a formal manner actually contains four arbitrary constants. If the integrability conditions are not satisfied identically, they represent additional differential equations which y must verify. The number of arbitrary constants in the general solution of (3) is reduced at least by one for every independent non-identically satisfied integrability condition. The above condition (8), for example, requires that

$$\alpha_0 c_4 + \beta_0 c_2 + \gamma_0 c_3 + \delta_0 c_1$$

shall be equal to zero, where $\alpha_0 \cdots \delta_0$ are the values of $\alpha \cdots \delta$ for $u = u_0, v = v_0$. This clearly reduces the number of arbitrary constants to three.

If the integrability conditions are satisfied identically, the most general analytic solution of (3) can depend, therefore, upon only four arbitrary constants which enter linearly. Such an analytic solution actually exists if the coefficients of (3) are holomorphic in the vicinity of $u = u_0, v = v_0$. This may be proved by the method which, since the time of CAUCHY, has always been employed for such investigations.* Of course, it may seem desirable to prove the existence of just a four-fold infinity of solutions for a system of form (3), for which $aa' - 1$ does not vanish and whose integrability conditions are satisfied identically under more general assumptions. For our present purpose, it suffices to restrict our considerations to systems of form (3) with analytic coefficients.

* Cf. for example GOURSAT, *Leçons sur l'intégration des équations aux dérivées partielles du premier ordre*, Paris, 1891, chapter I.

We proceed to consider the case

$$aa' - 1 = 0. \tag{9}$$

If we multiply both members of the second equation of (5) by a and add to the members of the first, taking account moreover of (3), we find

$$\begin{aligned}[a_v + b + a(a'_u + c') + a^2 b' + ca']y_{uv} + [b_v + a(b'_u + d') + abb' + cb']y_u \\ + [c_v + d + ac'_u + acb' + cc']y_v + [d_v + ad'_u + adb' + cd']y = 0.\end{aligned} \tag{10}$$

Unless all of the coefficients of this equation are zero, it becomes possible, by means of it, to reduce every expression of the form $\alpha y + \beta y_u + \gamma y_v + \delta y_{uv}$ to one involving only three terms, so that the general analytic solution of (3) could depend upon no more than three arbitrary constants. If, on the other hand, all of the coefficients of (10) are equal to zero, one of the partial derivatives of each order will remain arbitrary, and the general solution will contain an infinite number of arbitrary constants. This solution actually exists. The two equations are then said to be in *involution*;* the system is *involutory*.

We may recapitulate as follows. 1. *The system* (3) *of simultaneous linear differential equations has precisely four linearly independent solutions if* $aa' - 1$ *is different from zero and if the integrability conditions are identically satisfied.*

2. *If* $aa' - 1$ *vanishes, and the integrability conditions are satisfied, while not all of the coefficients of* (10) *are zero, the system* (3) *has less than four linearly independent solutions.*

3. *If, however, the first two conditions of No.* 2 *are satisfied, and if besides all of the coefficients of* (10) *are equal to zero, the general solution of system* (3) *contains an infinite number of arbitrary constants. The system is an involutory one.*

§ 2. *Geometric interpretation. The integrating surface.*

If $aa' - 1$ is different from zero and if all of the integrability conditions are satisfied identically, system (3) has just four linearly independent solutions. Denote four such solutions by y', y'', $y^{(3)}$, $y^{(4)}$. We shall have

$$y^{(k)} = f^{(k)}(u, v) \qquad (k = 1, 2, 3, 4).$$

Interpret $y^{(k)}$ as the homogeneous coördinates of a point P_y in space. As u and v assume all of their values, P_y will describe a surface S_y an *integrating surface* of the system. *This surface cannot degenerate into a curve.* For if

*Cf. GOURSAT, *Leçons sur l'intégration des équations aux dérivées partielles du second ordre à deux variables indépendantes*, Paris, 1896–98, vol. 2, chap. 6, where a more general case is treated according to BIANCHI.

S_y were to degenerate in that way, the ratios of y', y'', $y^{(3)}$, $y^{(4)}$, would become functions of a single variable $t = \phi(u, v)$. In that case $y', \cdots, y^{(4)}$ would be solutions of some linear partial differential equation of the first order, say

$$\alpha y_u + \beta y_v + \gamma y = 0, \tag{11}$$

besides satisfying (3). As a matter of fact let

$$\alpha w_u + \beta w_v = 0$$

be the partial differential equation satisfied by $t = \phi(u, v)$. Then if

$$\frac{y'}{y^{(4)}}, \quad \frac{y''}{y^{(4)}}, \quad \frac{y^{(3)}}{y^{(4)}},$$

were functions of t alone, they would satisfy the same equation, so that

$$\alpha(y_u^{(k)} y^{(4)} - y^{(k)} y_u^{(4)}) + \beta(y_v^{(k)} y^{(4)} - y^{(k)} y_v^{(4)}) = 0 \qquad (k = 1, 2, 3),$$

or

$$\alpha y_u^{(k)} + \beta y_v^{(k)} - \frac{\alpha y_u^{(4)} + \beta y_v^{(4)}}{y^{(4)}} y^{(k)} = 0 \qquad (k = 1, 2, 3).$$

Put

$$-\frac{\alpha y_u^{(4)} + \beta y_v^{(4)}}{y^{(4)}} = \gamma,$$

and it becomes clear that $y', y'', y^{(3)}, y^{(4)}$ would satisfy the same equation of form (11). Differentiate (11) with respect to u and v. If the two equations thus obtained coincide with the two equations of (3), $aa' - 1$ would be equal to zero, contrary to our assumption. (11) would be an intermediary integral of (3) whose general solution would depend upon an arbitrary function. If, on the other hand, the two equations obtained from (11) by differentiation were not identical with the two equations of (3), the most general function of u and v which satisfies all of these equations cannot depend upon more than two linearly independent functions. It is impossible, therefore, that the four linearly independent functions $y', \cdots, y^{(4)}$ should satisfy an equation of form (11), i. e., it is impossible that the integrating surface S_y should degenerate into a curve.

The integrating surface of system (3) is not unique. However, the most general system of linearly independent solutions of (3), in the case considered, is of the form

$$\eta^{(i)} = \sum_{k=1}^{4} c_{ik} y^{(k)} \qquad (i = 1, 2, 3, 4),$$

where the determinant of the constant coefficients,

$$|c_{ik}|,$$

does not vanish. *The most general integrating surface of* (3) *is, therefore, a projective transformation of any particular one.* It is for this reason that we shall make use of (3) as a basis for the projective theory of surfaces.

If $aa' - 1$ vanishes, while not all of the conditions for an involutory system are satisfied, the system (3) has only three or fewer linearly independent solutions. Its integrating surface, therefore, degenerates into a plane or a straight line. For the purposes of the theory of surfaces this case may, therefore, be left aside.

It remains to examine the case when the equations of (3) are in involution. That case must also be excluded from our considerations. For we wish to construct a *projective* theory of surfaces; but, if the equations (3) are in involution, they have more than four linearly independent solutions in common, so that the most general integrating surface of (3) will not be merely a projective transformation of any particular one. The question which requires investigation is this: what kind of surfaces are thus excluded from consideration?

If we put in (3)

$$\bar{y} = \lambda y$$

where λ is an arbitrary function of u and v, and if we introduce new independent variables by putting

$$\bar{u} = \mu(u, v), \qquad \bar{v} = \nu(u, v),$$

another system of differential equations of form (1) will be obtained from (3) with $\bar{y}, \bar{u}, \bar{v}$, in place of y, u, v. Moreover, it is clear that, if the equations (3) are in involution, the same must be true of the transformed equations and conversely. For if they were not, they could have only four linearly independent solutions and the same would have to be true of the original system of equations.

Let equations (3) be in involution. They have an infinite number of linearly independent solutions. Let $y', \cdots, y^{(4)}$ be four of these. Transform system (3) by putting $\bar{y} = \lambda y$ where $\lambda = 1/y^{(4)}$. The new system of differential equations is again an involutory system and has the four linearly independent solutions

$$1, \qquad x = \frac{y'}{y^{(4)}}, \qquad y = \frac{y''}{y^{(4)}}, \qquad z = \frac{y^{(3)}}{y^{(4)}},$$

where x, y and z may be considered as non-homogeneous coördinates of the point of the integral surface whose homogeneous coördinates were y', y'', $y^{(3)}$, $y^{(4)}$. Moreover x, y, z are functions of u and v. Introduce x and y as independent variables. This is possible unless x should happen to be a function of y alone. In that case we might introduce x and z as independent variables. For otherwise x, y and z would be functions of a single variable $t = \phi(u, v)$, so that $y', \cdots, y^{(4)}$, as we have seen above, would satisfy one and the same linear partial differential equation of the first order. In that case there would be no integral surface, but an integral curve; for our purpose that case must be excluded. Accordingly two of the three functions x, y, z of u and v will be indepen-

dent, say x and y. We may then introduce x and y as independent variables in place of u and v. The transformed system of equations will be again an involutory system with the independent variables x and y, and of which 1, x, y and $z = f(x, y)$ are four linearly independent solutions. This system of equations will certainly be of form (1), and in most cases may at once be written in the form (3) by solving for θ_{xx} and θ_{yy}, denoting the dependent variable by θ. The system is then of the form

$$\theta_{xx} = a\theta_{xy} + b\theta_x + c\theta_y + d\theta,$$
$$\theta_{yy} = a'\theta_{xy} + b'\theta_x + c'\theta_y + d'\theta,$$

and it must be satisfied by $\theta = 1$, x, y and $z = f(x, y)$, so that

$$b = c = d = b' = c' = d' = 0,$$

while $z = f(x, y)$ must satisfy the equations

$$\frac{\partial^2 z}{\partial x^2} = a\frac{\partial^2 z}{\partial x \partial y}, \qquad \frac{\partial^2 z}{\partial y^2} = a'\frac{\partial^2 z}{\partial x \partial y}, \tag{12}$$

which must, moreover, be in involution.

If, however, the system of form (1), which is obtained by the above process, cannot be solved for θ_{xx} and θ_{yy}, then as was shown in § 1, by a simple linear transformation of the independent variables x and y we may transform it into a system of form (3). This latter system must then have the solutions

$$1, \qquad \alpha x + \beta y, \qquad \gamma x + \delta y, \qquad z, \qquad \alpha\delta - \beta\gamma \neq 0,$$

where α, β, γ, δ are constants, whence we conclude in the same way as before that z must satisfy an involutory system of form (12). But the equations (12) are in involution if, and only if,

$$aa' = 1, \qquad \frac{\partial a}{\partial y} + a\frac{\partial a}{\partial x} = 0, \tag{13}$$

so that

$$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 0. \tag{14}$$

If x, y, z are cartesian coördinates it is well known that (14) is the condition for developable surfaces. But this condition is left invariant under projective transformations. Therefore, *the surfaces excluded from consideration are developables.* If the condition

$$\frac{\partial a}{\partial y} + a\frac{\partial a'}{\partial x} = 0 \tag{15}$$

were not satisfied, the surface would be a plane. All developables are thus

excluded. For let $z = f(x, y)$ be the equation of a developable. We may determine a from the equation

$$\frac{\partial^2 z}{\partial x^2} = a \frac{\partial^2 z}{\partial x \partial y}$$

if $\partial^2 z / \partial x \partial y \neq 0$. Since the surface is a developable, (14) must be satisfied, which gives

$$\frac{\partial^2 z}{\partial y^2} = \frac{1}{a} \frac{\partial^2 z}{\partial x \partial y};$$

i. e., z satisfies a system of form (12) for which $aa' = 1$. Moreover (15) must then also be satisfied unless z is a linear function of x and y, i. e., unless the surface is a plane. If, however,

$$\frac{\partial^2 z}{\partial x \partial y} = 0,$$

equation (14) shows that either $\partial^2 z / \partial x^2$ or $\partial^2 z / \partial y^2$ must vanish. Suppose we have

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial x^2} = 0.$$

This system is not of form (3), but may be put into that form by a linear transformation of the independent variables. It will be an involutory system as one may see even without making the transformation. For the most general solution of the above system is

$$z = c_1 x + \phi(y)$$

where $\phi(y)$ is an arbitrary function of y, and therefore contains an infinite number of arbitrary constants.

We may recapitulate as follows: 1. *Given a system of partial differential equations of form* (3), *whose integrability conditions are satisfied identically and for which* $aa' - 1$ *is not equal to zero. Its most general integrating surface is a projective transformation of any particular one; it does not degenerate into a curve and is not a developable.*

2. *If for such a system of equations, which is not involutory,* $aa' - 1$ *is equal to zero, its integrating surface degenerates into a plane or line.*

3. *If the system is involutory, its integral surfaces are either curves or developables and the most general integral curve or developable is not a mere projective transformation of any particular one.*

For the purpose of constructing a projective differential geometry of non-developable surfaces we may therefore confine ourselves to the first case. Moreover, *any* non-developable surface may be studied by means of such a system of equations. For let x, y, z be the non-homogeneous coördinates of a point on

any non-developable surface, and take x and y as independent variables. Then z will be a certain function of x and y, say

$$z = f(x, y).$$

We can determine two functions a and a' of x and y so that z will satisfy equations (12), provided that

$$\frac{\partial^2 z}{\partial x \partial y} \neq 0.$$

If, however,

$$\frac{\partial^2 z}{\partial x \partial y} = 0,$$

we may transform the independent variables by putting

$$\bar{x} = \alpha x + \beta y, \qquad \bar{y} = \gamma x + \delta y, \qquad \alpha\delta - \beta\gamma \neq 0,$$

where α, β, γ, δ are constants. These constants may always be chosen in such a way that $\partial^2 z / \partial \bar{x} \partial \bar{y}$ will not be zero, unless not only $\partial^2 z / \partial x \partial y$, but also $\partial^2 z / \partial x^2$ and $\partial^2 z / \partial y^2$ are equal to zero. In that case the surface would be a plane. Excepting this case it is, therefore, always possible to construct a system of form (12) for every surface. Since, moreover, the surface is not developable $aa' - 1$ will not vanish. If homogeneous coördinates be employed, and an arbitrary set of independent variables, a system of form (3) will be obtained for which $aa' - 1$ is different from zero, and which is not involutory.

Therefore, *the theory of a completely integrable non-involutory system of two partial differential equations of the second order, with two independent and one dependent variable, is identical with the projective differential geometry of non-degenerate, non-developable surfaces in three-dimensional space.*

§ 3. *General notions on invariants and covariants.*

Consider a system of form (3) whose integrability conditions are satisfied identically and for which $aa' - 1$ does not vanish. Let

$$y^{(k)} = f^{(k)}(u, v) \qquad (k = 1, 2, 3, 4), \tag{16}$$

be four linearly independent solutions of the system. Equations (16) determine an integral surface S_y of (3), and the most general integral surface of (3) will be a projective transformation of S_y.

But there are some, so to speak, accidental elements in the representation (16) of the surface S_y. In the first place, since the coördinates are homogeneous, multiplication of the four functions $y^{(k)}$ by an arbitrary function λ of u and v will not alter the surface. In the second place the independent variables may

be transformed arbitrarily. If, therefore, we make any transformation of the form

$$\bar{y} = \lambda(u, v)y, \qquad \bar{u} = \phi(u, v), \qquad \bar{v} = \psi(u, v), \tag{17}$$

$$\lambda \neq 0, \qquad \frac{\partial(\phi, \psi)}{\partial(u, v)} \neq 0,$$

the transformed system of differential equations will have the same integral surfaces as (3).

Those combinations of the coefficients of (3) which are unaltered in value by a transformation of form (17), will be called *invariants*. Such invariant functions as depend also upon y and its derivatives will be called *covariants*. Any projective property of a surface will be expressed by an invariant equation, or system of equations, i. e., by one which is left unchanged by transformations of the form (17). The covariants will determine other surfaces and other geometrical configurations which have a projective relation to the given one.

The determination of the invariants and covariants of system (3) under the transformations (17) constitutes, therefore, nothing more or less than a projective differential theory of surfaces. In this general form, however, it is exceedingly difficult. The calculations involved are so long and complicated as to become tedious and uninstructive. We have, therefore, decided to reduce (3) to a canonical form, a form which exists for every surface, so that there is no sacrifice in the generality of our considerations, in so far as they refer to any non-developable surface. It remains desirable, nevertheless, to construct directly an invariant theory for a system of form (3), or more generally still, for a system of form (1) which is not in the canonical form. This, however, is a question which we shall, for the present, leave untouched.

§ 4. *The intermediate form. The surface referred to its asymptotic curves.*

Transform the independent variables in (3) by putting

$$\bar{u} = \phi(u, v), \qquad \bar{v} = \psi(u, v).$$

Let $\bar{y}_u$, $\bar{y}_v$, etc., denote $\partial y/\partial \bar{u}$, $\partial y/\partial \bar{v}$, etc., and put

$$\phi_u = \frac{\partial \phi}{\partial u}, \qquad \phi_v = \frac{\partial \phi}{\partial v}, \qquad \phi_{uu} = \frac{\partial^2 \phi}{\partial u^2},$$

etc. Then we shall find

$$y_u = \phi_u \bar{y}_u + \psi_u \bar{y}_v, \qquad y_v = \phi_v \bar{y}_u + \psi_v \bar{y}_v, \tag{18}$$

and

$$\begin{aligned} y_{uu} &= \phi_u^2 \bar{y}_{uu} + 2\phi_u \psi_u \bar{y}_{uv} + \psi_u^2 \bar{y}_{vv} + \phi_{uu} \bar{y}_u + \psi_{uu} \bar{y}_v, \\ y_{uv} &= \phi_u \phi_v \bar{y}_{uu} + (\phi_u \psi_v + \psi_u \phi_v) \bar{y}_{uv} + \psi_u \psi_v \bar{y}_{vv} + \phi_{uv} \bar{y}_u + \psi_{uv} \bar{y}_v, \\ y_{vv} &= \phi_v^2 \bar{y}_{uu} + 2\phi_v \psi_v \bar{y}_{uv} + \psi_v^2 \bar{y}_{vv} + \phi_{vv} \bar{y}_u + \psi_{vv} \bar{y}_v. \end{aligned} \tag{19}$$

If these values be substituted into (3), if the resulting system of equations be solved for $\bar{y}_{uu}$ and $\bar{y}_{vv}$, and if the new coefficients be denoted by $\bar{a}, \bar{b}, \cdots, \bar{a}', \cdots, \bar{d}'$, we shall find

$$(20)\qquad \begin{aligned} \bar{a} &= \frac{a\psi_v^2 - 2\psi_u\psi_v + a'\psi_u^2}{\phi_u\psi_v + \phi_v\psi_u - a\phi_v\psi_v - a'\phi_u\psi_u}, \\ \bar{a}' &= \frac{a\phi_v^2 - 2\phi_u\phi_v + a'\phi_u^2}{\phi_u\psi_v + \phi_v\psi_u - a\phi_v\psi_v - a'\phi_u\psi_u}, \end{aligned}$$

the other coefficients being fractions with the common denominator

$$(21)\qquad \Delta = (\phi_u\psi_v - \phi_v\psi_u)(\phi_u\psi_v + \phi_v\psi_u - a\phi_v\psi_v - a'\phi_u\psi_u),$$

the first factor of which will not be equal to zero since ϕ and ψ are independent functions of u and v.

Let $\phi_v : \phi_u$ and $\psi_v : \psi_u$ be the two roots of the quadratic

$$at^2 - 2t + a' = 0,$$

so that, for example,

$$\frac{\phi_v}{\phi_u} = \frac{1 + \sqrt{1 - aa'}}{a}, \qquad \frac{\psi_v}{\psi_u} = \frac{1 - \sqrt{1 - aa'}}{a},$$

or

$$(22)\qquad (1 + \sqrt{1 - aa'})\phi_u - a\phi_v = 0, \qquad (1 - \sqrt{1 - aa'})\psi_u - a\psi_v = 0.$$

These equations are distinct since $1 - aa'$ does not vanish, so that the Jacobian of the functions ϕ and ψ will not vanish as a consequence of (22). Assume $a \neq 0$. Then neither ϕ_u nor ψ_u can vanish since, as a consequence of (22), ϕ_v or ψ_v would then also vanish, and this would reduce the Jacobian of ϕ and ψ to zero. The denominator of the expressions (20), or the second factor of Δ, becomes

$$2\frac{\phi_u\psi_u(1 - aa')}{a},$$

which is different from zero. The transformation indicated, therefore, makes $\bar{a} = \bar{a}' = 0$. We have proved this under the assumption that $a \neq 0$. If $a = 0$ and $a' \neq 0$ we may still make $\bar{a} = \bar{a}' = 0$ by a very slight modification of the above argument. If $a = a' = 0$ the original system has the required form without transformation.

Suppose this to be the case. Equations (20) show that this form will not be altered if we make either the transformation

$$\bar{u} = \phi(u), \qquad \bar{v} = \psi(v),$$

or

$$\bar{u} = \phi(v), \qquad \bar{v} = \psi(u),$$

where ϕ and ψ are functions of a single argument.

We may recapitulate as follows. *A system of form* (3), *whose integrating*

surfaces are not developables, may always be reduced to the form

$$(23)\qquad \begin{aligned} y_{uu} &= by_u + cy_v + dy, \\ y_{vv} &= b'y_u + c'y_v + d'y, \end{aligned}$$

which shall be called its intermediate form. The reduction requires the integration of the partial differential equation

$$(24)\qquad a'\left(\frac{\partial\chi}{\partial u}\right)^2 - 2\frac{\partial\chi}{\partial u}\frac{\partial\chi}{\partial v} + a\left(\frac{\partial\chi}{\partial v}\right)^2 = 0.$$

If the independent variables have been selected in such a way as to cause the system to assume this intermediate form, the most general transformation of the independent variables which does not disturb this form is either

$$(25)\qquad \bar{u} = \phi(u), \qquad \bar{v} = \psi(v),$$

or

$$(26)\qquad \bar{u} = \phi(v), \qquad \bar{v} = \psi(u),$$

where ϕ and ψ are arbitrary functions of their arguments.

It is easy to find the geometrical significance of this intermediate form. Consider a curve $v =$ const. on the surface S_y. Let λ and μ be arbitrary functions of u and v. Then the four values of $\lambda y^{(k)} + \mu y_u^{(k)}$ will be the coördinates of an arbitrary point on the tangent of this curve constructed at P_y; similarly $\lambda y^{(k)} + \mu y_u^{(k)} + \nu y_{uu}^{(k)}$ will represent the coördinates of an arbitrary point of the osculating plane of the same curve at P_y. An aggregate of the form $\rho y^{(k)} + \sigma y_u^{(k)} + \tau y_v^{(k)}$ represents a point in the plane tangent to the surface S_y at P_y. Equations (23) show that any arbitrary point of the plane which osculates the curve $v =$ const. at P_y, is a point of the plane which is tangent to the surface at P_y. The curve $v =$ const., therefore, has the property that its osculating planes are at the same time tangent planes of the surface. Such curves are called *asymptotic* curves. From our considerations one easily deduces that there are just two families of ∞^1 asymptotic curves upon the surface. They are the curves $u =$ const. and $v =$ const. determined by the above process.

Reduction of a system of form (3) *to its intermediate form is, therefore, equivalent to the determination of the asymptotic curves of its integrating surfaces.*

It is also clear why the transformations (25) or (26) leave the intermediate form unchanged. They merely transform the asymptotic curves of each family into themselves, or else interchange the two families.

§ 5. *The integrability conditions for a system in the intermediate form. The canonical form.*

Consider a system of the intermediate form

$$(27)\qquad \begin{aligned} y_{uu} + 2ay_u + 2by_v + cy &= 0, \\ y_{vv} + 2a'y_u + 2b'y_v + c'y &= 0. \end{aligned}$$

We proceed to develop its integrability conditions explicitly. By differentiation, we find from (27),

$$
(28)\qquad
\begin{aligned}
y_{uuu} &= p_1 y_{uv} + p_2 y_u + p_3 y_v + p_4 y,\\
y_{uuv} &= q_1 y_{uv} + q_2 y_u + q_3 y_v + q_4 y,\\
y_{uvv} &= r_1 y_{uv} + r_2 y_u + r_3 y_v + r_4 y,\\
y_{vvv} &= s_1 y_{uv} + s_2 y_u + s_3 y_v + s_4 y,
\end{aligned}
$$

where

$$
(29)\qquad
\begin{array}{llll}
p_1 = -2b, & p_2 = 4a^2 - 2a_u - c, & p_3 = 4ab - 2b_u, & p_4 = 2ac - c_u,\\
q_1 = -2a, & q_2 = 4a'b - 2a_v, & q_3 = 4bb' - 2b_v - c, & q_4 = 2bc' - c_v,\\
r_1 = -2b', & r_2 = 4aa' - 2a'_u - c', & r_3 = 4a'b - 2b'_u, & r_4 = 2a'c - c'_u,\\
s_1 = -2a', & s_2 = 4a'b' - 2a'_v, & s_3 = 4b'^2 - 2b'_v - c', & s_4 = 2b'c' - c'_v.
\end{array}
$$

The integrability conditions are obtained by writing

$$
\frac{\partial y_{uuu}}{\partial v} = \frac{\partial y_{uuv}}{\partial u}, \qquad \frac{\partial y_{uuv}}{\partial v} = \frac{\partial y_{uvv}}{\partial u}, \qquad \frac{\partial y_{uvv}}{\partial v} = \frac{\partial y_{vvv}}{\partial u}.
$$

It will be found that the first and last of these three conditions are fulfilled identically. We find further

$$
\begin{aligned}
\frac{\partial y_{uuv}}{\partial v} &= \left(\frac{\partial q_1}{\partial v} + q_1 r_1 + q_2\right) y_{uv} + \left(\frac{\partial q_2}{\partial v} + q_1 r_2 - 2a' q_3\right) y_u\\
&\qquad + \left(\frac{\partial q_3}{\partial v} + q_1 r_3 - 2b' q_3 + q_4\right) y_v + \left(\frac{\partial q_4}{\partial v} + q_1 r_4 - c' q_3\right) y,\\
\frac{\partial y_{uvv}}{\partial u} &= \left(\frac{\partial r_1}{\partial u} + r_1 q_1 + r_3\right) y_{uv} + \left(\frac{\partial r_2}{\partial u} + r_1 q_2 - 2a r_2 + r_4\right) y_u\\
&\qquad + \left(\frac{\partial r_3}{\partial u} + r_1 q_3 - 2b r_2\right) y_v + \left(\frac{\partial r_4}{\partial u} + r_1 q_4 - c r_2\right) y.
\end{aligned}
$$

In order that these two expressions may be equal for all values of y, y_u, y_v, y_{uv}, the corresponding coefficients in the two expressions must be identical. We thus find the following four conditions:

$$
(30)\qquad
\begin{aligned}
&a_v - b'_u = 0,\\
&a'_{uu} + c'_u - 2a' a_u - 2a a'_u - (a_{vv} + 2b' a_v - 2b a'_v - 4a' b_v) = 0,\\
&b_{vv} + c_v - 2b b'_v - 2b' b_v - (b'_{uu} + 2a b'_u - 2a' b_u - 4b a'_u) = 0,\\
&c'_{uu} - 4c a'_u - 2a' c_u + 2a c'_u - (c_{vv} - 4c' b_v - 2b c'_v + 2b' c_v) = 0.
\end{aligned}
$$

The conditions (30) *must be satisfied in order that the general solution of* (27) *may depend upon just four arbitrary constants, for this is the only case of*

interest from the point of view of the theory of surfaces, as was explained in § 2. These integrability conditions (30) are symmetrically constructed from the coefficients of (27) in such a way that the operation which consists in interchanging simultaneously the letters in the two rows of the symbol

$$\begin{pmatrix} a & a' & c & u \\ b' & b & c' & v \end{pmatrix},$$

converts the first and fourth equations into themselves and interchanges the other two.

From the significance of the conditions (30), it is clear that they must form an invariant system of equations. In other words, if the variables in (27) are changed arbitrarily, but in such a way as not to disturb the form of the system, the integrability conditions for the transformed system of equations will be fulfilled if they are fulfilled for the original system, and conversely.

On account of the first condition of (30) we may write

$$a = p_u, \qquad b' = p_v. \tag{31}$$

Transform (27) by putting

$$y = \lambda \bar{y},$$

where λ is a function of u and v. We find a new system of the same form for $\bar{y}$ whose coefficients are

$$\bar{a} = a + \frac{\lambda_u}{\lambda}, \qquad \bar{b} = b, \qquad \bar{c} = c + \frac{\lambda_{uu} + 2a\lambda_u + 2b\lambda_v}{\lambda},$$

$$\bar{a}' = a', \qquad \bar{b}' = b' + \frac{\lambda_v}{\lambda}, \qquad \bar{c}' = c' + \frac{\lambda_{vv} + 2a'\lambda_u + 2b'\lambda_v}{\lambda}.$$

Owing to (31) we may make $\bar{a} = \bar{b}' = 0$ by putting

$$\lambda = e^{-p},$$

so that

$$\frac{\lambda_u}{\lambda} = -p_u, \qquad \frac{\lambda_v}{\lambda} = -p_v,$$

$$\frac{\lambda_{uu}}{\lambda} = -p_{uu} + p_u^2, \qquad \frac{\lambda_{vv}}{\lambda} = -p_{vv} + p_v^2,$$

whence

$$\begin{aligned} \bar{c} &= c - a_u - a^2 - 2bb' = f, \\ \bar{c}' &= c' - b'_v - b'^2 - 2aa' = g. \end{aligned} \tag{32}$$

It becomes possible, therefore, to put the system (27) *into the form*

$$\begin{aligned} y_{uu} + 2by_v + fy &= 0, \\ y_{vv} + 2a'y_u + gy &= 0, \end{aligned} \tag{33}$$

which is characterized by the conditions $a = b' = 0$. We shall speak of (33) as the *canonical form* of the given system of differential equations. It may be obtained from the intermediate form in the way just indicated. *The first of the four integrability conditions* (30) *is satisfied identically. The other three assume the simplified form:*

$$
\begin{aligned}
& a'_{uu} + g_u + 2ba'_v + 4a'b_v = 0, \\
(34) \qquad & b_{vv} + f_v + 2a'b_u + 4ba'_u = 0, \\
& g_{uu} - f_{vv} - 4ga'_u - 2a'f_u + 4fb_v + 2bg_v = 0.
\end{aligned}
$$

§ 6. *Seminvariants and semi-covariants.*

Consider again the system (27) of differential equations, and let it be transformed by putting

$$(35) \qquad y = \lambda \bar{y},$$

where λ is an arbitrary function of u and v. Those functions of the coefficients of (27), of y, and of the derivatives of these quantities, which are left unchanged by every such transformation, shall be called *seminvariants* or *semi-covariants* according as they do not, or do contain y and its derivatives.

We find from (35)

$$
\begin{aligned}
& y_u = \lambda \bar{y}_u + \lambda_u \bar{y}, \qquad y_v = \lambda \bar{y}_v + \lambda_v \bar{y}, \\
& y_{uu} = \lambda \bar{y}_{uu} + 2\lambda_u \bar{y}_u + \lambda_{uu} \bar{y}, \\
(36) \qquad & y_{uv} = \lambda \bar{y}_{uv} + \lambda_u \bar{y}_v + \lambda_v \bar{y}_u + \lambda_{uv} \bar{y}, \\
& y_{vv} = \lambda \bar{y}_{vv} + 2\lambda_v \bar{y}_v + \lambda_{vv} \bar{y},
\end{aligned}
$$

which gives, upon substitution into (27),

$$
\begin{aligned}
& \bar{y}_{uu} + 2\bar{a}\bar{y}_u + 2\bar{b}\bar{y}_v + \bar{c}\bar{y} = 0, \\
& \bar{y}_{vv} + 2\bar{a}'\bar{y}_u + 2\bar{b}'\bar{y}_v + \bar{c}'\bar{y} = 0,
\end{aligned}
$$

where

$$
\begin{aligned}
(37) \qquad & \bar{a} = a + \frac{\lambda_u}{\lambda}, \qquad \bar{b} = b, \qquad \bar{c} = c + \frac{\lambda_{uu} + 2a\lambda_u + 2b\lambda_v}{\lambda}, \\
& \bar{a}' = a', \qquad \bar{b}' = b' + \frac{\lambda_v}{\lambda}, \qquad \bar{c}' = c' + \frac{\lambda_{vv} + 2a'\lambda_u + 2b'\lambda_v}{\lambda}.
\end{aligned}
$$

We find the following seminvariants

$$
\begin{aligned}
(38) \qquad & a', \qquad b, \\
& f = c - a_u - a^2 - 2bb', \qquad g = c' - b'_v - b'^2 - 2aa'.
\end{aligned}
$$

The derivatives of these four quantities are also seminvariants. *Every semin-*

variant is a function of a', b, f, g and of their derivatives. That this is true may be, most simply, seen as follows: A seminvariant has the same value for system (27) as for any system obtained from it by a transformation of form (35). Make the transformation of form (35), which reduces (27) to its canonical form according to the method of § 5. The resulting system (33) has precisely a', b, f and g as its coefficients. Every seminvariant can, therefore, contain only these quantities and their derivatives, as was to be shown.

Semi-covariants contain y, y_u, y_v, y_{uv}, etc., in addition to the coefficients of (27). Since all higher derivatives of y may be expressed in terms of the four quantities

$$y,\ y_u,\ y_v,\ y_{uv},$$

we may confine our search for semi-covariants to such as involve them. We find, from (36),

$$\begin{aligned} y &= \frac{1}{\lambda}\bar{y}, \\ \bar{y}_u &= \frac{1}{\lambda}\left(y_u - \frac{\lambda_u}{\lambda}y\right), \qquad \bar{y}_v = \frac{1}{\lambda}\left(y_v - \frac{\lambda_v}{\lambda}y\right), \\ \bar{y}_{uv} &= \frac{1}{\lambda}\left[y_{uv} - \frac{\lambda_u}{\lambda}y_v - \frac{\lambda_v}{\lambda}y_u + \left(\frac{2\lambda_u\lambda_v}{\lambda^2} - \frac{\lambda_{uv}}{\lambda}\right)y\right], \end{aligned} \tag{39}$$

so that y itself is a *relative* semi-covariant. By making use of (37) we find three other relative semi-covariants

$$\begin{aligned} z &= y_u + ay, \qquad \rho = y_v + b'y, \\ \sigma &= y_{uv} + b'y_u + ay_v + \tfrac{1}{2}(a_v + b'_u + 2ab')y; \end{aligned} \tag{40}$$

in fact these quantities satisfy the equations

$$\bar{y} = \lambda y, \qquad \bar{z} = \lambda z, \qquad \bar{\rho} = \lambda\rho, \qquad \bar{\sigma} = \lambda\sigma,$$

so that their ratios are absolute semi-covariants. The four semi-covariants y, z, ρ, σ are clearly independent. Moreover equations (40) show that y, y_u, y_v, y_{uv} may be expressed in terms of y, z, ρ, σ. Since every relative semicovariant can be expressed in terms of y, y_u, y_v, y_{uv}, it may be also expressed in terms of y, z, ρ, σ. By reducing to the canonical form it becomes obvious that *all semi-covariants can be expressed as functions of y, z, ρ, σ and of seminvariants.*

§ 7. *Invariants and covariants.*

In order to find invariants we introduce transformations of the independent variables which leave the intermediate form of the system of partial differential equations unaltered. These are either of form (25) or (26). Since (26) may be obtained from (25) by the simple transformation

$$u = \bar{v}, \qquad v = \bar{u},$$

which merely interchanges the two families of asymptotic lines, it becomes an easy matter to find the invariants under the general transformation if those under transformation (25) have been found. It is desirable, moreover, to keep the two families of asymptotic curves separate. For these reasons we confine ourselves to the transformation (25) or as we shall now write it

$$\bar{u} = \alpha(u), \qquad \bar{v} = \beta(v), \tag{41}$$

denoting the new independent variables by $\bar{u}$ and $\bar{v}$ where α is a function of u alone, and β is a function of the single variable v. Upon making the transformation (41), we find

$$\begin{gathered} y_u = \alpha_u \frac{\partial y}{\partial \bar{u}}, \qquad y_{uu} = \alpha_u^2 \frac{\partial^2 y}{\partial \bar{u}^2} + \alpha_{uu} \frac{\partial y}{\partial \bar{u}}, \\ y_v = \beta_v \frac{\partial y}{\partial \bar{v}}, \qquad y_{vv} = \beta_v^2 \frac{\partial^2 y}{\partial \bar{v}^2} + \beta_{vv} \frac{\partial y}{\partial \bar{v}}, \\ y_{uv} = \alpha_u \beta_v \frac{\partial^2 y}{\partial \bar{u} \partial \bar{v}}. \end{gathered} \tag{42}$$

Equations (27) are transformed into a similar system, whose coefficients are

$$\begin{gathered} \bar{a} = \frac{1}{\alpha_u}(a + \tfrac{1}{2}\eta), \qquad \bar{b} = \frac{\beta_v b}{\alpha_u^2}, \qquad \bar{c} = \frac{c}{\alpha_u^2}, \\ \bar{a}' = \frac{\alpha_u a'}{\beta_v^2}, \qquad \bar{b}' = \frac{1}{\beta_v}(b' + \tfrac{1}{2}\zeta), \qquad \bar{c}' = \frac{c'}{\beta_v^2}, \end{gathered} \tag{43}$$

where

$$\eta = \frac{\alpha_{uu}}{\alpha_u}, \qquad \zeta = \frac{\beta_{vv}}{\beta_v}. \tag{44}$$

These equations show that a' and b are (relative) invariants as well as seminvariants.

Since invariants are also seminvariants, they can only be functions of a', b, f, g and of their derivatives. It becomes necessary, therefore, to investigate how the transformation (41) affects these seminvariants. Invariants are such functions of seminvariants as are left unaltered by this transformation.

We find, from (43),

$$\begin{aligned} &\frac{\partial \bar{a}}{\partial \bar{u}} = \frac{1}{\alpha_u^2}[a_u - \eta a + \tfrac{1}{2}(\eta_u - \eta^2)], && \frac{\partial \bar{a}}{\partial \bar{v}} = \frac{a_v}{\alpha_u \beta_v}, \\ &\frac{\partial \bar{b}'}{\partial \bar{u}} = \frac{b'_u}{\alpha_u \beta_v}, && \frac{\partial \bar{b}'}{\partial \bar{v}} = \frac{1}{\beta_v^2}[b'_v - \zeta b' + \tfrac{1}{2}(\zeta_v - \zeta^2)], \\ &\frac{\partial \bar{a}'}{\partial \bar{u}} = \frac{1}{\beta_v^2}(a'_u + \eta a'), && \frac{\partial \bar{a}'}{\partial \bar{v}} = \frac{\alpha_u}{\beta_v^3}(a'_v - 2\zeta a'), \\ &\frac{\partial \bar{b}}{\partial \bar{u}} = \frac{\beta_v}{\alpha_u^3}(b_u - 2\eta b), && \frac{\partial \bar{b}}{\partial \bar{v}} = \frac{1}{\alpha_u^2}(b_v + \zeta b). \end{aligned} \tag{45}$$

From the expressions (38) for f and g, and the above equations, we find

$$\bar{f} = \frac{1}{\alpha_u^2}(f - b\zeta - \tfrac{1}{2}\mu), \qquad \bar{g} = \frac{1}{\beta_v^2}(g - a'\eta - \tfrac{1}{2}\nu), \tag{46}$$

where

$$\mu = \eta_u - \tfrac{1}{2}\eta^2, \qquad \nu = \zeta_v - \tfrac{1}{2}\zeta^2. \tag{47}$$

We have further

$$\begin{aligned} \frac{\partial^2 \bar{a}'}{\partial \bar{u}^2} &= \frac{1}{\alpha_u \beta_v^2}[a'_{uu} + \eta a'_u + \eta_u a'], \\ \frac{\partial^2 \bar{a}'}{\partial \bar{u}\,\partial \bar{v}} &= \frac{1}{\beta_v^3}[a'_{uv} + \eta a'_v - 2\zeta a'_u - 2\eta\zeta a'], \\ \frac{\partial^2 \bar{a}'}{\partial \bar{v}^2} &= \frac{\alpha_u}{\beta_v^4}[a'_{vv} - 5\zeta a'_v - 2\zeta_v a' + 6\zeta^2 a'], \end{aligned} \tag{48a}$$

and

$$\begin{aligned} \frac{\partial^2 \bar{b}}{\partial \bar{u}^2} &= \frac{\beta_v}{\alpha_u^4}[b_{uu} - 5\eta b_u - 2\eta_u b + 6\eta^2 b], \\ \frac{\partial^2 \bar{b}}{\partial \bar{u}\,\partial \bar{v}} &= \frac{1}{\alpha_u^3}[b_{uv} + \zeta b_u - 2\eta b_v - 2\eta\zeta b], \\ \frac{\partial^2 \bar{b}}{\partial \bar{v}^2} &= \frac{1}{\alpha_u^2 \beta_v}[b_{vv} + \zeta b_v + \zeta_v b]. \end{aligned} \tag{48b}$$

By means of these equations it is easy to verify that

$$\bar{h} = \frac{\beta_v^2 h}{\alpha_u^6}, \qquad \bar{k} = \frac{\alpha_u^2 k}{\beta_v^6}, \tag{49}$$

where

$$h = b^2(f + b_v) - \tfrac{1}{4}bb_{uu} + \tfrac{5}{16}b_u^2, \qquad k = a'^2(g + a'_u) - \tfrac{1}{4}a'a'_{vv} + \tfrac{5}{16}a'^2_v, \tag{50}$$

so that we have found four invariants a', b, h and k. We shall show how other invariants may be derived from these four by certain differentiation processes. Put

$$A = a'b^2, \qquad B = a'^2 b, \qquad H = a'h, \qquad K = bk \tag{51}$$

so that

$$\bar{A} = \frac{A}{\alpha_u^3}, \qquad \bar{B} = \frac{B}{\beta_v^3}, \qquad \bar{H} = \frac{H}{\alpha_u^5}, \qquad \bar{K} = \frac{K}{\beta_v^5}, \tag{52}$$

it being assumed that a' and b are different from zero.

Introduce two symbols of operation

$$U = a'\frac{\partial}{\partial u}, \qquad V = b\frac{\partial}{\partial v}. \tag{53}$$

Then clearly

$$U(B), \qquad V(A), \qquad U(K), \qquad V(H) \tag{54}$$

are also invariants. In fact

$$(55)\quad \overline{V}(\overline{A}) = \frac{V(A)}{\alpha_u^5}, \quad \overline{U}(\overline{B}) = \frac{U(B)}{\beta_v^5}, \quad \overline{V}(\overline{H}) = \frac{V(H)}{\alpha_u^7}, \quad \overline{U}(\overline{K}) = \frac{U(K)}{\beta_v^7},$$

so that it becomes clear that every repetition of these two processes gives rise to an invariant.

Further, from two invariants λ and μ for which

$$\overline{\lambda} = \frac{1}{\alpha_u^l}\lambda, \qquad \mu = \frac{1}{\alpha_u^m}\mu,$$

we can always form another, its Wronskian. In fact we find

$$\overline{\lambda^m \mu^{-l}} = \lambda^m \mu^{-l},$$

whence, by logarithmic differentiation,

$$\overline{m\frac{\lambda_u}{\lambda} - l\frac{\mu_u}{\mu}} = \frac{1}{\alpha_u}\left(m\frac{\lambda_u}{\lambda} - l\frac{\mu_u}{\mu}\right),$$

so that the function

$$(56)\qquad (\mu, \lambda_u) = m\mu\lambda_u - l\lambda\mu_u$$

is an invariant for which

$$(57)\qquad \overline{(\mu\lambda_u)} = \frac{(\mu\lambda_u)}{\alpha_u^{l+m+1}}.$$

We shall speak of $(\mu\lambda_u)$ as the Wronskian of μ and λ with respect to u.

By combining the Wronskian process with the operations U and V, it clearly becomes possible to deduce an infinity of invariants from A, B, H and K. We proceed to show that any invariant is a function of those obtained in this manner.

In the first place, the equations which define A, B, H and K, show that a', b, f and g may be expressed in terms of A, B, H, K and of their derivatives. Since all invariants are functions of a', b, f, g, and of their derivatives, any invariant may be expressed as a function of A, B, H, K and of the derivatives of these quantities.

Introduce infinitesimal transformations by putting

$$(58)\qquad \alpha(u) = u + \phi(u)\delta t \qquad \beta(v) = v + \psi(v)\delta t,$$

where δt is an infinitesimal. Then

$$(59)\qquad \begin{aligned} &\alpha_u = 1 + \phi_u\delta t, && \alpha_v = 0, \\ &\beta_u = 0, && \beta_v = 1 + \psi_v\delta t, \\ &\alpha_u\beta_v - \alpha_v\beta_u = 1 + (\phi_u + \psi_v)\delta t. \end{aligned}$$

The infinitesimal transformations of A, B, H and K become

$$(6)\qquad \delta A = -3\phi_u A\delta t,\ \delta B = -3\psi_v B\delta t,\ \delta H = -5\phi_u H\delta t,\ \delta K = -5\psi_v K\delta t.$$

The infinitesimal transformation, δF, of any function of A, B, etc., being known, we may easily find the infinitesimal transformations δF_u and δF_v of its first derivatives. In fact we shall have

$$(61)\qquad \begin{aligned} \delta F_u &= \frac{\partial \bar{F}}{\partial \bar{u}} - \frac{\partial F}{\partial u} = \frac{\partial(F+\delta F)}{\partial u}\,\frac{1}{1+\phi_u\delta t} - \frac{\partial F}{\partial u} = \frac{\partial(\delta F)}{\partial u} - \phi_u F_u \delta t, \\ \delta F_v &= \frac{\partial \bar{F}}{\partial \bar{v}} - \frac{\partial F}{\partial v} = \frac{\partial(F+\delta F)}{\partial v}\,\frac{1}{1+\psi_v\delta t} - \frac{\partial F}{\partial v} = \frac{\partial(\delta F)}{\partial v} - \psi_v F_v \delta t. \end{aligned}$$

Thus we find

$$(62)\qquad \begin{aligned} \delta A_u &= -(3\phi_{uu}A + 4\phi_u A_u)\delta t, & \delta A_v &= -(3\phi_u + \psi_v)A_v\delta t, \\ \delta B_u &= -(\phi_u + 3\psi_v)B_u\delta t, & \delta B_v &= -(3\psi_{vv}B + 4\psi_v B_v)\delta t, \\ \delta H_u &= -(5\phi_{uu}H + 6\phi_u H_u)\delta t, & \delta H_v &= -(5\phi_u + \psi_v)H_v\delta t, \\ \delta K_u &= -(\phi_u + 5\psi_v)K_u\delta t, & \delta K_v &= -(5\psi_{vv}K + 6\psi_v K_v)\delta t. \end{aligned}$$

All absolute invariants θ which involve only A, B, H, K and their first derivatives must, therefore, satisfy the system of partial differential equations obtained by forming the infinitesimal transformation $\delta\theta$ of an arbitrary function of these arguments and equating to zero the coefficients of ϕ_{uu}, ψ_{vv}, ϕ_u and ψ_v. If we are looking for relative invariants we may dispense with the latter two equations, as the following consideration will show. Consider the special transformation for which

$$(63)\qquad \alpha(u) = cu, \qquad \beta(v) = c'v,$$

so that

$$\alpha_u = c, \qquad \beta_v = c',$$

where c and c' are arbitrary constants. Then

$$\bar{A} = c^{-3}A, \qquad \bar{B} = (c')^{-3}B, \quad \text{etc.},$$

$$\bar{A}_u = c^{-4}A_u, \quad \text{etc.}$$

In general, let P be a combination of the quantities A, B, H, K and of their derivatives, and suppose that the effect of the transformation (63) is to convert it into $\bar{P}$, where

$$\bar{P} = c^{-p}c'^{-p'}P.$$

We shall then say that P has an α-weight equal to p and a β-weight equal to p'. Thus the weights of A and B are $(+3, 0)$ and $(0, +3)$ respectively. It is clear that differentiation with respect to u and v diminishes the α- or β-

weight respectively by unity. The weights of a product are the sums of the respective weights of the factors.

We shall say of a sum that it is *isobaric* of weights w, w' if each of its terms has the weights w, w'. It is clear that every absolute invariant is isobaric of weights $0, 0$. *Every relative invariant must be isobaric.*

We obtain an infinitesimal transformation of the form (63) if we let ϕ_u and ψ_v in (60) and (62) represent constants. Under this assumption, if F is any isobaric function of weights w and w', the infinitesimal transformation of F will be of the form

$$\delta F = -(w\phi_u + w'\psi_v)F\delta t. \tag{64}$$

If ϕ_u and ψ_v are not constants, the expression for δF will contain other terms which will have ϕ_{uu}, ψ_{vv}, etc. as factors. If, however F is an irreducible (relative or absolute) invariant, the equation $\delta F = 0$ must be a consequence of $F = 0$, which implies that if F is an invariant of weights w, w', the equation (64) must be satisfied even if ϕ_u and ψ_v are not constants.*

Suppose then that we form the general expression for δF, where F depends upon A, B, H, K and the derivatives of these quantities up to any order n, and equate to zero the coefficients of all of the derivatives of the arbitrary functions ϕ and ψ, including ϕ_u and ψ_v. We thus obtain a complete system S of partial differential equations, whose solutions are the *absolute* invariants up to the nth order. If we omit the two equations of this system obtained by equating to zero the coefficients of ϕ_u and ψ_v, we obtain a system S' of partial differential equations satisfied by all relative invariants. Moreover, only such solutions of this system S' will be relative invariants for which (64) is satisfied, i. e., only isobaric solutions of the system are admissible as invariants. The relative invariants are certainly not all of weight zero. We know in fact that A, B, H and K are independent relative invariants, both of whose weights are not zero. Any relative invariant can, however, by multiplication with properly chosen powers of A and B be converted into an absolute invariant. Consequently there are two more independent relative invariants than absolute invariants. If the system S' were not a complete system, the difference between the number of relative and absolute invariants could not be equal to two. *The system S' is, therefore, a complete system and every isobaric solution of S' is a relative invariant.*

All relative invariants which involve only A, B, H, K and their first derivatives are, therefore, isobaric solutions of the complete system of two equations

$$3A\frac{\partial\theta}{\partial A_u} + 5H\frac{\partial\theta}{\partial H_u} = 0, \qquad 3B\frac{\partial\theta}{\partial B_v} + 5K\frac{\partial\theta}{\partial K_v} = 0,$$

* For the coefficients of ϕ_{uu}, ψ_{vv}, etc. cannot vanish in consequence of $F = 0$, if we assume F to be irreducible, since their coefficients are of weights lower than w, w'. The complete discussion would be the same as that in the author's treatise *Projective Differential Geometry*, etc., p. 22.

which involves (in part with zero coefficients) twelve independent variables. It has therefore ten independent solutions. These may be selected in the form

$$(65a) \qquad A,\ B,\ H,\ K,\ V(A),\ U(B),\ V(H),\ U(K),$$

together with the two Wronskians

$$(65b) \qquad (A, H_u), \qquad (B, K_v).$$

For they are clearly independent. In fact, taken in this order, each of these functions involves one variable which does not occur in any of the preceding ones.

All relative invariants which involve also the second derivatives of A, B, H, K will be isobaric solutions of a complete system of four independent equations with twenty-four independent variables. They are, therefore, twenty in number. Ten of these we have already found. The other ten may be taken in the form

$$(66) \qquad \begin{array}{l} V^2(A),\ \ V^2(H),\ \ V(A, H_u),\ \ \big(A, V(A)_u\big),\ \ \big(A, (A, H_u)_u\big), \\ U^2(B),\ \ U^2(K),\ \ U(B, K_v),\ \ \big(B, U(B)_v\big),\ \ \big(B, (B, K_v)_v\big); \end{array}$$

for it is apparent that these are independent.

We have now determined all invariants up to and including those of the second order; they are isobaric functions of the twenty invariants (65) and (66). We make the following remarks. Let an invariant be called an α- or a β-invariant if only one of its two weights is different from zero, so that the β-weight of an α-invariant is zero, etc. Our invariants have been so chosen that half of them are α- and half of them β-invariants. Consider the five α-invariants of the second order. They are those in the first row of (66). Two of them, the first and the fourth do not contain H, and contain only two of the three derivatives of the second order of A, viz., A_{vv} and A_{uv}. The other three α-invariants contain each a different one of the three second derivatives of H. A similar classification may be made of the β-invariants. Suppose we had determined all of the invariants up to and including those of the $n-1$th order. To determine those of the nth order we should have to solve a system of partial differential equations containing two more equations and $4(n+1)$ more variables than the system for the invariants of order $n-1$. There will be, therefore, $4n+2$ invariants which involve the nth derivatives of A, B, H, K. Similarly there will be just $4n-2$ invariants of the $n-1$th order.

Let us suppose that what we have found to be true for the invariants of the second order is also true for those of order $n-1$, i. e., let us suppose that $2n-1$ of them are α- and $2n-1$ of them β-invariants. Denote these α-invariants of the $n-1$th order by

$$I_1,\ I_2,\ \cdots,\ I_{n-1}; \qquad I'_1,\ I'_2,\ \cdots,\ I'_n.$$

The notation indicates a division of these α-invariants into two classes; those of the first class do not depend upon H and its derivatives, those of the second class do. Let

$$P^{(n-1)}_{u_r,\, v_s} \qquad (r+s=n-1),$$

denote the derivative of P taken r times with respect to u and s times with respect to v. Then we shall assume that $I_1, I_2, \cdots, I_{n-1}$ contain in this order $A^{(n-1)}_{u_0,\, v_{n-1}}, A^{(n-1)}_{u_1,\, v_{n-2}}, \cdots, A^{(n-1)}_{u_{n-2},\, v_1}$, and that no function I_k contains $A^{(n-1)}_{u_{n-1},\, v_0}$.

We assume further that the functions $I'_1, I'_2, \cdots, I'_n$ taken in this order contain respectively $H^{(n-1)}_{u_0,\, v_{n-1}}, H^{(n-1)}_{u_1,\, v_{n-2}}, \cdots, H^{(n-1)}_{u_{n-1},\, v_0}$.

The $2n-1$ functions

$$V(I_k), \qquad V(I'_k) \tag{67}$$

will then be α-invariants of the nth order. Those of the first set contain all of the derivatives of A of the nth order except $A^{(n)}_{u_{n-1},\, v_1}, A^{(n)}_{u_n,\, v_0}$. Those of the second set contain all of the derivatives of H of the nth order except $H^{(n)}_{u_n,\, v_0}$.

Consider the Wronskian of A and I_{n-1} with respect to u. It will be an α-invariant which will contain the derivative $A^{(n)}_{u_{n-1},\, v_1}$. Similarly the Wronskian of A and I'_n will be an α-invariant which contains $H^{(n)}_{u_n,\, v_0}$. If we add these two α-invariants to the $2n-1$ of (67), we shall have $2n+1$ independent α-invariants of order n which can be divided into two classes precisely in the same way as was assumed to be the case for those of the $n-1$th order. Since this classification was actually possible for the invariants of 2, it follows that this same classification will actually hold for any order n.

We have seen, therefore, that the Wronskian process and the process V enables us to obtain $2n+1$ independent α-invariants of order n from A and H. Similarly, by means of the Wronskian process and the process U we may obtain $2n+1$ independent β-invariants of order n from B and K. It only remains to show that these $4n+2$ invariants are also independent of each other. There can be no question about the independence of those which contain H and K. But the α-invariants of the first class contain derivatives of B as well as of A, and the β-invariants of the first class contain derivatives of A as well as of B. However, the derivatives of B in the α-invariants of order n are only of order $n-1$, so that no relation is possible between the α- and β-invariants.

The Wronskian, the U-process and the V-process suffice, therefore, to deduce all invariants from the four fundamental ones A, B, H and K. There are $2(n^2+2n+2)$ independent invariants of order equal to or less than n.

In order to find a system of covariants we investigate the result of the transformation

$$\bar{u} = \alpha(u), \qquad \bar{v} = \beta(v),$$

upon the four independent semi-covariants (40). We find

$$(68)\qquad \begin{gathered} \bar{y} = y, \qquad \bar{z} = \frac{1}{\alpha_u}(z + \tfrac{1}{2}\eta y), \qquad \bar{\rho} = \frac{1}{\beta_v}(\rho + \tfrac{1}{2}\zeta y), \\ \bar{\sigma} = \frac{1}{\alpha_u \beta_v}(\sigma + \tfrac{1}{2}\eta\rho + \tfrac{1}{2}\zeta z + \tfrac{1}{4}\eta\zeta y). \end{gathered}$$

It now becomes an easy matter to verify that the following four functions are covariants

$$(69)\qquad y, \qquad 6Az + A_u y, \qquad 6B\rho + B_v y, \qquad \sigma - a\rho - b'z + ab'y.$$

They are clearly independent. *Every other covariant may be expressed in terms of the covariants* (69) *and of invariants.*

The most general transformation which leaves the canonical form invariant may be found as follows. We have shown that a system of differential equations of form (3) may be reduced to the canonical form

$$y_{uu} + 2by_v + fy = 0,$$
$$y_{vv} + 2a'y_u + gy = 0.$$

Make successively the transformations

$$y = \lambda\bar{y}, \qquad \bar{u} = \alpha(u), \qquad \bar{v} = \beta(v).$$

The resulting system will again be in the canonical form if, and only if,

$$\frac{\lambda_u}{\lambda} + \frac{1}{2}\frac{\alpha_{uu}}{\alpha_u} = 0, \qquad \frac{\lambda_v}{\lambda} + \frac{1}{2}\frac{\beta_{vv}}{\beta_v} = 0,$$

i. e., if

$$\lambda = \frac{g(v)}{\sqrt{\alpha_u}} = \frac{h(u)}{\sqrt{\beta_v}},$$

where $g(v)$ and $h(u)$ are functions of the single arguments indicated. We find, therefore,

$$\lambda = \frac{C}{\sqrt{\alpha_u \beta_v}},$$

where C is an arbitrary constant.

Therefore, *the most general transformation which leaves the canonical form invariant is*

$$(70)\qquad \bar{y} = C\sqrt{\alpha_u \beta_v}\, y, \qquad \bar{u} = \alpha(u), \qquad \bar{v} = \beta(v),$$

where α and β are arbitrary functions of u and v respectively, and where C is an arbitrary constant.

§ 8. *Principle of duality. Adjoined systems. The fundamental theorem.*

Given a system of partial differential equations in the intermediate form

$$(71)\qquad \begin{aligned} y_{uu} + 2ay_u + 2by_v + cy &= 0, \\ y_{vv} + 2a'y_u + 2b'y_v + c'y &= 0, \end{aligned}$$

whose integrability conditions are fulfilled identically. If

$$y^{(k)} = f^{(k)}(u, v) \qquad (k = 1, 2, 3, 4),$$

are four independent solutions of (71), we interpret them as the homogeneous coördinates of a point P_y of a surface S. Let p_y be the plane which is tangent to S at P_y. Its equation will be

$$(72)\qquad \begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y' & y'' & y^{(3)} & y^{(4)} \\ y'_u & y''_u & y^{(3)}_u & y^{(4)}_u \\ y'_v & y''_v & y^{(3)}_v & y^{(4)}_v \end{vmatrix} = 0.$$

The coördinates of the plane p_y will be proportional to the minors of x_1, x_2, x_3, x_4 in this determinant. Denote these minors by $\lambda', \lambda'', \lambda^{(3)}, \lambda^{(4)}$. We shall have

$$\lambda' = \begin{vmatrix} y'' & y^{(3)} & y^{(4)} \\ y''_u & y^{(3)}_u & y^{(4)}_u \\ y''_v & y^{(3)}_v & y^{(4)}_v \end{vmatrix}, \qquad \lambda'' = -\begin{vmatrix} y' & y^{(3)} & y^{(4)} \\ y'_u & y^{(3)}_u & y^{(4)}_u \\ y'_v & y^{(3)}_v & y^{(4)}_v \end{vmatrix}, \text{ etc.}$$

Let us convert the columns into rows and write only a representative row. We may then write for each of the four λ's, disregarding signs, an expression of the form

$$(73)\qquad \lambda = D(y, y_u, y_v),$$

which stands for a determinant of the third order of the matrix

$$|y^{(k)},\ y^{(k)}_u,\ y^{(k)}_v| \qquad (k = 1, 2, 3, 4).$$

We find from (73), making use of (71),

$$(74)\qquad \begin{aligned} \lambda_u &= D(y, y_u, y_{uv}) - 2a\lambda, \\ \lambda_v &= D(y, y_{uv}, y_v) - 2b'\lambda. \end{aligned}$$

Therefore, making use of (28) and (29),

$$(75)\qquad \begin{aligned} \lambda_{uu} &= -2a\lambda_u + (4bb' - 2b_v - c - 2a_u)\lambda - 4aD(y, y_u, y_{uv}) - 2bD(y, y_v, y_{uv}), \\ \lambda_{vv} &= -2b'\lambda_v + (4aa' - 2a'_u - c' - 2b'_v)\lambda - 2a'D(y, y_{uv}, y_u) - 4b'D(y, y_{uv}, y_v). \end{aligned}$$

But from (74),

$$D(y,\ y_u,\ y_{uv}) = \lambda_u + 2a\lambda,$$

$$D(y,\ y_{uv},\ y_v) = \lambda_v + 2b'\lambda.$$

Substituting these values in (75) we find

$$(76)\qquad \begin{aligned} &\lambda_{uu} + 6a\lambda_u - 2b\lambda_v + (2a_u + 2b_v + c + 8a^2 - 8bb')\lambda = 0, \\ &\lambda_{vv} - 2a'\lambda_u + 6b'\lambda_v + (2b'_v + 2a'_u + c' + 8b'^2 - 8aa')\lambda = 0. \end{aligned}$$

The coördinates of the plane p_y, therefore, satisfy the equations (76) *which are of the same form as* (71).

This system of equations (76) may be simplified. Consider the determinant,

$$(77)\qquad \Delta = \begin{vmatrix} y'_{uv} & y'_u & y'_v & y' \\ y''_{uv} & y''_u & y''_v & y'' \\ y^{(3)}_{uv} & y^{(3)}_u & y^{(3)}_v & y^{(3)} \\ y^{(4)}_{uv} & y^{(4)}_u & y^{(4)}_v & y^{(4)} \end{vmatrix} = D(y_{uv}, y_u, y_v, y).$$

We find

$$(78)\qquad \frac{\partial\Delta}{\partial u} = -4a\Delta, \qquad \frac{\partial\Delta}{\partial v} = -4b'\Delta.$$

But we have also

$$\frac{\partial a}{\partial v} = \frac{\partial b'}{\partial u},$$

according to the first equation of (30). We may put, therefore,

$$a = \frac{\partial p}{\partial u}, \qquad b' = \frac{\partial p}{\partial v},$$

so that (78) shows Δ to have the value

$$(79)\qquad \Delta = Ce^{-4p},$$

where C is a non-vanishing constant. In fact, if Δ were equal to zero there would be relations possible of the form

$$\alpha y^{(k)}_{uv} + \beta y^{(k)}_u + \gamma y^{(k)}_v + \delta y^{(k)} = 0 \qquad (k=1, 2, 3, 4),$$

with non-vanishing coefficients. But this is impossible in the case under consideration in which the general solution of system (71) depends upon four arbitrary constants.

The coördinates of the plane p_y were proportional to λ', λ'', $\lambda^{(3)}$, $\lambda^{(4)}$. Let us define them as being equal to

$$Y^{(k)} = \frac{\lambda^{(k)}}{\sqrt{\Delta}} \qquad (k=1, 2, 3, 4).$$

These four functions $Y^{(k)}$ will satisfy a system of differential equations obtained from (76) by the transformation

$$\lambda = \sqrt{\Delta}\, Y = e^{-2p} Y. \tag{80}$$

The resulting system of equations is

$$\begin{aligned} &Y_{uu} + 2aY_u - 2bY_v + (c + 2b_v - 4bb')Y = 0, \\ &Y_{vv} - 2a'Y_u + 2b'Y_v + (c' + 2a'_u - 4aa')Y = 0. \end{aligned} \tag{81}$$

This shall be called *the system adjoined to* (71). Denote its coefficients by $2\bar{a}$, $2\bar{b}$, etc. We have

$$\begin{aligned} &\bar{a} = a, && \bar{b} = -b, && \bar{c} = c + 2b_v - 4bb', \\ &\bar{a}' = -a', && \bar{b}' = b', && \bar{c}' = c' + 2a'_u - 4aa', \end{aligned}$$

whence

$$\begin{aligned} &a = \bar{a}, && b = -\bar{b}, && c = \bar{c} + 2\bar{b}_v - 4\bar{b}\bar{b}', \\ &a' = -\bar{a}', && b' = \bar{b}', && c' = \bar{c}' + 2\bar{a}'_u - 4\bar{a}\bar{a}', \end{aligned}$$

so that *the relation between* (71) *and* (81) *is reciprocal, i. e., each is the adjoint of the other.* The seminvariants of (81) are

$$\bar{a}', \ \bar{b}, \ \bar{f}, \ \bar{g},$$

which are respectively equal to the seminvariants

$$-a', \ -b, \ f, \ g$$

of (81).

System (81) *represents the same surface as* (71) *in plane- instead of in point-coördinates. But we may also interpret* Y', Y'', $Y^{(3)}$, $Y^{(4)}$ *as point-coördinates. In that case every integral surface of* (81) *would be dualistic to every integral surface of* (71).

It is clear, therefore, that *dualistic properties of any integral surface of* (71) *are analytically characterized by an invariant system of equations which remains unaltered if the signs of* a' *and* b *are changed.*

If we think of a surface as being at the same time described by a point and enveloped by its tangent planes we must consider the systems (71) and (81) simultaneously. A surface S may be *identically* self-dual; i. e., there may exist a dualistic transformation which converts it into itself in the particular way that every point P_y is converted into the plane p_y tangent to S at P_y and conversely. In that case (71) and (81) will be identical, so that

$$a' = b = 0.$$

The surface is then a quadric. In fact, we always have

$$y' Y' + y'' Y'' + y^{(3)} Y^{(3)} + y^{(4)} Y^{(4)} = 0,$$

and in this case $Y^{(k)}$ would have to be proportional to $y^{(k)}$ so that there will be a quadratic relation between $y' \cdots, y^{(4)}$. We may also see this in another way. If $a' = 0$, equations (71) show that the curves $u =$ const. are straight lines. If $b = 0$, the curves $v =$ const. are straight lines. But a quadric is the only surface which has two separate families of straight lines upon it.

We have, therefore, the following further result. *If either a' or b is equal to zero, the surface S is a ruled surface. If both a' and b vanish it is a quadric. Quadrics, moreover, are the only identically self-dual surfaces.*

We shall close this paper with the enunciation of a theorem which may be regarded as the *fundamental theorem* of the projective differential geometry of curved surfaces. Any non-developable surface being given

$$y^{(k)} = f^{(k)}(u, v) \qquad (k = 1, 2, 3, 4),$$

it determines a system of form (3) which may always be reduced to the intermediate or even the canonical form, from which the invariants a', b, h and k may be computed as functions of u and v. If on the other hand a', b, h and k are given as arbitrary functions of u and v, subject of course to the integrability conditions (34), the coefficients of the canonical form will be uniquely determined if a' and b are different from zero. In other words,

If the four invariants a', b, h, k are given as arbitrary functions of two variables u and v, subject to the integrability conditions (34), *and if moreover a' and b are not equal to zero, they determine a non-developable, non-ruled surface except for a projective transformation.*

It is not necessary to attempt to remove the restrictions of this theorem, as the author has already constructed a projective differential geometry of ruled and developable surfaces upon a different basis.

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A METHOD FOR CONSTRUCTING THE FUNDAMENTAL REGION OF A DISCONTINUOUS GROUP OF LINEAR TRANSFORMATIONS*

BY

J. I. HUTCHINSON

Two methods at present exist for the construction of the fundamental region of a discontinuous group.† The one applies only to groups which are capable of extension by reflection and consists in constructing first the region for the extended group (which can be done by a direct process) and then combining two such regions that are adjacent. The other method uses the process of the continuous reduction of a quadratic form. It is, however, tedious in application, and this fact appears to place a narrow restriction on its range of practical effectiveness. In the following pages I propose a much more simple and effective method, applicable to groups which leave a given Hermitian form invariant.

§ 1. *Groups of transformations of a single variable.*

Let the complex variable be denoted by z and its conjugate by $\bar{z}$. Also let the equation of the fixed circle for the given group be

$$F \equiv \alpha z\bar{z} + \gamma z + \bar{\gamma}\bar{z} + \beta = 0\,, \tag{1}$$

in which the discriminant $\delta = \gamma\bar{\gamma} - \alpha\beta$ is positive (α, β being real) and α is negative. The circle is then real and its interior will be defined by the inequality $F > 0$.

Let $F = 0$ be transformed into itself by the substitution

$$z' = \frac{az + b}{cz + d}, \qquad ad - bc = 1\,. \tag{2}$$

The conditions for this are

$$\begin{gathered} \alpha(a\bar{a} - 1) + \gamma a\bar{c} + \bar{\gamma}\bar{a}c + \beta c\bar{c} = 0\,, \\ \alpha a\bar{b} + \gamma a\bar{d} + \bar{\gamma}c\bar{b} \times \beta c\bar{d} = \gamma\,, \\ \alpha b\bar{b} + \gamma b\bar{d} + \bar{\gamma}\bar{b}d + \beta(d\bar{d} - 1) = 0\,. \end{gathered} \tag{3}$$

Since $F = 0$ is also transformed into itself by the inverse of (2), the conditions

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† See FRICKE-KLEIN, *Automorphe Functionen*, I, p. 539.

(3) may be replaced by the equivalent ones,

$$
(3') \qquad \begin{aligned} \alpha(d\bar{d}-1)-\gamma d\bar{c}-\bar{\gamma}\bar{d}c+\beta c\bar{c}&=0,\\ -\alpha d\bar{b}+\gamma d\bar{a}+\bar{\gamma}c\bar{b}-\beta c\bar{a}&=\gamma,\\ \alpha b\bar{b}-\gamma b\bar{a}-\bar{\gamma}a\bar{b}+\beta(a\bar{a}-1)&=0. \end{aligned}
$$

The transformation (2) changes the function F into F' such that

$$
(4) \qquad F' = F/\Delta\bar{\Delta},
$$

$$
(5) \qquad \Delta = cz + d.
$$

Suppose that for a given point z inside the circle (1) we have $\Delta\bar{\Delta} < 1$; thence follows $F' > F$, and the effect of the substitution has been to increase the value of the positive function F. If there exists a second substitution changing the point z' into z'' so that $F'' > F'$, we apply it to the preceding one and so proceed until F has reached its maximum value for the series of congruent points $z, z', z'', \cdots$. Such a maximum value will be attained after a finite number of steps provided that the substitutions employed all belong to a group G which is "properly discontinuous" in the z-plane. I will name the point so determined *the maximum point congruent to* z.

There may be several points congruent to z for which F has a maximum value. The number of these is equal to the order* ν of the subgroup g of G defined by $c = 0$, since under this condition it follows from (3′) that $\Delta\bar{\Delta} = 1$. The totality of maximum congruent points obtained by varying z throughout the interior of F will form a region R bounded by certain circles, orthogonal to $F = 0$, whose equations are

$$
(6) \qquad \Delta\bar{\Delta} = 1.
$$

The region R is divided by means of g into ν congruent regions $R_1, R_2, \cdots, R_\nu$. Any one of the regions R_i may then be taken as the fundamental region for the given group G.

It only remains to show how to select from among the circles (6) those which bound the region R. For this purpose we observe that when F increases, the point z approaches the center C of the circle $F = 0$. Accordingly the boundaries of R are to be selected from those Δ-circles (6) which pass nearest to C. The point of $\Delta\bar{\Delta} = 1$ nearest to C is

$$
(7) \qquad z = \frac{d\lambda + \sqrt{\lambda\bar{\lambda}}}{-c\lambda},
$$

in which

$$
(8) \qquad \lambda = \gamma\bar{c} - \alpha\bar{d}.
$$

* This order may be infinite as in the case of the modular group when it has the real axis for its fixed circle.

The value of F at this point is

$$F = 2(\alpha + \sqrt{\lambda\bar{\lambda}})/c\bar{c}. \tag{9}$$

This is evidently the greatest value attained by F on the given Δ-circle, and it may be found by the usual process for determining the maximum value of a function.

From (8) we deduce

$$\gamma\bar{\gamma}c\bar{c} = \alpha\gamma\bar{c}d + \alpha\bar{\gamma}c\bar{d} - \alpha^2 d\bar{d} + \lambda\bar{\lambda};$$

also, from the first equation (3′),

$$\alpha\beta c\bar{c} = \alpha\gamma d\bar{c} + \alpha\bar{\gamma}c\bar{d} - \alpha^2 d\bar{d} + \alpha^2,$$

whence, by subtraction,

$$\delta c\bar{c} = \lambda\bar{\lambda} - \alpha^2.$$

Using this result in (9) we have for the maximum value of F on (6),

$$F = \frac{2\delta}{\sqrt{\lambda\bar{\lambda}} - \alpha}.$$

From this it follows that the greatest values attained by F at the points defined by (7) are those for which the absolute values of λ are the least.

It is evident, then, that the boundary circles for the region R occur among those determined by the smallest values of $|\lambda|$. There does not appear to be a sufficiently simple and general criterion by which to select out the required circles from among those found in this way, but a graphical construction will quickly indicate which ones are to be used.

The coefficients a, b, d are expressible in terms of λ, c in the form

$$a = \frac{\lambda + \bar{\gamma}c}{-\alpha}, \qquad d = \frac{\bar{\gamma}c - \bar{\lambda}}{\alpha}, \qquad b = \frac{\bar{\gamma}(\bar{\lambda} - \bar{\gamma}c) + \delta\bar{c} - \lambda\bar{\gamma}}{\alpha^2}, \tag{10}$$

the numbers λ, c being connected by the relation

$$\lambda\bar{\lambda} = \alpha^2 + \delta c\bar{c} \tag{11}$$

which is the condition that the determinant shall equal 1. Equation (11) is also the condition that the circle $\Delta\bar{\Delta} = 1$ shall be orthogonal to $F = 0$.

To give a clearer idea of how the preceding method works out in practise, we consider the following examples in which a, b, c, d are restricted to complex integers.

Example I. $\alpha = 1$, $\gamma = 0$, $\beta = -5$.

For $c = 0$ a substitution of period 2 is obtained having the origin as fixed point. Hence the region R consists of two congruent regions R_1, R_2. For $|c| > 0$ the minimum value of $\lambda\bar{\lambda}$ is 26, and hence $\lambda = \pm 5 \pm i$ or $\pm 1 \pm 5i$;

also $c\bar{c} = 5$ and therefore $c = \pm 2 \pm i$ or $\pm 1 \pm 2i$. The equation of the Δ-circle is $(cz - \bar{\lambda})(\bar{c}\bar{z} - \lambda) = 1$, which may be written

$$\left(z - \frac{\bar{\lambda}}{c}\right)\left(\bar{z} - \frac{\lambda}{\bar{c}}\right) = \frac{1}{c\bar{c}},$$

and hence its center is the point $z = \bar{\lambda}/c$ and its radius is $1/\sqrt{c\bar{c}}$. With the values just obtained for λ, c we construct the corresponding circles $1, 1, \cdots$, Fig. 1, using only those which occur in the upper half plane in which we take

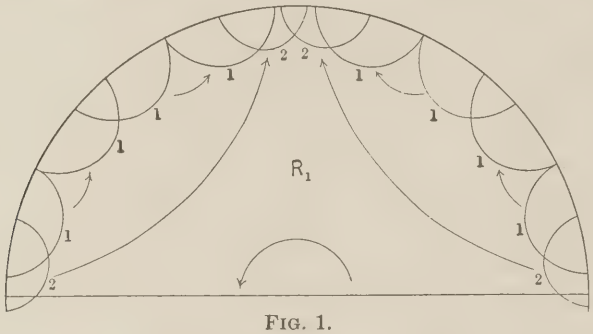

FIG. 1.

R_1. These do not form a closed region and we accordingly take the next higher value of $\lambda\bar{\lambda}$ which is 41, whence $c\bar{c} = 8$. The circles $2, 2, \cdots$ are determined by these values, and now the region above the real axis being closed we have in R_1 thus determined the fundamental region for the given group.*

Example II. $\alpha = -3$, $\beta = \gamma = 1$.

It is useful to observe from the relation $\lambda + \bar{\lambda} = -\alpha(a + d)$ that $\lambda + \bar{\lambda}$ is divisible by α. For $c = 0$ the only substitution of the group is identity. The values of λ and c first to be considered are those for which $\lambda\bar{\lambda} = 13$, $c\bar{c} = 1$. Among these the only ones which give different substitutions with complex integral values for a, b, d are $\lambda = 3 + 2\epsilon i$, $c = \epsilon i$, $\epsilon = \pm 1$. These determine the two circles 1, 2, Fig. 2. The next higher values of λ and c are derived from $\lambda\bar{\lambda} = 25$, $c\bar{c} = 4$, whence we obtain the circles 3, 4, 5. We exclude 4, 5 since they lie inside 1, 2. As the region about the center C is not yet closed we take the next case $\lambda\bar{\lambda} = 45$, $c\bar{c} = 9$, which determines the circles 6, 7, 8, 9, the last two of which are to be excluded. As the region R is now closed the operation is completed.

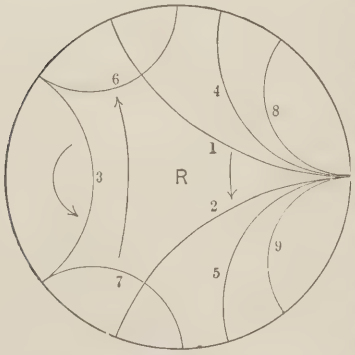

FIG. 2.

*Cf. FRICKE-KLEIN, *Automorphe Functionen*, I, p. 479.

§ 2. *Transformations on two variables.*

It is evident that the general method explained above is at once applicable to groups of substitutions on any number of variables, the region R being then defined by certain inequalities of the form $\Delta\bar{\Delta} > 1$. For the sake of simplicity I will confine myself in the present paper to the case of a group G of linear transformations on two complex variables x, y, the general substitution being written in the form

$$x' = (ax + by + c)/\Delta, \qquad y' = (dx + ey + f)/\Delta \tag{12}$$

in which $\Delta = px + qy + r$. I will further suppose that the group G leaves the hypersphere

$$F \equiv 1 - x\bar{x} - y\bar{y} = 0 \tag{13}$$

invariant. The conditions on the coefficients are

$$\begin{aligned} a\bar{a} + d\bar{d} - p\bar{p} &= 1, & a\bar{b} + d\bar{e} - p\bar{q} &= 0, \\ b\bar{b} + e\bar{e} - q\bar{q} &= 1, & a\bar{c} + d\bar{f} - p\bar{r} &= 0, \\ c\bar{c} + f\bar{f} - r\bar{r} &= -1, & b\bar{c} + e\bar{f} - q\bar{r} &= 0. \end{aligned} \tag{14}$$

Multiply the two equations (12) by $\bar{a}$, $\bar{d}$ respectively and add; then subtract $\bar{p}$ from both members of the resulting equation and use (14). This gives

$$\bar{a}x' + \bar{d}y' - \bar{p} = x/\Delta;$$

similarly,

$$\bar{b}x' + \bar{e}y' - \bar{q} = y/\Delta \qquad \text{and} \qquad \bar{c}x' + \bar{f}y' - \bar{r} = -1/\Delta.$$

Hence the inverse of (12) may be written

$$\begin{gathered} x = (\bar{a}x' + \bar{d}y' - \bar{p})/\Delta', \qquad y = (\bar{b}x' + \bar{e}y' - \bar{q})/\Delta', \\ \Delta' = -\bar{c}x' - \bar{f}y' + \bar{r}. \end{gathered} \tag{15}$$

As the inverse substitution also leaves the hypersphere invariant we have the conditions

$$\begin{aligned} a\bar{a} + b\bar{b} - c\bar{c} &= 1, & a\bar{d} + b\bar{e} - c\bar{f} &= 0, \\ d\bar{d} + e\bar{e} - f\bar{f} &= 1, & a\bar{p} + b\bar{q} - c\bar{r} &= 0, \\ p\bar{p} + q\bar{q} - r\bar{r} &= -1, & d\bar{p} + e\bar{q} - f\bar{r} &= 0. \end{aligned} \tag{14'}$$

From the fourth and fifth of these equations we deduce

$$a = k(\bar{f}\bar{q} - \bar{e}\bar{r}), \qquad b = k(\bar{d}\bar{r} - \bar{f}\bar{p}), \qquad c = k(\bar{d}\bar{q} - \bar{e}\bar{p}).$$

In order to evaluate the unknown k, we substitute these expressions for a, b, c

in the determinant

$$\begin{vmatrix} a & b & c \\ d & e & f \\ p & q & r \end{vmatrix} = N$$

and obtain

$$k(-A\bar{A} - B\bar{B} + C\bar{C}) = N,$$

in which A, B, C are the minors corresponding to a, b, c. But since the inverse of the substitution (12) is

$$x = \frac{Ax' + Dy' + P}{Cx' + Fy' + R}, \qquad y = \frac{Bx' + Ey' + Q}{Cx' + Fy' + R},$$

it follows that $A\bar{A} + B\bar{B} - C\bar{C} = 1$ and therefore $k = -N$. Hence,

$$\begin{aligned} a &= N(\bar{e}\bar{r} - \bar{f}\bar{q}) = N\bar{A}, \\ b &= N(\bar{f}\bar{p} - \bar{d}\bar{r}) = N\bar{B}, \\ c &= N(\bar{e}\bar{p} - \bar{d}\bar{q}) = -N\bar{C}, \end{aligned} \tag{16}$$

in which d, e, $\cdots$ satisfy the three relations

$$\begin{aligned} p\bar{p} + q\bar{q} - r\bar{r} &= -1, \\ d\bar{d} + e\bar{e} - f\bar{f} &= 1, \\ d\bar{p} + e\bar{q} - f\bar{r} &= 0. \end{aligned} \tag{17}$$

We next proceed to determine the maximum point of the *spread*

$$\Delta\bar{\Delta} = 1, \qquad \Delta = px + qy + r, \tag{18}$$

that is, the point at which the $F = 1 - x\bar{x} - y\bar{y}$ takes its maximum value when x, y are restricted to the locus (18). Regarding $\bar{y}$ as a dependent variable on account of (18) we differentiate F with respect to x, $\bar{x}$, y and obtain as conditions for a maximum

$$\bar{x} + y\frac{\partial \bar{y}}{\partial x} = 0, \qquad x + y\frac{\partial \bar{y}}{\partial \bar{x}} = 0, \qquad \bar{y} + y\frac{\partial \bar{y}}{\partial y} = 0.$$

By differentiating $\Delta\bar{\Delta} = 1$ and substituting the expressions thus obtained for the derivatives, we deduce

$$x = \frac{\bar{p}}{\bar{q}}y, \qquad \bar{x} = \frac{p\bar{\Delta}}{\bar{q}\Delta}y, \qquad \bar{y} = \frac{q\bar{\Delta}}{\bar{q}\Delta}y. \tag{19}$$

By substitution of these results in the second equation (18) we obtain

$$\Delta = (r\bar{r} - 1)\bar{y}/\bar{q},$$

and by substituting in (10),

$$\bar{y} = \frac{q\bar{r}}{\bar{q}r} y, \qquad \bar{x} = \frac{p\bar{r}}{\bar{q}r} y, \qquad x = \frac{\bar{p}}{\bar{q}} y.$$

Substitute now in $\Delta\bar{\Delta} = 1$ and solve for y. This gives

$$y = \frac{\bar{q}(-rr \pm \sqrt{r\bar{r}})}{\bar{r}(r\bar{r} - 1)}.$$

Therefore

$$F = \frac{-2 \pm 2\sqrt{r\bar{r}})}{r\bar{r} - 1}.$$

From the first equation (17) it is evident that $r\bar{r}$ cannot be less than 1. In order that the point determined may be inside the hypersphere it is necessary to take the lower sign. Hence, *the point*

$$x = \frac{\bar{p}(\sqrt{r\bar{r}} - r\bar{r})}{\bar{r}(r\bar{r} - 1)}, \qquad y = \frac{\bar{q}(\sqrt{r\bar{r}} - r\bar{r})}{\bar{r}(r\bar{r} - 1)}$$

is the maximum point for the spread $\Delta\bar{\Delta} = 1$ *and gives to* F *the value*

$$F = \frac{2}{\sqrt{r\bar{r}} + 1}. \tag{20}$$

A glance at (20) shows that the largest maximum values are attained by F when $r\bar{r}$ has its smallest values. The corresponding Δ-spreads are those of (18) which pass nearest to the center of $F = 0$, and the boundaries for the region R are to be selected from among these.

§ 3. *Klein's group of* $(n+1)!$ *collineations.*

The working out of the fundamental region for an infinite discontinuous group of the class just treated appears to involve a considerable amount of necessary detail and for that reason I reserve for a future paper the discussion of particular groups. In the present article I will only notice, by way of illustration, the fundamental region R for KLEIN's group of $(n+1)!$ collineations defined by the homogeneous equations of transformation on the variables y_i,

$$y_i' = y_{a_i} \qquad (i = 1, 2, \cdots, n+1), \tag{21}$$

in which $a_1, a_2, \cdots, a_{n+1}$ is any permutation of the numbers $1, 2, \cdots, n+1$. We first make the group non-homogeneous by introducing the variables $x_k = y_k / y_{n+1}$ $(k = 1, 2, \cdots, n)$. Equations (21) then become

$$x_k' = x_{a_k} / x_i \qquad (k = 1, 2, \cdots, n), \tag{22}$$

in which x_{a_k} of the numerator is to be replaced by 1 when $a_k = i$. We have, then, $\Delta = x_i$ and the inequality conditions $\Delta\bar{\Delta} > 1$ reduce to

$$x_i\bar{x}_i - 1 > 0 \qquad (i = 1, 2, \cdots, n). \tag{23}$$

Apply now to (23) the different transformations of the group. We obtain additional inequalities of the form

$$x_i\bar{x}_i - x_j\bar{x}_j > 0 \qquad (i, j = 1, 2, \cdots, n). \tag{24}$$

A complete set of consistent inequalities of this type will be obtained by subjecting i and j to the condition $i > j$.

Returning now to the homogeneous variables, the conditions (23) and (24) become

$$y_i\bar{y}_i - y_j\bar{y}_j > 0, \qquad i > j \qquad (i, j = 1, 2, \cdots, n+1). \tag{25}$$

These inequalities (some of which are superfluous) define a fundamental region R for the given group. For, since any substitution of the group produces a permutation of the y_i, it follows that at least one of the inequalities (25) is reversed. The only case in which this is not immediately apparent is that of the substitution which produces the cyclic permutation $(y_1, y_2, \cdots, y_{n+1})$. The inequalities (25) are permuted except that in place of $y_2\bar{y}_2 - y_1\bar{y}_1 > 0$ we now have $y_1\bar{y}_1 - y_{n+1}\bar{y}_{n+1} > 0$. But if we add this last to the $n-1$ inequalities $y_{i+1}\bar{y}_{i+1} - y_i\bar{y}_i > 0\,(i = 2, 3, \cdots, n)$, we obtain $y_1\bar{y}_1 - y_2\bar{y}_2 > 0$, which is a reversal of the first inequality (25). Hence every point within R is transformed into a point outside R by each substitution of the group. On the other hand, every point outside R can be transformed into a point within. For, the coördinates of an arbitrarily fixed point can evidently satisfy a set of conditions of the form

$$y_{a_i}\bar{y}_{a_i} - y_{a_j}\bar{y}_{a_j} > 0, \qquad i > j \qquad (i, j = 1, 2, \cdots, n+1). \tag{26}$$

By choosing the a_i as all possible permutations of $1, 2, \cdots, n+1$ we obtain $(n+1)!$ different sets of conditions (26), including (25). It is also clear that no other set of conditions except those enumerated would be a consistent one. For if in the $n(n+1)$ inequalities (26) no variable occurred in the first term in more than n of the formulas, it would be necessary for two of the inequalities to have the form $y_k\bar{y}_k - y_l\bar{y}_l > 0$, $y_l\bar{y}_l - y_k\bar{y}_k > 0$, which are contradictory. Suppose, then, that y_{a_1} occurs in the first term of $n+1$ of the inequalities (26). A similar argument will show that some other variable, say y_{a_2}, must occur in the first term of n of the remaining inequalities, and so on. We accordingly have the arrangement of (26). But since there are $(n+1)!$ substitutions in the group and no substitution, except 1, leaves a particular set unaltered, it follows that any given set (26) can be transformed into (25) by a certain substitution of the group, and therefore any point outside R has a congruent point inside.

After excluding superfluous conditions it is evident that the $n(n+1)$ inequalities (25) reduce to *

$$y_{i+1}\bar{y}_{i+1} - y_i\bar{y}_i > 0 \qquad (i=1, 2, \cdots, n). \tag{27}$$

The boundary

$$y_i\bar{y}_i - y_{i+1}\bar{y}_{i+1} = 0 \tag{28}$$

is divided by the flat space $y_i = y_{i+1}$ into two parts which are interchanged by the substitution

$$y'_i = y_{i+1}, \qquad y'_{i+1} = y_i, \qquad y'_k = y_k \quad (k=1, 2, \cdots, i-1, i+2, \cdots, n+1),$$

while the $n-1$ remaining inequalities (27) are unaltered. Hence each of the boundaries (28) is paired with itself.

* The fundamental region for this group, when the variables y_i are restricted to real values, is given by E. H. Moore, *Concerning Klein's group of* $(n+1)!$ *n-ary collineations*, American Journal of Mathematics, vol. 22 (1900), p. 336. When the variables are real the conditions (27) become $y_{i+1}^2 - y_i^2 > 0$. For positive values of the y's these reduce to exactly the conditions given by Professor Moore, p. 338; for negative values of the variables the two sets of conditions do not coincide, but they are equivalent and one may be reduced to the other by an "allowable change" of the fundamental region.

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OBLIQUE REFLECTIONS AND UNIMODULAR STRAINS*

BY

EDWIN BIDWELL WILSON

Introduction

The object of the present communication is to study the group of unimodular strains about a fixed point in its relation to products of involutory strains.† The interest in this group is not merely geometric; rather is it mechanical. The group of strains is fundamental in the theory of elasticity and as such has long been studied by students of both theoretical and applied mathematics. Indeed the different types of strain, such as simple and complex shears, tonics and cyclotonics, have been carefully classified ‡ so that the present investigation has something well known to which to attach itself. The restriction to unimodular or equivoluminal strains is necessitated by the fact that an involutory strain obviously cannot alter (except for sign) the volume of any portion of space, and hence the most general strain that can be obtained as a product of successive involutory strains must leave volume unchanged in magnitude. It may be worth while to note that the ether is a body which possesses this unimodular elastic property, and that the instantaneous strain at a point in an incompressible fluid is similarly characterized.

As the use of multiple algebras in geometric investigations has not been very wide spread, it may be proper at this juncture to enter somewhat upon the question of what algebra should be chosen for the present purpose. Four, at least, suggest themselves immediately. They are: GRASSMANN's point analysis, PEANO's vector analysis, quaternions, and GIBBS's vector analysis.§ Any of these,

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† Geometric investigations on the relation of involutory transformations to various groups in which they occur have been carried on during the last fifteen years by numerous authors. For a general résumé of this work see an article on *Involutory transformations in the projective group and in its subgroups* by the present writer in the Annals of Mathematics, ser. 2, vol. 7 (1907), pp. 77–86.

‡ See, for example, THOMSON AND TAIT, *Natural Philosophy* (new edition), vol. 1, pt. 1, pp. 139–185. Also GIBBS, *Elements of Vector Analysis* (1881–84); *Scientific Papers*, vol. 2 (1906), pp. 17–90.

§ For GRASSMAN's analysis one may consult his first *Ausdehnungslehre* (1844). MEHMKE, *Vergleich der Vektoranalysis americanischer Richtung und derjenigen deutsch-italienischer Richtung*, Jahresberichte der deutschen Math.-Ver., vol. 13 (1904), pp. 217–228, gives some

and others as well, might serve. The first two are more truly projective or strain algebras than the last two. They do not depend so essentially on relations of perpendicularity, which of course are not preserved by strains. Hence they would appear more germane to the subject in hand.* The physicists, however, make great use of relations of perpendicularity in connection with the study of strains — the strain invariants are estimated on perpendicular axes, the strain quadric is referred to such axes, and so on. In short it does appear desirable to take account of perpendicularity.† This is done alike by quaternions and the vector analysis of GIBBS. The former seems nevertheless to be not quite so well suited to the questions in hand as is the latter; for the linear vector function of the quaternion analysis is not treated primarily as an independent multiple quantity with an independent set of algebraic relations but rather as an operator meaningless apart from the vector to which it is applied. Moreover strains have been carefully treated in GIBBS's work‡ and the treatment is perhaps more easily available for reference than in any other which uses multiple algebra. Hence, everything considered, GIBBS's algebra will be chosen in this instance, although any one of the others would doubtless exhibit at some stage of the work its peculiar advantages over the rest.

It may be taken as known that there are three and only three types of real involutory strains:§ 1°, a strain which reverses in direction every vector issuing from the origin; 2°, a strain which reverses in direction vectors parallel to a given line and leaves unchanged vectors parallel to a given plane which does not contain the line; 3°, a strain which reverses in direction vectors parallel to a given plane and leaves unchanged vectors parallel to a given line which does not lie in the plane. The first of these three types of involutory strain will be called *central reflection;* the second, *planar reflection* parallel to a line; the third, *linear reflection* parallel to a plane. The last two types are, so to speak, *dual* in nature — one through a plane parallel to a line, the other through a line parallel to a plane.

These transformations may be expressed as dyadics.‖ Let the dyadics be

account of the methods of PEANO. See also PEANO, *Formulario Matematico* (editio v) (1906), pp. 165–201. The real difficulty with these algebras is not so much their inappropriateness as the undeveloped state of their linear vector function.

* For this side of the question consult MEHMKE, loc. cit.

† For this side of the question consult PRANDTL, *Ueber die physikalische Richtung in der Vektoranalysis*, Jahresberichte etc, vol. 13 (1904), pp. 436–449. There are some general remarks on both sides of the question by myself in the *Verhandlungen des* III. *internationalen Mathematiker-Kongresses* (1904), pp. 202–215.

‡ In the pamphlet referred to above and more especially in his *Vector Analysis* edited by myself (1901), chap. 6 (Charles Scribner's Sons, New York). This work will be referred to as *Vector Analysis* in the remaining footnotes or portions of the text.

§ For a full discussion of the types of involutory strains in the projective group and its subgroups see my *Involutory transformations*, etc., previously cited.

‖ *Vector Analysis*, p. 332.

used as prefactors so that

$$\rho' = \Phi \cdot \rho$$

represents the transformation of the position vector ρ into the position vector ρ'. Consider the dyadic Φ to be expressed in terms of any three noncomplanar vectors α, β, γ and their reciprocals α', β', γ' as antecedents and consequents respectively. The identical transformation is represented by the idemfactor which in this case takes the form *

$$I = \alpha\alpha' + \beta\beta' + \gamma\gamma'.$$

Central reflection which merely effects a reversal of all directions is represented by the negative of the idemfactor, that is by

$$-I = -(\alpha\alpha' + \beta\beta' + \gamma\gamma'),$$

and this is the only way in which it may be represented so long as the antecedents are to form a set reciprocal to the consequents.

For oblique planar reflection in the plane of α, β and along the direction of γ, the dyadic

$$\Phi = \alpha\alpha' + \beta\beta' - \gamma\gamma' = I - 2\gamma\gamma'$$

evidently suffices. The analytical form of the dyadic will not be nearly so simple unless the vectors α, β, γ are chosen in the plane and along the direction of the reflection. There, is however, a form which at least in appearance is a little more general than the above and which we will adopt in the future. Assume for trial the form

$$\Phi = I - 2\epsilon\zeta.$$

Any vector ρ perpendicular to ζ becomes

$$\rho' = \Phi \cdot \rho = I \cdot \rho - 2\epsilon\zeta \cdot \rho = \rho.$$

Hence the plane perpendicular to ζ and all components parallel to this plane are left unchanged. Any vector parallel to ϵ becomes

$$\rho' = \Phi \cdot x\epsilon = (1 - 2\zeta \cdot \epsilon)x\epsilon,$$

and this amounts to reversal of direction when and only when $\zeta \cdot \epsilon = 1$.

Hence the general form of a dyadic which shall represent reflection in a plane perpendicular to ζ and along the direction ϵ is

$$\Phi = I - 2\epsilon\zeta, \qquad \zeta \cdot \epsilon = 1.$$

If for any reason some other form of the dyadic should be desired, it is only necessary to express I, ϵ, ζ in terms of any convenient antecedents and consequents.

In like manner a simple inspection shows that the dyadic

* *Vector Analysis*, p. 288, et seq.

$$\Phi = -\alpha\alpha' - \beta\beta' + \gamma\gamma' = 2\gamma\gamma' - I$$

represents an oblique linear reflection through the line whose direction is γ and parallel to the plane of α, β. This admits of the more general form

$$\Phi = 2\epsilon\zeta - I, \qquad \epsilon\cdot\zeta = 1.$$

It is noteworthy that both the linear and the planar reflections have a fundamental *line* and *plane*, although the part played by the line and plane is related in a sort of dual fashion in the two cases. To sum up the essential points in the analysis we state

THEOREM 1. *The three forms of reflection in the group of strains with their corresponding dyadics are*

$$\begin{array}{lll} \textit{central}, & \Phi = -I, & \\ \textit{planar}, & \Phi = I - 2\epsilon\zeta, & \epsilon\cdot\zeta = 1, \\ \textit{linear}, & \Phi = 2\epsilon\zeta - I, & \epsilon\cdot\zeta = 1, \end{array}$$

where ϵ represents the line that enters into the reflection whether the reflection takes place along it or through it, and where ζ is perpendicular to, that is, represents the plane of the reflection whether the reflection takes place through this plane or parallel to it.*

The analytical forms given above have been obtained from the known geometric properties of the reflections which have been proved to be the sole involutory strains. If it were not the intention to make use of the general theory of involutory collineations before entering upon the detail of this paper, the special case of involutory strains which concerns us might have been treated directly as an algebraic problem, and in the following way. Let Φ represent any involutory strain. Then

$$\rho' = \Phi\cdot\rho, \qquad \rho = \Phi\cdot\rho' = \Phi\cdot\Phi\cdot\rho = \Phi^2\cdot\rho.$$

Hence, as the relation

$$\rho = I\cdot\rho = \Phi^2\cdot\rho$$

holds for all vectors ρ, it follows that

$$I = \Phi^2.$$

In other words the geometric problem of determining all involutory strains is coextensive with the algebraic problem of finding all dyadics Φ which satisfy the equation $\Phi^2 = I$, that is, which are square roots of the idemfactor.

To solve the equation $\Phi^2 = I$, assume the solution in the form

$$\Phi = \Psi + I.$$

Then

$$I = \Phi^2 = (\Psi + I)\cdot(\Psi + I) = \Psi\cdot\Psi + 2\Psi\cdot I + I.$$

* *Vector Analysis*, p. 46.

Hence

$$\Psi\cdot\Psi + 2\Psi\cdot I = \Psi\cdot(\Psi + 2I) = 0\,.$$

Now if the product of two dyadics is zero one of them must be zero or one of them must be linear while the other is at most merely planar.* If $\Psi = 0$, then $\Phi = I$, which is involutory although trivially so. If $\Psi + 2I = 0$, then $\Phi = -I$, which is the central reflection. Next suppose that Ψ is linear and write it as $-2\epsilon\zeta$. Then

$$\Phi = I - 2\epsilon\zeta \qquad \text{and} \qquad \Psi + 2I = 2(I - \epsilon\zeta)\,.$$

The latter must be planar. The condition of planarity is that the third of the dyadic vanish. Hence

$$(I - \epsilon\zeta)_3 = I_3 - \epsilon\cdot I\cdot\zeta = 1 - \epsilon\cdot\zeta = 0\,,\dagger$$

or $\epsilon\cdot\zeta = 1$ as before. In like manner if $\Psi + 2I$ were linear, assume

$$\Psi + 2I = +2\epsilon\zeta, \qquad \Phi = 2\epsilon\zeta - I.$$

Then $\Psi = 2(\epsilon\zeta - I)$ must be planar and the application of the condition shows that $\epsilon\cdot\zeta = 1$ again in this case. Hence the problem has been solved by purely algebraic means. The only square roots of the idemfactor are of the form

$$\pm I, \qquad \pm(I - 2\epsilon\zeta) \text{ with } \epsilon\cdot\zeta = 1\,.$$

The dyadic $\Phi_2 = \frac{1}{2}\Phi\times\Phi$ gives the transformation of surfaces.‡ If Φ_2 be calculated it is seen to be

$$\Phi_2 = I \qquad \text{if} \qquad \Phi = \pm I,$$

or

$$\Phi_2 = 2\zeta\epsilon - I \qquad \text{if} \qquad \Phi = \pm(I - 2\epsilon\zeta)\,.$$

The change of volume is given by Φ_3. The calculation § of this shows that volumes are reversed in sign by central or planar reflection, but are left unchanged by linear reflection. The results may be stated as

THEOREM 2. *In central reflection surfaces (regarded as vectors) are unchanged while volumes change sign. In planar and linear reflection surfaces (regarded as vectors) suffer linear reflection through a line perpendicular to the plane of the original reflection Φ and along a plane perpendicular to the line of the reflection Φ. Volumes change or keep their sign according as Φ is planar or linear.*

* *Vector Analysis*, pp. 282–288. This is but a particular case of a theorem in the general theory of matrices and due to SYLVESTER, namely: The nullity of a product of two matrices is not less than the greater of the nullities of the factors nor greater than the sum of the nullities of both factors. See WHITEHEAD, *Universal Algebra*, Vol. 1 (1899), p. 253.

† *Vector Analysis*, p. 331, Exs. 19 and 20. These formulas are important.

‡ *Vector Analysis*, p. 333. The best method of calculating ϕ_2 in this case is by the relation (68) of p. 313; for as ϕ is involutory, $\phi_2 = \pm\phi_c^{-1}$.

§ *Vector Analysis*, p. 333. The value of ϕ_3 may be calculated as above.

1. *The special case in the composition of two reflections.*

One of the first questions to arise is: When are two reflections commutative? It is obvious that whatever be the dyadic Φ,

$$-I\cdot\Phi = \Phi\cdot(-I) = -\Phi$$

and hence that central reflection is commutative not merely with other reflections, but with any strain whatsoever. Consider therefore two reflections neither of which is central. They may be written as

$$\pm(I-2\epsilon\zeta) \qquad \text{and} \qquad \pm(I-2\eta\xi),$$

where the sign is plus if the reflection is planar, and minus if it is linear. The condition that the reflections be commutative is

$$\pm(I-2\epsilon\zeta)\cdot(I-2\eta\xi) = \pm(I-2\eta\xi)\cdot(I-2\epsilon\zeta),$$

which, on expansion, reduces to the equation

$$\zeta\cdot\eta\epsilon\xi = \xi\cdot\epsilon\eta\xi.$$

This equation is satisfied, first if ϵ and η are parallel (as they both may be considered as passing through the origin, they may be taken as actually collinear) and ζ and ξ are parallel at the same time; second, if $\zeta\cdot\eta = 0$ and $\zeta\cdot\epsilon = 0$. The latter case is obvious, and the former becomes so when it is remembered that a single dyad possesses but five degrees of freedom, of which four are here satisfied by the relations of collinearity and the fifth by the equality in magnitude which results from the homogeneity of the equation. The geometric relation of the reflections in the first case needs no discussion; in the second case the equation $\zeta\cdot\eta = 0$ shows that η is perpendicular to ζ. Now ζ represents the plane of the reflection. Hence the line η lies in the plane perpendicular to ζ, that is, in the plane of the reflection. To sum up:

THEOREM 3. *That two reflections be commutative it is necessary and sufficient that:* 1°, *one of them be central;* 2°, *their lines and planes be respectively coincident; or* 3°, *the lines of each lie in the plane of the other.*

To find the result of the product of two commutative reflections note first that the product of a central reflection into a planar or linear reflection gives the other one of the two. For the product of two reflections, neither of which is central, we have

$$\pm(I-2\epsilon\zeta)\cdot(I-2\eta\xi) = \pm(I-2\epsilon\zeta-2\eta\xi+4\zeta\cdot\eta\epsilon\xi).$$

In case ϵ and ζ are respectively parallel to η and ξ they may be taken equal to them, for the relations $\epsilon\cdot\zeta = \eta\cdot\xi = 1$ show that the angle between the pairs of vectors ϵ, ζ and η, ζ are the same and hence that the vectors are either equal or opposite, which amounts to the same thing in this problem. Hence the product

is $\pm I$. In case $\zeta\cdot\eta = \xi\cdot\epsilon = 0$ the product is

$$\pm(I - 2\epsilon\zeta - 2\eta\xi)$$

with the relations

$$\epsilon\cdot\zeta = 1, \qquad \eta\cdot\xi = 1, \qquad \epsilon\cdot\xi = 0, \qquad \zeta\cdot\eta = 0.$$

Hence

$$\zeta\times\xi\cdot\epsilon\times\eta = \zeta\cdot\epsilon\xi\cdot\eta - \xi\cdot\epsilon\xi\cdot\eta = 1.$$

The vectors ϵ, η, $\zeta\times\xi$ and ζ, ξ, $\epsilon\times\eta$ therefore satisfy the relations for reciprocal systems. If they be chosen as α, β, γ and α', β', γ', the product takes the form

$$\pm(I - 2\alpha\alpha' - 2\beta\beta') = \mp(I - 2\gamma\gamma').$$

The geometric meaning of the results may be summed up in a theorem in which it will be convenient to put some additional but obvious statements concerning the converse problem.

THEOREM 4. *Two commutative reflections compound into the identical transformation when and only when they are identical with one another. And conversely the identical transformation may be resolved in ∞^4 ways into the product of reflections which are identical. Two commutative reflections compound into a central reflection when and only when one is planar and the other is linear with the same fundamental line and plane. Conversely the resolution may be effected in ∞^4 ways into such reflections. Two commutative reflections compound into a planar reflection when and only when they are either central and linear or planar and linear; in the former case the line and plane remain the same, in the latter case the plane of the resultant reflection is the plane of the two lines of the component reflections and the line of the resultant reflection is the line of intersection of the planes of the components. Conversely the resolution of a planar reflection into two commutative reflections may be accomplished in ∞^2 ways by choosing the lines of the desired component reflections at random in the plane of the given reflection and taking the planes of the components through these lines and the line of the given reflection. Two commutative reflections compound into a linear reflection when and only when they are either central and planar or both planar or both linear; in the first case the line and plane remain the same, in the latter cases the line is the intersection of the planes of the components and the plane is the plane through the two lines of the components. A converse similar to the above holds good.*

The analysis has shown that the product of two commutative reflections is a reflection and has furnished specific information as to what reflection the product is. If it had been a matter of no importance to determine the resultant reflection, the fact that it was a reflection could have been inferred as a special case of the general theorem: If the product of two involutory transformations

is commutative, the product is itself involutory. This general theorem follows from the symbolic equations

$$t_1 t_2 = t_2 t_1 = t_2^{-1} t_1^{-1} = (t_1 t_2)^{-1},$$

where t_1 and t_2 are involutory. A similar set of equations,

$$t_1 t_2 = (t_1 t_2)^{-1} = t_2^{-1} t_1^{-1} = t_2 t_1,$$

shows that conversely: If the product of two involutory transformations is involutory, the product is commutative. This enables us to omit the word commutative from the above theorem. Hence

THEOREM 5. *The word commutative may be deleted from theorem* 4; *for the compositions and resolutions there indicated exist when and only when the component reflections are commutative, that is, when and only when they satisfy the conditions of theorem* 3.

By *definition* two reflections, planar or linear, will be said to be *complanar* when their fundamental planes coincide, and *collinear* when their fundamental lines coincide, irrespective of whether the reflections belong to the same type or not. It has been seen that two reflections which are both complanar and collinear compound into the identical transformation or into the central reflection according as they are of the same or of different types. The next question is concerning the composition of reflection which are either complanar or collinear, but not both.

The product of two complanar planar reflections may be written as

$$(I - 2\epsilon\zeta)\cdot(I - 2\eta\zeta), \qquad \epsilon\cdot\zeta = \eta\cdot\zeta = 1.$$

This is immediately reducible to

$$I + 2(\epsilon - \eta)\zeta.$$

The effect of applying this dyadic to any vector ρ is

$$\rho' = \rho + 2(\epsilon - \eta)\zeta\cdot\rho.$$

The condition $\epsilon\cdot\zeta = \eta\cdot\zeta$ gives $(\epsilon - \eta)\cdot\zeta = 0$. Hence $\epsilon - \eta$ is perpendicular to ζ, or, in other words, lies in the plane represented by ζ. Hence the transformation consists in adding to any vector a component in the definite direction $\epsilon - \eta$ and of magnitude proportional to the component of that vector perpendicular to the plane represented by ζ. Points in the plane represented by ζ are unchanged in position.

Now by definition a special simple shear* is a transformation which leaves a plane fixed point for point and which moves points not in this plane along lines

* *Vector Analysis*, p. 363. Note that in this case and those which follow, the coefficient of I is 1. This is necessitated by the fact that for any transformation compounded of reflections there is no dilatation.

parallel to a given direction in the plane by an amount proportional to their distance above the plane. Thus a special simple shear is completely specified when its fixed plane, a direction in the plane, and a number which gives the amount of motion in that direction per unit distance from the plane is given. This amount is easy to determine in the above case. Analytically it is $2(\epsilon - \eta)$: for ζ may be taken as a unit vector perpendicular to the common plane, and then, since ρ must be taken of unit length in the direction of ζ if the amount of shift at unit distance from the plane be desired, we have $\xi \cdot \rho = 1$. Now the relations $\epsilon \cdot \zeta = \eta \cdot \zeta = 1$ show that ϵ and η terminate in a plane at unit distance above the fixed plane. Hence

THEOREM 6. *The product of two complanar planar reflections is a special simple shear parallel to the common plane and in the direction of the intersection of this plane with the plane of the two lines of the reflections and of amount equal (per unit distance from the fixed plane) to twice the distance from the point where the line of the first* reflection cuts a plane at a unit distance from the fixed plane to the point where the line of the second* reflection cuts this plane.*†

Two remarks will be sufficient to lead to two more theorems. In the first place the identity

$$(I - 2\epsilon\zeta)\cdot(I - 2\eta\zeta) = (2\epsilon\zeta - I)\cdot(2\eta\zeta - I)$$

shows that no new work need be done on complanar linear reflections. In the second place, it should be noted that any simple shear of the type above considered may be written in the form

$$I + \gamma\beta, \qquad \gamma \cdot \beta = 0.$$

Now if two reflections be chosen with a common plane perpendicular to β and if the choice of the lines ϵ and η be made to conform to the relations

$$\gamma = 2(\epsilon - \eta), \qquad \epsilon \cdot \beta = \eta \cdot \beta = 1,$$

it is evident that the shear has been resolved into two complanar planar reflections. Moreover the choice of the direction of either ϵ or η is arbitrary — the choice of either one confines the other to a perfectly definite direction in the plane of the one first chosen and of γ. Hence

THEOREM 7. *The product of two complanar linear reflections is a special simple shear parallel to the common plane in the direction of the intersection of this*

* It should be noted that the *first* and *second* reflections are respectively $I - 2\eta\zeta$ and $I - 2\epsilon\zeta$, that is, the order of the reflections is necessarily the opposite of that in which the product of the dyadics is written. The reason for this is obvious.

† The value of this amount may be expressed trigonometrically by direct translation from the vector formulas. If $I - 2\epsilon\zeta$ be written as $I - 2\epsilon_1\zeta_1 \sec(\epsilon\zeta)$, where ϵ_1 and ζ_1 are unit vectors, the evaluation of the amount of shear gives

$$\sqrt{\sec^2(\epsilon\zeta) + \sec^2(\eta\zeta) - 2\sec(\epsilon\zeta)\sec(\eta\zeta)\cos(\epsilon\eta)}.$$

plane with the plane of the two lines of the reflections and of amount equal (per unit distance from the fixed plane) to twice the distance from the point where the line of the first reflection cuts a plane at unit distance from the fixed plane to the point where the line of the second reflection cuts this plane.

THEOREM 8. *A special simple shear may be resolved in ∞^2 ways into the product of two planar or two linear reflections which are complanar with the fixed plane of the shear, and the line of one reflection may be chosen at random.* The line of the other reflection will then be uniquely determined by the conditions stated in theorems 6 and 7.*

The discussion may be carried on in similar manner for the case of collinear reflections. The product

$$(I-2\epsilon\zeta)\cdot(I-2\epsilon\xi) \qquad \epsilon\cdot\zeta=\epsilon\cdot\xi=1,$$

reduces to the form

$$I+2\epsilon(\xi-\zeta).$$

The vector ϵ is unchanged because of the relations

$$I\cdot\epsilon=\epsilon, \qquad \xi\cdot\epsilon-\zeta\cdot\epsilon=0.$$

The vector $\xi\times\zeta$ is likewise unchanged because of the relations

$$I\cdot\xi\times\zeta=\xi\times\zeta, \qquad \xi\cdot\xi\times\zeta=\zeta\cdot\xi\times\zeta=0.$$

Hence points in the plane of ϵ and $\xi\times\zeta$ remain fixed. This is the plane of the shear which the product obviously represents. The result of applying the shear to any vector ρ is

$$\rho'=I\cdot\rho+2\epsilon(\xi-\zeta)\cdot\rho=\rho+2\epsilon(\xi-\zeta)\cdot\rho.$$

Hence the direction of the shear is ϵ. The vector which is perpendicular to the plane of the shear is $\epsilon\times(\xi\times\zeta)$ or $\xi-\zeta$. The relations $\epsilon\cdot\zeta=\epsilon\cdot\xi=1$ enable one to read off the amount of the shear as twice the distance between the points where the vectors ξ and ζ meet a plane at unit distance from the origin and perpendicular to ϵ. It is, however, more convenient to note that

$$\sigma=\frac{\xi-\zeta}{|\xi-\zeta|}-\frac{(\xi-\zeta)\cdot\xi}{|\xi-\zeta|}\epsilon+y\xi\times\zeta$$

and

$$\tau=\frac{\xi-\zeta}{|\xi-\zeta|}-\frac{(\xi-\zeta)\cdot\zeta}{|\xi-\zeta|}\epsilon+y\xi\times\zeta$$

* The only exception to the at-random-ness of the choice of this line is that it shall not lie in the plane of the shear. It has been stated previously, however, that this plane is the plane of the reflections, and hence the line could not lie in this plane. Frequently this same state of affairs occurs. We shall therefore make the convention that at random, applied to the choice of a line or plane of a reflection, means at random except for such positions as make the line and plane of the reflection occupy this position of coincidence, which would render the reflection meaningless.

are two vectors terminating in a plane at unit distance from the fixed plane of the shear, that they lie respectively in the planes represented by ξ and ζ, and that their difference is precisely the magnitude and direction of the shear. Hence, adding a few obvious propositions, we may state

THEOREM 9. *The product of two collinear reflections, be they both planar or both linear, is a special simple shear parallel to the plane that contains their common line and the line of intersection of their planes and in the direction of their common line and of amount equal (per unit distance from the fixed plane) to twice the distance from the point where a line parallel to and at a unit distance above the fixed plane cuts the plane of the first reflection, to the point where it cuts the plane of the second reflection. And conversely any such shear may be resolved in ∞^2 ways into the product of two collinear reflections; for the common line may be chosen as the direction of the shear, and the line of intersection of the planes may be taken as any line in the plane of the shear, and the two planes of the reflections may be so set as to cut off one half the amount of the shear upon a line parallel to the line of the shear at unit distance from the plane of the shear.*

If two complanar or collinear reflections one of which is planar and the other linear be compounded the result is of the form

$$-(I-2\epsilon\zeta)\cdot(I-2\eta\zeta) \qquad \text{or} \qquad -(I-2\epsilon\zeta)\cdot(I-2\epsilon\xi).$$

Apart from the initial negative sign this is merely a shear, as before. The negative sign reverses all the directions in space. A transformation which is in all respects like a shear except that all directions in space have been reversed may be called a *perverted** shear. Then

THEOREM 10. *The relation of two complanar or collinear reflections, of which one is planar and the other linear, to the perverted shear is the same as the relation of two like complanar or collinear reflections to the (unperverted) shear. See theorems* 6–9.

There have been found a number of ways in which a special simple shear may be resolved into the product of two reflections. The question arises whether all the possibilities have been enumerated. The corresponding question in the case of resolving a reflection into two reflections was answered in theorem 5 by means of general equations in the theory of involutory transformations. This method is not available here. Collineations may, however, be classified, in the simpler cases, as essentially (projectively) different according as they have similar or dissimilar characteristic equations. This method has been carried out in detail for strains in the *Vector Analysis*, pages 356–367. The facts are in part these: Any dyadic Φ satisfies identically a definite cubic equation †

* Analogously to the nomenclature of GIBBS. *Vector Analysis*, p. 337.

† In general if ϕ denote a dyadic representing a strain in n dimensions and if $\phi_s, \phi_{2s}, \cdots, \phi_n$ be its invariants, the dyadic satisfies identically the equation of the nth order

$$\phi^n - \phi_s\phi^{n-1} + \phi_{2s}\phi^{n-2}\cdots + (-1)^n\phi_n I = (\phi - a_1 I)\cdot(\phi - a_2 I)\cdot\cdots\cdot(\phi - a_n I) = 0$$

$$\Phi^3 - \Phi_S\Phi^2 + \Phi_{2S}\Phi - \Phi_3 = 0$$

or

$$(\Phi - aI)\cdot(\Phi - bI)\cdot(\Phi - cI) = 0,$$

where the quantities a, b, c are roots of the cubic equation

$$x^3 - \Phi_S x^2 + \Phi_{2S} x - \Phi_3 = 0.$$

If two or three roots of this equation are equal the dyadic may satisfy an equation of lower degree. The equation of lowest degree which the dyadic satisfies is called the *characteristic* equation. There are six distinct types of characteristic equation for strains* (if no distinction is made between real and imaginary roots of the scalar cubic). To test a given strain it is merely necessary to find what type of characteristic equation it has.

In the case of the special simple shear the characteristic equation is of the second order and takes the form

$$(\Phi - I)^2 = 0.$$

The roots of the cubic are all equal to unity (for we are considering merely the case where there is no stretching modulus), and the cubic is

$$x^3 - 3x^2 + 3x - 1 = 0.$$

For the perverted shear the roots are all equal to -1 instead of $+1$ and the characteristic equation becomes

$$(\Phi + I)^2 = 0,$$

while the scalar cubic is

$$x^3 + 3x^2 + 3x + 1 = 0.$$

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where $a_1, a_2, \cdots, a_n$ are the roots of a scalar equation of the nth degree. To classify the strain it is convenient to classify first all different sorts of sets of roots which the scalar n-ic can have, according to their multiplicity, and then to subclassify according to the form of the characteristic equation. Thus in the case $n = 4$ the different types are: a, b, c, d; a, b, c, c; a, a, c, c; a, a, a, b; a, a, a, a. In the first case the characteristic equation is determined as a product of four factors: $\phi - aI$, $\phi - bI$, $\phi - cI$, $\phi - dI$. Not so, however, in the other cases. In the second case, the factor $\phi - cI$ may occur twice, leaving the equation of the nth degree, or it may occur only once, reducing the degree the $(n-1)$st. Similarly in the third case there may be four factors to the characteristic equation, or one of the factors may drop out, or two of them may drop out, reducing the equation to $(\phi - aI)\cdot(\phi - cI)$. There are thus three types of strain corresponding to this set of roots for the scalar n-ic. If all the types be counted up, it appears that there are 13 types of characteristic equation at most. That each of these types exists, is easily shown by actually writing down the expression for it. This classification is not complete except when $n = 2$ or 3; for larger values of n additional criteria must be adduced. Thus when $n = 4$, there are 14 types of strain, but only 13 types of characteristic equations, and when $n = 5$ these numbers are 27 and 24 respectively. This difficulty need not concern us here.

* *Vector Analysis*, p. 366. The first two types there given differ only by the fact that in one case two of the roots are conjugate imaginaries. Geometrically this is an essential difference analytically it is unimportant.

This case we will not take up in detail. The entire treatment resembles too closely that for the (unperverted) shear.

Consider any reflection

$$\Phi = \pm(I - 2\epsilon\zeta), \quad \Phi_2 = 2\zeta\epsilon - I, \quad \Phi_S = +1, \quad \Phi_{2S} = -1, \quad \Phi_3 = +1.$$

The invariants Φ_S, Φ_{2S}, Φ_3 which are the coefficients of the scalar cubic may be computed for a product of dyadics by means of the formulas*

$$(\Psi\cdot\Omega)_2 = \Psi_2\cdot\Omega_2, \qquad (\Phi\cdot\Omega)_3 = \Phi_3\Omega_3.$$

Hence the scalar cubic for the product

$$\Phi = \pm(I - 2\epsilon\zeta)\cdot(I - 2\eta\xi) = \pm(I - 2\epsilon\zeta - 2\eta\xi + 4\zeta\cdot\eta\epsilon\xi)\dagger$$

has the form

$$x^3 \mp (4\zeta\cdot\eta\epsilon\cdot\xi - 1)x^2 + (4\epsilon\cdot\xi\zeta\cdot\eta - 1)x \mp 1 = 0.$$

If this product is to be a shear of the type we have been considering, it is necessary that

$$\Phi_S = 3 = \pm(4\zeta\cdot\eta\epsilon\cdot\xi - 1), \qquad \Phi_{2S} = 3 = 4\epsilon\cdot\xi\zeta\cdot\eta - 1, \qquad \Phi_3 = 1 = \pm 1.$$

The first or last of these relations shows that the two reflections must be of the same type. That is, the upper sign must hold throughout. Since the upper sign holds, form the difference

$$\Phi - I = 2(\epsilon\zeta + \eta\xi - 2\epsilon\xi\zeta\cdot\eta).$$

To multiply out the expression $(\Psi - I)^2$ and hence obtain the condition for a simple shear would be long—a shorter method is desirable. Now if the product $\Psi\cdot\Omega$ of two dyadics is zero, either Ψ or Ω must be linear. Hence in this case $\Phi - I$ is linear. For this the necessary and sufficient condition is that $(\Phi - I)_2$ vanish.‡ But

$$(\Phi - I)_2 = 4\epsilon \times \eta\zeta \times \xi = 0.$$

Hence either ϵ and η are collinear or ζ and ξ are collinear; that is, the reflections are either collinear or complanar. It has already been seen that they cannot be complanar and collinear. Hence

THEOREM 11. *The necessary and sufficient condition that a special simple shear be resoluble into two reflections is that the reflections be of the same type, either linear or planar, and that they be complanar or collinear, but not both. The resolution may then be accomplished in ∞^2 ways as specified in theorems* 6–9; *similar results hold for the perverted shear.*

* *Vector Analysis*, p. 312.

† Here and throughout the following work the upper signs belong together and the lower signs belong likewise together, forming only two possible sets.

‡ *Vector Analysis*, p. 315.

Another case where the roots of the scalar cubic are all equal to unity is that of the complex shear of which the characteristic equation is

$$(\Phi - I)^3 = 0.$$

This shear may be reduced to the canonical form *

$$I + \alpha\beta' + \beta\gamma'.$$

The geometric properties of the transformation are these: There is a direction, the direction α, which is unchanged. Vectors parallel to α do not change either in magnitude or in direction. Through the line there is a plane, the plane of α and β, in which points move parallel to α by an amount proportional to their distance from the fixed line α. Through the line there is a second plane, the plane of α and γ, the points of which move parallel to β by an amount proportional to their distance from the line α. It should be noted, however, that the points of any plane passing through the line α, say the plane of α and δ, move parallel to a certain direction in the plane of α and β, namely the direction

$$\alpha\beta' \cdot \delta + \beta\gamma' \cdot \delta.$$

Thus the reduction of the complex shear to the canonical form may be accomplished in a single infinity of ways. A complex strain may therefore be specified by giving its fixed line a fixed plane through this line and a number which expresses the shift parallel to the line (per unit distance from the line), and a second plane through the fixed line with the direction and amount of shift (per unit distance from the line) of the points is this plane.

In distinction from the complex shear, there is the perverted complex shear, which is the combination of the shear with reversal of all directions in space. The characteristic equation of the perverted complex shear is

$$(\Phi + I)^3 = 0$$

and its scalar cubic is

$$x^3 + 3x^2 + 3x + 1 = 0.$$

This case is so similar to that of the (unperverted) shear that it need not be taken up in detail.

Consider then the product of two reflections. Just as before, it appears that the reflections must be of the same type, either planar or linear, if their product is to be a complex shear. The upper sign therefore holds. The value of the invariant Φ_{28} is

$$\Phi_{28} = 3 = 4\epsilon \cdot \xi\zeta \cdot \eta - 1.$$

* *Vector Analysis*, pp. 365–7.

Hence

$$\zeta\cdot\eta\epsilon\cdot\xi = 1.$$

But the relations

$$\epsilon\cdot\zeta = \eta\cdot\xi = 1 \qquad \text{and} \qquad \epsilon\times\eta\cdot\zeta\times\xi = \epsilon\cdot\zeta\eta\cdot\xi - \eta\cdot\xi\epsilon\cdot\xi$$

show that the condition reduces to

$$\epsilon\times\eta\cdot\zeta\times\xi = 0.$$

And in this equation neither $\epsilon\times\eta$ nor $\zeta\times\xi$ can vanish, or the product would reduce to the case of a simple shear previously considered. Hence

THEOREM 12. *The necessary and sufficient condition that the product of two reflections be a complex shear is that the reflections be of the same type and that the lines of the two reflections be complanar with the intersection of the planes of the reflections without the reflections themselves being either complanar or collinear. An analogous result holds for perverted complex shears, the only difference being that the reflections must be of different type.*

Further to discuss the transformation write the product in the form

$$\Phi = I - 2(\eta - \zeta\cdot\eta\epsilon)\xi - 2\epsilon(\zeta - \zeta\cdot\eta\xi).$$

Here

$$(\eta - \zeta\cdot\eta\epsilon)\cdot\xi = \eta\cdot\xi - \zeta\cdot\eta\epsilon\cdot\xi = 0,$$

$$\epsilon\cdot(\zeta - \zeta\cdot\eta\xi) = \epsilon\cdot\zeta - \zeta\cdot\eta\epsilon\cdot\xi = 0,$$

$$(\eta - \zeta\cdot\eta\epsilon)\cdot(\zeta - \zeta\cdot\eta\xi) = 0.$$

Hence the vectors $\eta - \zeta\cdot\eta\epsilon$, ϵ, ξ, $\zeta - \zeta\cdot\eta\xi$ satisfy the relations which are characteristic of the vectors α, β, β', γ' that enter into the canonical expression for the complex shear (with the exception of the relation $\beta\cdot\beta' = 1$ that depends on the magnitudes of the vectors). The vector $\eta - \zeta\cdot\eta\epsilon$ which corresponds to α is the fixed direction of the shear. By inspection, however, the vector $\zeta\times\xi$ is unchanged—as is geometrically obvious. And in fact it is evident, on expanding, that

$$(\eta - \zeta\cdot\eta\epsilon)\times(\zeta\times\xi) = 0.$$

The direction ϵ, which is the line of the second reflection, is the other fundamental line in the fixed plane. In particular a vector $a\zeta\times\xi + b\epsilon$ becomes

$$a\zeta\times\xi + b\epsilon + 2b\left(\epsilon - \frac{1}{\zeta\cdot\eta}\eta\right).$$

Hence the amount of motion of the points in the fixed plane is

$$2b\left(\epsilon - \frac{1}{\zeta\cdot\eta}\eta\right).$$

Moreover, since $\epsilon\cdot\zeta = 1$, ϵ may be regarded as terminating in the plane determined by ζ, and if η be regarded as terminating in this same plane, $\eta\cdot\zeta = 1$. Then the additional term which represents the motion in the fixed plane reduces to $2b(\epsilon - \eta)$, with the interpretation that the points move parallel to the fixed line $\zeta\times\xi$ by an amount equal (per unit distance from that line) to twice the distance from the point where the first line η cuts a line at a unit distance from that line to the point where the second line ϵ cuts the same line.

The expression for the product shows that any vector ρ in the plane of $\zeta\times\xi$ and ξ suffers no change parallel to the fixed line $\zeta\times\xi$ but is altered by the amount $-2\epsilon\zeta\cdot\rho$ in the direction ϵ. By virtue of the relation $\epsilon\cdot\zeta = 1$ the equation

$$-\epsilon\zeta\cdot\rho = (\epsilon\times\rho)\times\zeta - \rho$$

is identical and shows that the change $-2\epsilon\zeta\cdot\rho$ is twice the distance from the extremity of ρ to the point where a line parallel to ϵ and passing through this extremity cuts the plane represented by ζ. To sum up we have

THEOREM 13. *The product of two planar or two linear reflections which are neither complanar nor collinear but so related that their two lines and the line of intersection of their two planes are complanar gives rise to a complex shear of which the fixed line is the line of intersection of the two planes, and of which the fixed plane is the common plane of the lines* ϵ, η, $\zeta\times\xi$. *The amount of the shift parallel to the fixed line and in the fixed plane is equal (per unit distance from the fixed line) to twice the distance from the intersection of the first line* η *to the intersection of the second line* ϵ *with a line lying in the fixed plane at a unit distance from the fixed line. The plane in which there is no shift parallel to the fixed line is the plane of the first reflection, and points in this plane are shifted parallel to* ϵ *by an amount* equal (per unit distance from the fixed line) to twice the distance from a point in the plane of the first reflection and at a unit distance from the fixed line to the point in the plane of the second reflection where a line through this point and parallel to* ϵ *cuts the plane of the second reflection. An analogous result for the perverted complex shear.*

THEOREM 14. *Conversely any complex shear may be resolved into the product of two planar or two linear reflections by choosing the lines of the two reflections in the fixed plane of the shear and subject to the sole restriction that the distance of the*

* The deduction of the trigonometric relations that express the magnitude of the shear is left to the reader.

points where the first and second line respectively cut the line at a unit distance from the fixed line of the shear shall be equal to one-half the amount of shift (per unit distance from the fixed line) in the fixed plane; and by choosing the two planes of the reflections to pass through the fixed line of the shear in such a way that the first plane coincides with the plane whose points are shifted parallel to the line of the second reflection and that the second plane be distant from the first by an amount measured along a line parallel to the second line and through a point at a unit distance from the fixed line equal to one-half the amount of the shift due to the shear. This resolution may therefore be accomplished in ∞^1 ways and not in ∞^2 ways.

The case in which the lines of the reflections lay in the planes, each of the other, gave rise to commutative reflections which compound into another reflection. If the scalar cubic and the characteristic equation had been called into play, it would have appeared that this case corresponds to the roots $-1, -1, +1$ or $-1, +1, +1$ and to the characteristic equations

$$(\Phi - I)\cdot(\Phi + I) = 0 \qquad \text{or} \qquad (\Phi + I)\cdot(\Phi + I) = 0,$$

according as the resulting reflection was linear or planar. There arises then the question: What if only one of the lines lie in the plane of the other reflection? In this case either $\zeta\cdot\eta = 0$ or $\xi\cdot\epsilon = 0$; but not both. There is no need of giving the details of the computation. The result shows that the characteristic equation is of the third order and takes the form

$$(\Phi - I)\cdot(\Phi + I)^2 = 0 \qquad \text{or} \qquad (\Phi + I)\cdot(\Phi - I)^2 = 0,$$

according as the roots are the first or second of the above set.

This sort of equation betokens a simple shear* (non-special). The general shear of this type is

$$a\alpha\alpha' - p(\beta\beta' + \gamma\gamma') - \gamma\beta' \qquad \text{or} \qquad a\alpha\alpha' + p(\beta\beta' + \gamma\gamma') + \gamma\beta'.$$

Here, in the first case, there is a fixed direction α. Through this direction pass two planes, of which one, the plane of α and γ, is fixed as a whole, though if vectors are resolved parallel to α and γ the γ-components are stretched in the ratio $-p : 1$ and the α-components in the ratio $a : 1$. The points of the other plane, that of α and β, have a compound motion. The vectors which denote the points are stretched along α and β and take on an increment parallel to γ of amount proportional to their components along β. The plane of β and γ is stretched in the ratio $p : 1$ and the points move out of it by an amount proportional to their β components, the motion being in the direction γ. The vector β may be chosen at will in the plane of β and γ. In the case of the above roots the shear takes the form

* *Vector Analysis*, pp. 366-7.

$$\alpha\alpha' - (\beta\beta' + \gamma\gamma') - \gamma\beta' \quad \text{or} \quad -\alpha\alpha' + (\beta\beta' + \gamma\gamma') + \gamma\beta'.$$

The latter is merely a perversion of the first.

If the line of the first reflection lies in the plane of the second, $\zeta \cdot \eta = 0$, and the product takes the form (we shall consider, as usual, the first or unperverted case)

$$\Phi = I - 2\epsilon\zeta - 2\eta\xi.$$

It is obvious that the line $\zeta \times \xi$ of intersection of the two planes plays the rôle of the fixed line α; the line η, namely that which lies in the plane of other reflection, plays the rôle of γ, parallel to which the shear takes place; the plane of the two lines ϵ and η, that of the plane of β and γ. The amount of the shear suffered by ϵ is

$$-2\eta\xi \cdot \epsilon, \qquad \text{but} \qquad (\epsilon - \xi \cdot \epsilon\eta) \cdot \xi = 0;$$

which shows that this amount is twice the distance from the plane of ϵ and α to the plane ξ, if that distance be measured parallel to η.

If it is the line of the second reflection that lies in the plane of the first, the product becomes

$$\Phi = I - 2\epsilon\zeta - 2\eta\xi + 4\zeta \cdot \eta\epsilon\xi, \qquad \epsilon \cdot \xi = 0.$$

The line of intersection $\zeta \times \xi$ of the two planes still corresponds to the fixed line α. The line ϵ is now the line parallel to which the shear takes place. Thus in either case it is the line which lies in the plane of the other reflection that determines the direction of the shear. The plane corresponding to the plane of β and γ is still the plane of the two lines. The amount of shear suffered by η is

$$2\epsilon\zeta \cdot \eta, \qquad \text{but} \qquad (\eta - \epsilon\zeta \cdot \eta)\zeta = 0;$$

which shows that this amount* is twice the distance from the plane ζ to the plane of η and α, if the distance be measured parallel to ϵ.

Conversely any simple shear all of whose roots are numerically equal to unity (but not all equal algebraically) may be resolved in ∞^1 ways into the product of two reflections. This may be accomplished by choosing the planes of the reflections through the fixed line and one of them coincident with the fixed plane. The line of the other reflection must then be chosen in this plane and along the direction of the shear parallel to the plane. The line of the other reflection may then be taken at will in a definite plane, after which the final plane is wholly determined by the magnitude of the shear — but differently according as

* The trigonometric values are again left to the reader.

it is the line of the first reflection which is to lie in the plane of the second or vice versa. It will not be necessary to specify all these relations in a theorem or to prove, as may readily be done, that these are the only possible resolutions. It will be sufficient to state

THEOREM 15. *The necessary and sufficient condition that two reflections compound into a simple (non-special) shear of which the roots are numerically equal to unity is that the line of one of the reflections lie in the plane of the other. And conversely any such shear may be resolved into two reflections in* ∞^1 *ways.*

2. *The general case of the composition of two or three reflections.*

The general product of two reflections and the scalar cubic associated therewith have been given on page 282. The form of the cubic leads to

THEOREM 16. *If a dyadic* Φ *is the product of two roots of the idemfactor the conditions*

$$\Phi_{2S} = \pm \Phi_S, \qquad \Phi_3 = \pm 1$$

hold. The scalar cubic is a reciprocal equation with one of the three roots equal to ± 1 *and with the product of the other two roots equal to* $+1$.

As one of the roots of the cubic is ± 1, the equation may be written

$$(x \mp 1)\left[x^2 \mp (\Phi_{2S} - 1)x + 1\right] = 0.$$

The condition that the roots of the second factor be equal is

$$\Phi_{2S} - 1 = \pm 2, \qquad \Phi_{2S} = 3 \text{ or } -1.$$

The three roots are then some combination selected from ± 1, ± 1, ± 1. Geometrically these cases correspond to the identical transformation, the reflections, and the two shears — all of which have been treated in detail in § 1. If the coefficient of the first degree term in the quadratic factor be numerically decreased from the value which gives the equal roots, the roots of the resulting quadratic will be imaginary, whereas if that coefficient be increased the roots will be real. Hence the inequalities

$$|\Phi_{2S} - 1| > 2, \quad \Phi_{2S} > 3 \quad \text{or} \quad \Phi_{2S} < -1, \quad \epsilon\cdot\xi\zeta\cdot\eta < 0 \quad \text{or} \quad \epsilon\cdot\xi\zeta\cdot\eta > 1$$

obtain when two roots of the cubic are real, and the dyadic reduces to a tonic; while the inequalities

$$|\Phi_{2S} - 1| < 2, \qquad -1 < \Phi_{2S} < 3, \qquad 0 < \epsilon\cdot\xi\zeta\cdot\eta < 1$$

hold when two roots are imaginary and the dyadic reduces to the cyclotonic.

Just as in the case of shears, it will be necessary to distinguish between perverted and unperverted tonics and cyclotonics. The same convention will be adopted, namely, that if the value of Φ_3 is $+1$ so that one of the roots of the cubic is $+1$ the transformation is unperverted, but when Φ_3 is -1 so that one root of the cubic is -1 the transformation will be called perverted. In any transformation which is the product of two reflections there is one line which either remains unchanged both in magnitude and in direction or is merely reversed in direction without change of magnitude. This is the line of intersection of the two planes; and in the former case the transformation is unperverted, while in the latter case it is perverted.

Let the attention be hereafter confined to the unperverted types. The computation of the second of the product gives

$$\Phi_2 = I - 2\zeta\epsilon - 2\xi\eta + 4\xi\cdot\epsilon\zeta\eta.$$

This shows that areas in the plane denoted by $\epsilon \times \eta$, that is, in the plane of the two lines ϵ and η are unchanged. Now the entire result of the transformation may be expressed in terms of the transformation along $\zeta \times \xi$ which is the identical transformation and the transformation in the plane of ϵ and η. Thus it becomes possible to read off the general results in the present case from those obtained previously.* They may be stated as

THEOREM 17. *The product of two planar or two linear reflections which relative to one another have none of the special relations discussed in § 1, is a transformation which leaves the line of intersection of the two planes unchanged whether in magnitude or in direction and which leaves area in the plane of the two lines invariant. The fixed lines of the transformation in this plane divide harmonically the two pairs of lines formed by associating the line of each reflection with the line in which its plane cuts this fixed plane. The transformation is tonic or cyclotonic according as its fixed lines are real or imaginary, that is, according as the two pairs of lines do not or do separate each other. The result for the product of two reflections of which one is planar and the other linear is the same except that the transformation is perverted. And conversely any tonic or cyclotonic which leaves one line invariant in magnitude and direction and which has an invariant plane not passing through this line and subject to the invariancy of areas may be resolved into the product of two reflections both planar or both linear by choosing the invariant line as the intersection of the two planes and by arranging the lines of the reflections and the lines in which their planes cut the fixed plane so as to give the required*

* If the statements of the following theorem 17 are not entirely clear from the very brief reasoning given here, a reference to my *A generalized conception of area: applications to collineations in the plane*, Annals of Mathematics, ser. 2, vol. 5 (1903), pp. 29–45, will doubtless make them evident. It seems undesirable to increase the dimensions of the present discussion by practically repeating portions of this earlier work.

transformation in this plane. As this may be done in ∞^1 ways, the above resolution may be accomplished in ∞^1 ways. A similar result holds for perverted tonics and cyclotonics.*

In the general classification of strains there are seven types if distinction between reals and imaginatives is made. In the classification of unimodular strains there are these seven types, but the additional relation $\Phi_3 = \pm 1$ limiting the product of the roots of the scalar cubic to ± 1 modifies the result somewhat. If the roots are all alike and if the transformation is real (which has been constantly the point of view of this discussion) the roots are either plus or minus one. This gives the identical transformation or the central reflection (the perverted identical transformation) if the characteristic equation is $\Phi \mp I = 0$; the simple special shear or its perversion, if the characteristic equation is $(\Phi \mp I)^2 = 0$; the complex shear or its perversion, if the equation is $(\Phi \mp I)^3 = 0$. If two of the roots are alike, they must be $+1, +1, -1$ or $+1, -1, -1$ or $a, a, 1/a^2$. The first two cases give the planar and linear reflections respectively, if the characteristic equation is $(\Phi \pm I)\cdot(\Phi \mp I) = 0$; the simple shear, if the equation is

$$(\Phi + I)\cdot(\Phi - I)^2 = 0 \qquad \text{or} \qquad (\Phi - I)\cdot(\Phi + I)^2 = 0.$$

Finally if the roots are $\pm 1, k, 1/k$ the transformation is a tonic or cyclotonic resoluble into two reflections, and if the roots are not of this type the result is a tonic or cyclotonic not so resoluble.

It should be noted that theorem 16 states that if a transformation is resoluble into two reflections

$$\Phi_{2S} = \mp \Phi_S, \qquad \Phi_3 = \mp 1,$$

according as it is perverted or unperverted. Now conversely if these relations hold, the cubic becomes a reciprocal equation and the roots cannot be of the form

$$a, a, \pm 1/a^2 \qquad \text{or} \qquad a, b, c \text{ (not } \pm 1, k, 1/k).$$

That is, the roots must bring the transformation under one of the several heads which has already been found to be resoluble into two reflections. Hence theorem 16 may be completed in

THEOREM 18. *The necessary and sufficient condition that a strain be resoluble into two reflections is that the invariants of the strain satisfy the relations*

$$\Phi_{2S} = \pm \Phi_S, \qquad \Phi_3 = \pm 1,$$

that is, the scalar cubic shall be reciprocal.

* It is interesting to note the fact, which might be stated as a theorem, that if a strain is resoluble into two reflections, it is so resoluble in ∞^1, ∞^2, or ∞^4 ways according as its characteristic equation is of the 3rd, the 2nd, or the 1st order.

To return to the geometry relative to the cyclotonic and tonic. The cyclotonic of which the characteristic equation is

$$(\Phi - aI)\cdot(\Phi^2 - 2p\cos q\Phi + p^2 I) = 0$$

may be put in the form *

$$a\alpha\alpha' + p\cos q(\beta\beta' + \gamma\gamma') + p\sin q(\gamma\beta' - \beta\gamma').$$

The scalar cubic takes the form

$$(x - a)(x^2 - 2p\cos q\cdot x + p^2) = 0.$$

Now it has been shown that if the transformation is the product of two reflections the scalar cubic may be written as

$$(x \mp 1)[x^2 \mp (\Phi_{2S} - 1)x + 1] = 0.$$

Comparing corresponding coefficients shows that in this case the cyclotonic (confining the attention to the unperverted type) reduces to

$$p=1,\ \alpha\alpha' + \cos\cos^{-1}[(\Phi_{2S}-1)/2](\beta\beta' + \gamma\gamma') + \sin\cos^{-1}[(\Phi_{2S}-1)/2](\gamma\beta' - \beta\gamma').$$

This is a *cyclic* dyadic and represents the transformation which has been called an elliptic rotation through the angle $\cos^{-1}[(\Phi_{2S} - 1)/2]$. If $m = 2\pi/\cos^{-1}[(\Phi_{2S} - 1)/2]$ this dyadic may be regarded as the mth root of the idemfactor.† Hence

THEOREM 19. *The necessary and sufficient condition that a cyclotonic be resoluble into the product of two reflections is that it be merely cyclic. The angle of the elliptic rotation is* $\cos^{-1}[(\Phi_{2S} - 1)/2] = \cos^{-1}(2\zeta\cdot\eta\xi\cdot\epsilon - 1)$.

THEOREM 20. *Any root ‡ of the idemfactor may be written in* ∞^1 *ways as the product of two square roots.*

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* *Vector Analysis*, pp. 355, 361, 366.

† *Vector Analysis*, pp. 348–350.

‡ It has not been shown that even if the cyclic dyadic is a root of the idemfactor, every root must conversely be of this type. To do this proceed as follows: First the determinant of the idemfactor is unity, and the determinant of a pth root of the factor must therefore be a pth root of unity. If the strain is to be real, this pth root must be ± 1 (we may consider p as a prime, if we desire, and then the root would become $+1$). Next the equation $(\sqrt[p]{I})_2 = (\sqrt[p]{I})_3(\sqrt[p]{I})_c^{-1}$ shows that the values of $(\sqrt[p]{I})_{2S}$ and $(\sqrt[p]{I})_S$ are such as to make the scalar cubic reciprocal. Now if the roots of this equation are all real they must be all equal to 1, and there must be a complete set of fixed directions. Hence the root of the idemfactor reduces to $\pm I$, according as p is odd or even. But if two of the roots of the scalar cubic are conjugate imaginaries the pth root of I takes on cyclic form either unperverted or perverted. Thus it is proved that every root of the idemfactor is of this type.

It is not difficult to write down the ellipse in which the rotation takes place. It will be more convenient, as the rest of the work is in space, to write down the cylinder which has this ellipse for director curve and the line of intersection of the two planes for generator. The three directions ϵ, $\zeta \times \xi$, $(\epsilon \times \eta) \times \zeta$, of which the last represents the line of intersection of the ζ-plane with the plane of ϵ and η, are three conjugate directions in the quadric surface.* The three directions reciprocal to these are ζ, $\epsilon \times (\zeta \times \xi)$, $\epsilon \times \eta$ and the quadric may therefore be written in the form †

$$a\zeta\zeta + b\epsilon \times (\zeta \times \xi)\epsilon \times (\zeta \times \xi) + c\epsilon \times \eta\epsilon \times \eta.$$

The condition that the quadric be a cylinder parallel to $\zeta \times \zeta$ necessitates the vanishing of c, and the fact that η and $\xi \times (\epsilon \times \eta)$ are conjugate directions gives

$$\eta \cdot [a\zeta\zeta + b\epsilon \times (\zeta \times \xi)\epsilon \times \cdot (\zeta \times \xi)] \cdot \xi \times (\epsilon \times \eta) = 0.$$

If the ratio of a to b is determined from this condition, the final form of the cylinder turns out to be

$$\rho \cdot [(1 - \epsilon \cdot \xi\zeta \cdot \eta)\epsilon \cdot \xi\zeta\zeta + \eta \cdot \zeta\epsilon \times (\zeta \times \xi)\epsilon \times (\zeta \times \xi)] \cdot \rho = \text{const.}$$

The above deduction of the equation of the conic in which the points of planes parallel to the plane of the two lines ϵ and η move is in no wise restricted to the case of elliptic rotation. If the transformation were a tonic which leaves area invariant the motion of the points of the plane would be in hyperbolas, and the above equation would be that of the hyperbolic cylinders of which the director curves are these hyperbolas and of which the elements are parallel to $\zeta \times \xi$. In fact the form of the equation shows that the type is elliptic or hyperbolic according as

$$(1 - \epsilon \cdot \xi\zeta \cdot \eta)\epsilon \cdot \zeta \qquad \text{and} \qquad \zeta \cdot \eta$$

have the same or opposite signs — which is equivalent to the criterion established to distinguish between the cyclotonic and tonic. This suggests the question of whether it may not be possible to explain the transformation in case it is a tonic as a sort of hyperbolic rotation, and thus introduce a greater symmetry into the treatment of the tonic and cyclotonic.

It should be remembered that in elliptic rotation the dyadic

$$\Phi = \alpha\alpha' + \cos q(\beta\beta' + \gamma\gamma') + \sin q(\gamma\beta' - \beta\gamma')$$

used as a prefactor advances a radius vector in an ellipse of which β and γ are

* This follows from the harmonic property stated in theorem 17 and from the harmonic properties of conjugate directions.

† *Vector Analysis*, p. 378.

conjugate diameters through a sector, of which the area is to the area of the whole ellipse as q is to 2π.* In order to extend this statement to the case of the hyperbola it is necessary to replace the area of the ellipse by the area of the rectangle constructed on the semi-axes of the ellipse. Then the radius vector advances through a sector, the area of which is to the area of the rectangle as q is to 2. From analogy we state

THEOREM 21. *A dyadic of the form*

$$\Phi = \alpha\alpha' + \cosh q(\beta\beta' + \gamma\gamma') + \sinh q(\gamma\beta' + \beta\gamma')$$

used as a prefactor advances a radius vector in a hyperbola of which the vectors β and γ are conjugate semi-diameters through a sector, of which the area is to the area of the rectangle constructed on the semi-axes (or the parallelogram on any conjugate semi-diameters) as q is to 2.

As the hyperbola of which β and γ are conjugate radius vectors may be projected into a rectangular hyperbola in such a manner that the said conjugate direction become the axes of the rectangular hyperbola and all areas remain the same, and as the statement of the problem admits a transformation of similitude with the origin as center, it will be sufficient to prove the theorem in the case of the hyperbola $x^2 - y^2 = 1$ and the dyadic

$$\mathbf{ii} + \cosh q(\mathbf{jj} + \mathbf{kk}) + \sinh q(\mathbf{kj} + \mathbf{jk}).$$

Any radius vector in the hyperbola may be written as

$$\rho = \mathbf{i}\cosh p + \mathbf{j}\sinh p$$

and then

$$\rho = \Phi\cdot\rho = \mathbf{i}\cosh(p+q) + \mathbf{j}\sinh(p+q).$$

The area of the sector of the hyperbola from the horizontal to any inclination is

$$\tfrac{1}{2}\int_0^\theta \rho\cdot\rho d\theta = \tfrac{1}{2}\int_0^p (\cosh^2 p + \sinh^2 p)\, d\tan^{-1}\tanh p = \tfrac{1}{2}p.$$

Hence the theorem is proved. It might be noted that this tonic has the analytic form of a $2\pi/(q\sqrt{-1})$th root of the idemfactor.

The criterion for a versor† is $\Phi\cdot\Phi_c = I$, $\Phi_3 = 1$. To apply this test to ascertain under what conditions the product of two reflections is a versor would be less simple than to proceed from the point of view of the elliptic rotation. If the ellipse becomes a circle the rotation becomes versorial, and conversely.

* *Vector Analysis*, p. 349.

† *Vector Analysis*, p. 335.

Hence for the versor ϵ, η, ζ, ξ lie in a plane perpendicular to the axis of version. Moreover as the conjugate diameters of a circle are perpendicular to each other, the lines of each reflection must be normal to its plane of the reflection. There are similar results for the perversor. Hence

THEOREM 22. *The necessary and sufficient condition that the product of two reflections be a versor is that the reflections be orthogonal and of the same type; if one is planar and the other linear, the result is a perversor.*

The actual expression for the elliptic or hyperbolic rotation has not yet been given in canonical form in terms of the elements which enter into the reflections of which the transformation is compounded. There is no difficulty in doing this. In fact the vectors which correspond to the α, β, γ of the canonical form may be chosen as $\zeta \times \xi$, ϵ, $(\epsilon \times \eta) \times \zeta$. If m denote the quantity $\xi \cdot \epsilon\zeta \cdot \eta$ the reciprocals are

$$\alpha' = \frac{\epsilon \times \eta}{m-1}, \qquad \beta' = \zeta, \qquad \gamma' = \frac{\epsilon \times (\zeta \times \xi)}{m-1}.$$

From these the canonical form may be written down whether for the elliptic or hyperbolic type by following the models on pages 291–3. The fixed lines of the transformation which are the asymptotes of the ellipse or hyperbola are along the directions

$$\eta + [\sqrt{(m-1)m} \pm m]\,\epsilon.$$

We pass on to the case of the resolution of the collineation or strain with a fixed point and a unit modulus into three reflections. Let Ω represent the strain. If the modulus is $\Omega_3 = +1$, all three of the reflections are linear or two of them are planar and the third is linear; whereas if the modulus is $\Omega_3 = -1$, two are linear and one is planar or all three are planar. Suppose the case to be that of the first of these four. It is required to find a reflection $2\epsilon\zeta - I$ such that

$$[\Omega \cdot (2\epsilon\zeta - I)]_{2S} = [\Omega \cdot (2\epsilon\zeta - I)]_S.$$

By use of the relation $(\Omega \cdot \Phi)_2 = \Omega_2 \cdot \Phi_2$ this reduces to

$$2\epsilon \cdot \Omega_2 \cdot \zeta - \Omega_{2S} = 2\epsilon \cdot \Omega_C \cdot \zeta - \Omega_S.$$

But we have always $\Omega_3 I = \Omega_C \cdot \Omega_2$, and hence $\Omega_2 = \Omega_C^{-1}$. That is to say,

$$2\zeta \cdot (\Omega - \Omega^{-1}) \cdot \epsilon = \Omega_S - \Omega_S^{-1}, \qquad \Omega_S^{-1} = (\Omega^{-1})_S \neq (\Omega_S)^{-1},$$

is the equation which must be satisfied by ϵ and ζ.

The other three possible cases may be reduced to these. For in the second case Ω_3 is still equal to $+1$, but the first reflection applied may be planar.

Then the product $\Omega\cdot(I-2\epsilon\zeta)$ would have a negative modulus and the condition that it be resoluble into two reflections would be

$$[\Omega\cdot(I-2\epsilon\zeta)]_{2S}=-[\Omega\cdot(I-2\epsilon\zeta)]_S,$$

which is precisely the same as before. If on the other hand $\Omega_3=-1$, we may effect the resolution of $-\Omega$ as above. But the reciprocal of $-\Omega$ is $-\Omega^{-1}$. Thus the condition is unchanged. The form of the condition may be changed by noting that

$$\zeta\cdot\epsilon=\zeta\cdot I\cdot\epsilon=1 \qquad \text{and} \qquad \Omega_S-\Omega_S^{-1}=\zeta\cdot(\Omega_S-\Omega_S^{-1})I\cdot\epsilon$$

and hence

$$\zeta\cdot[(\Omega-\Omega^{-1})-\tfrac{1}{2}(\Omega_S-\Omega_S^{-1})I]\cdot\epsilon=0.$$

We may therefore state

THEOREM 23. *In order that ϵ and ζ may serve as line and plane, respectively, for the first of three reflections into which a strain Ω may be resolved, it is necessary and sufficient that they satisfy the equation*

$$\zeta\cdot\Psi\cdot\epsilon=0, \qquad \Psi=(\Omega-\Omega^{-1})-\tfrac{1}{2}(\Omega_S-\Omega_S^{-1})I.$$

First consider the cases in which Ω is itself resoluble into two reflections. Here

$$\Omega_{2S}=\Omega_3\Omega_{CS}^{-1}=\Omega_3\Omega_S^{-1}=\pm\Omega_S \qquad \text{and hence} \qquad \Omega_S-\Omega_S^{-1}=0,$$

and the condition reduces to

$$\zeta\cdot(\Omega-\Omega^{-1})\cdot\epsilon=0.$$

The dyadic $\Omega-\Omega^{-1}$ is then always incomplete; for

$$(\Omega-\Omega^{-1})_3=\Omega_3-\Omega_2:\Omega^{-1}+\Omega:\Omega_2^{-1}-\Omega_3^{-1}=\Omega:\Omega_2^{-1}-\Omega_2:\Omega^{-1}.$$

But

$$\Omega:\Omega_2^{-1}=(\Omega\cdot\Omega_{2C}^{-1})_S=(\Omega^2)_S \qquad \text{and} \qquad \Omega_2:\Omega^{-1}=(\Omega_2^2)_S.$$

Hence the question resolves itself into the question whether the square of a transformation resoluble into two reflections is itself so resoluble. The answer is affirmative, as may be seen by investigating the squares of the various types that have been obtained: but though this method leads to a number of interesting relations, it is too long to take up here. The general expression for any dyadic may be written* as

$$\Omega=a\alpha\alpha'+g\alpha\beta'+f\alpha\gamma'+b\beta\beta'+e\beta\gamma'+c\gamma\gamma'$$

in a triply infinite system of ways. The coefficients of the diagonal terms are

* This general form may be recognized at once by checking it against the various types we have noted as possible in the classification of strains. It is, however, merely the special case in three dimensions of a general form to which a collineation may be reduced. This form is not so

the roots of the scalar cubic. The square of this dyadic is, as far as concerns the terms in the diagonal,

$$\Omega^2 = a^2\alpha\alpha' + b^2\beta\beta' + c^2\gamma\gamma' + 3 \text{ terms.}$$

Now if one of the quantities a, b, c is numerically unity and the other two are reciprocals, the same is true of a^2, b^2, c^2. Hence

THEOREM 24. *If Ω is resoluble into the product of two reflections the dyadic $\Omega - \Omega^{-1}$ is incomplete, and the condition on ϵ and ζ becomes*

$$\zeta \cdot \Phi \cdot \epsilon = 0, \qquad \Phi = \Omega - \Omega^{-1}, \qquad \Phi_S = 0,$$

and conversely.

It is possible to write $\Omega - \Omega^{-1}$ as $\Omega^{-1} \cdot (\Omega^2 - I)$, and all the degrees of incompleteness must occur in the factor $\Omega^2 - I$. If this vanishes, Ω is itself a reflection; if it is linear, all three roots of the cubic are numerically unity and the transformation is one of the simple shears, special or non-special; if it is merely planar, the transformation is a complex shear or an elliptic or hyperbolic rotation. In the first of these three cases ϵ may be chosen at pleasure and so may ζ. The condition is always satisfied. In the second case, if ϵ is taken perpendicular to the consequent of Φ, ζ may be chosen at pleasure, whereas if ϵ is not so chosen, ζ must be selected from the vectors perpendicular to the antecedent of Φ and not perpendicular to ϵ. It would be impossible to make this choice if ϵ had been chosen parallel to the antecedent of Φ, that is, if $\Phi \cdot \epsilon = x\epsilon$ and $x \neq 0$. But in this case, as Φ is linear, $\Phi \cdot \epsilon = \epsilon\Phi_S = 0$. Hence any choice of ϵ is admissible. In the last case any choice for ζ (not perpendicular to ϵ) is permissible if ϵ be taken perpendicular to the plane of the consequents of Φ. But if ϵ be not so taken, then ζ is restricted to plane perpendicular to $(\Omega - \Omega^{-1}) \cdot \epsilon$. There may be two independent values for ϵ, for which $(\Omega - \Omega^{-1}) \cdot \epsilon$ becomes parallel to ϵ without vanishing, and the indicated resolution then becomes impossible. A brief examination of this case shows that the

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widely used as it would be if authors were less willing to assume the "general case" of a complete system of fixed points when discussing collineations. (It may be noted in passing that the group of translations does not contain a single transformation which can be said to belong to the "general case.") The general form to which any collineation may be reduced is

$$\begin{array}{ll}
\rho x_1' = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & \cdots\cdots\cdots\cdots \\
\rho x_2' = \qquad a_{22}x_2 + \cdots + a_{2n}x_n & \rho x_{n-1}' = a_{n-1\,n-1}x_{n-1} + a_{n-1\,n}x_n \\
\cdots\cdots\cdots\cdots & \rho x_n' = \qquad a_{nn}x_n
\end{array}$$

where only the terms on one side of the diagonal occur. The proof of the reduction is easy. The collineation has one fixed linear space S_{n-1}. Take this as the space $x_n = 0$ of reference. In this space there must be one fixed subspace S_{n-2}. Choose the second space $x_{n-1} = 0$ of reference to pass through this space S_{n-2}. In the space S_{n-2} there is likewise a subspace S_{n-3} which is fixed. Pass the third space $x_{n-2} = 0$ of reference through this S_{n-3}. And so on. It is not hard to show that with this choice of spaces of reference the collineation reduces to the type here given.

only directions ϵ for which $(\Omega - \Omega^{-1})\cdot\epsilon$ is parallel to ϵ without vanishing are the fixed directions of the rotations. These results and some others which it is not difficult to verify may be stated in the following theorems:

THEOREM 25. *An elliptic or hyperbolic rotation may be resolved into three reflections by taking the line of the first reflection at random (with the exception of the two fixed lines in the plane of the rotation) and taking the plane through a specified line; or by assuming the plane of the first reflection at random and choosing the line in a specified plane. There is one exceptional choice of the line and one of the plane such that the choice of the other remains arbitrary.*

THEOREM 26. *A complex shear may be resolved into three reflections by taking the line of the first reflection at random and taking the plane through a specified line, etc., as in theorem* 25.

THEOREM 27. *A simple shear of the types considered above, whether special or non-special, may be resolved into the product of three reflections by taking the line of the first reflection at random and taking the plane through a specified line; or by assuming the plane of the first reflection at random and choosing the line in a specified plane. If, exceptionally, the line be chosen in a certain plane or the plane through a certain line, the subsequent choice of the other remains arbitrary.*

In the discussion of the case where Ω is not resoluble into two reflections, the behavior of the dyadic

$$\Psi = (\Omega - \Omega^{-1}) - \tfrac{1}{2}(\Omega_S - \Omega^{-1}_S)I$$

is of fundamental importance. If Ω be expressed in the reduced form

$$\Omega = a\alpha\alpha' + b\beta\beta' \pm \frac{1}{ab}\gamma\gamma' + 3 \text{ terms}$$

the dyadic Ψ takes the form

$$-\Psi = \left[\tfrac{1}{2}\left(a + b \pm \frac{1}{ab} - \frac{1}{a} - \frac{1}{b} \mp ab\right) - \left(a - \frac{1}{a}\right)\right]\alpha\alpha'$$
$$+ [\quad]\beta\beta' + [\quad]\gamma\gamma' + 3 \text{ terms},$$

and will be complete unless one of the coefficients of the three diagonal terms vanishes. Suppose the first one to vanish. Then

$$-\left(a - \frac{1}{a}\right) + \left(b - \frac{1}{b}\right) \pm \left(\frac{1}{ab} - ab\right) = 0$$

or

$$-a^2b + b + ab^2 - a \pm 1 \mp a^2b^2 = 0.$$

This expression is always factorable into three factors chosen from among the six $(b \pm 1)(a \pm 1)(ab \pm 1) = 0$. These are just the conditions that one of the roots be numerically unity, that is, that the strain Ω be resoluble into two reflections. Hence

THEOREM 28. *The necessary and sufficient condition that Ω be resoluble into two reflections is that*

$$\Psi = (\Omega - \Omega^{-1}) - \tfrac{1}{2}(\Omega_S - \Omega_S^{-1})I$$

be incomplete.

In case Ω is not resoluble into two reflections, it is (by virtue of this theorem) impossible to find any direction for the line ϵ which shall allow the plane ζ to be arbitrary. The question arises whether there may not be directions for ϵ which shall make it impossible to find any ζ whatsoever. The case of impossibility can only arise when $\Psi \cdot \epsilon$ is parallel to ϵ. That is, in case ϵ is one of the fixed directions of Ψ. But as every direction is fixed in the transformation kI, it is sufficient to state that ϵ is a fixed direction for $\Omega - \Omega^{-1}$, and a reference to the expression of this dyadic in reduced form will show that this is equivalent to saying that ϵ is a fixed direction of Ω. Now there are only two types of strain which we have to consider and which are not resoluble into two reflections. They are where the roots are a, a, $\pm 1/a^2$, which is a simple shear or special tonic, and where the roots are a, b, c and none of them numerically unity — the tonic or cyclotonic. In the case of the tonic or cyclotonic there are just three directions of impossibility (and for the cyclotonic two of these are imaginary), for the simple shear there are two such directions of which one is the fixed direction of the shear and the other the direction parallel to which the shearing takes place; for the special tonic there is a whole plane of such directions, the plane of the equal roots a, a.

THEOREM 29. *Any unimodular strain not resoluble into two reflections is resoluble into three reflections of which the first has an arbitrary line (provided it be not collinear with any of the fixed directions of the strain) and a plane which must pass through a definite line but is otherwise unlimited except that it should not pass through the line of the reflection.*

It may be observed in general as a result of the foregoing theorems 25–29 that to resolve a strain into three reflections we have

THEOREM 30. *In all cases the line of the first reflection may be chosen at random (except along such fixed directions of the strain as correspond to stretching) and the plane may be any plane which passes through a specified line but does not contain the line of the reflection.*

The proportions which this article has attained and this natural close make it unwise to continue here and now with the more detailed relations of oblique reflections to the unimodular strains.

YALE UNIVERSITY, NEW HAVEN, CONN.,
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ON THE INTRODUCTION OF CONVERGENCE FACTORS INTO SUMMABLE SERIES AND SUMMABLE INTEGRALS*

BY

CHARLES N. MOORE

The object of this paper is to develop certain general theorems about convergence factors. In the case of series we shall mean by convergence factors a set of functions of a parameter which, when introduced as factors of the successive terms of the series, cause a divergent series to converge, or a series which is already convergent to converge more rapidly, throughout a certain range of values of the parameter. In the case of integrals we shall mean by a convergence factor a function of the variable of integration and a new parameter which, when introduced as a factor of the integrand, causes a divergent integral to converge, or an integral which is already convergent to converge more rapidly, throughout a certain range of values of the parameter.

Although the name convergence factor is of recent origin, the subject itself, in the simple case of a convergent series, goes back to ABEL and virtually takes its rise in his well known theorem on the continuity of a power series. The successive powers of x in the terms of a power series are, in fact, convergence factors of a simple nature, though not ordinarily regarded as such.

The first attempt to extend the theory of convergence factors to general types of divergent series was made by FROBENIUS.† The class of series which he considered will be designated in this paper as summable series‡ and may be defined as follows: Given a series

$$u_0 + u_1 + u_2 + \cdots,$$

we represent by s_n the sum of its first $(n+1)$ terms and form

$$S_n = \frac{s_0 + s_1 + s_2 + \cdots + s_n}{n+1}.$$

If the limit

$$\lim_{n=\infty} S_n$$

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† Crelle's Journal, vol. 89 (1880), p. 262.

‡ Many authors use this term in a more general sense.

exists, we say that the series is summable and attach to it the value of that limit. It is a well known and easily established theorem that every convergent series is summable, and that the process of summation leads to the value of the series as ordinarily defined.

The theorem proved by FROBENIUS in the article above mentioned includes only the simple class of convergence factors first considered by ABEL, i. e., successive powers of the parameter. A more general theorem was proved by FEJÉR,* in which the convergence factors have the form $\phi(0), \phi(t), \phi(2t), \cdots$. A similar theorem concerning still more general convergence factors was stated without proof by HARDY at the close of a recent paper in the Proceedings of the London Mathematical Society.† The present investigation had been begun and the essential results of §§ 1 and 2 obtained and communicated to Professor BÔCHER before the appearance of HARDY's paper.

The principal results of the present paper are contained in Theorems I, II, IV, and V. The method of proof in these four theorems is a development of that used by RIEMANN in proving a certain lemma (cf. RIEMANN, *Mathematische Werke*, 2d ed., p. 246). This lemma appears as a special case of Theorem II if we take $f_n(\alpha) = \sin^2 n\alpha / n^2\alpha^2$.

§ 1. *Convergence factors in summable series.*

THEOREM I. *If the series*

$$u_0 + u_1 + u_2 + \cdots$$

is summable and has the value S, then the series

$$F(\alpha) = u_0 + u_1 f_1(\alpha) + u_2 f_2(\alpha) + \cdots \tag{1}$$

will be absolutely convergent and continuous for all positive values of α, and will approach S as its limit when $\alpha = +0$, provided the convergence factors $f_1(\alpha), f_2(\alpha), \cdots$ satisfy the following conditions:

$$(a) \qquad f_n(\alpha) \text{ is continuous} \qquad (\alpha \geqq 0),$$

$$(b) \qquad |f_n(\alpha)| < \frac{N}{n^{2+\rho}\alpha^{2+\rho}} \qquad (\alpha > 0),$$

$$(c) \qquad f_n(0) = 1,$$

* Mathematische Annalen, vol. 58 (1904), p. 62.

† Ser. 2, vol. 4 (1906), part 4; p. 247. HARDY'S results are somewhat similar to Theorems I and II below, but the conditions imposed on the convergence factors are decidedly different, and his theorems are on the whole less far reaching. His requirement, in the case of summable series, that the differences between two successive convergence factors be positive, will, I think, be found unnecessary if a method of proof analogous to the proof of Theorem I of this article be followed.

$$(d) \qquad f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha) \geqq 0^* \qquad (0 \leqq \alpha \leqq c;\ 0 \leqq n\alpha \leqq c;\ n = 0, 1, \cdots),\dagger$$

$$(e) \qquad |f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha)| < \frac{L}{n^{2+\rho}\alpha^\rho} \qquad (\alpha > 0),$$

where N, ρ, c, and L are positive constants.

We must first derive the following further condition which is satisfied by the convergence factors,

$$(f) \qquad |f_n(\alpha) - f_{n+1}(\alpha)| < \frac{L_1}{n^{1+\rho}\alpha^\rho} \qquad (\alpha > 0),$$

where L_1 is a positive constant.

We have from condition (b)

$$\lim_{n=\infty} [f_n(\alpha) - f_{n+1}(\alpha)] = 0 \qquad (\alpha > 0),$$

and consequently

$$f_n(\alpha) - f_{n+1}(\alpha) = \sum_{n=n}^{\infty} \{ [f_n(\alpha) - f_{n+1}(\alpha)] - [f_{n+1}(\alpha) - f_{n+2}(\alpha)] \}$$
$$= \sum_{n=n}^{\infty} \{ f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha) \}.$$

Hence by virtue of condition (e)

$$(2) \qquad \begin{aligned} |f_n(\alpha) - f_{n+1}(\alpha)| &< \sum_{n=n}^{\infty} \frac{L}{n^{2+\rho}\alpha^\rho} < \int_{n-1}^{\infty} \frac{L}{x^{2+\rho}\alpha^\rho}\,dx = \frac{L}{(1+\rho)(n-1)^{1+\rho}\alpha^\rho} \\ &= \frac{\dfrac{L}{1+\rho}}{n^{1+\rho}\alpha^\rho}\left(\frac{n}{n-1}\right)^{1+\rho} < \frac{2^{1+\rho}\dfrac{L}{1+\rho}}{n^{1+\rho}\alpha^\rho} \qquad (\alpha > 0;\ n \geqq 2). \end{aligned}$$

For $n = 1$ we have from condition (b)

$$(3) \qquad |f_1(\alpha) - f_2(\alpha)| < \frac{2N}{\alpha^{2+\rho}} \leqq \frac{2N}{\alpha^{1+\rho}} \qquad (\alpha \geqq 1),$$

and from condition (a)

$$(4) \qquad |f_1(\alpha) - f_2(\alpha)| < \frac{M}{\alpha^{1+\rho}} \qquad (0 < \alpha < 1),$$

where M is the greatest numerical value of $[f_1(\alpha) - f_2(\alpha)]$ in the interval $0 \leqq \alpha \leqq 1$. Choosing as L_1 the greatest of the thre equantities M, $2N$, and $2^{1+\rho}L/(1+\rho)$ we get (f) by combining (2), (3), and (4).

We shall next prove that the series (1) is absolutely convergent for every positive value of α. Let

$$s_n = \sum_{n=0}^{n} u_n,$$

---

* We may substitute for the inequality in (d) the condition

$$(d') \qquad f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha) \leqq 0.$$

If (d) (or (d')) holds for all values of n and α, (e) is unnecessary.

† For $n = 0$ we take here $f_0(\alpha) = 1$.

$$S_n = \frac{s_0 + s_1 + \cdots + s_n}{n+1}.$$

If for the sake of uniformity we let $S_{-2} = S_{-1} = 0$, we have

$$u_n = (n+1)S_n - 2nS_{n-1} + (n-1)S_{n-2} \qquad (n=0, 1, 2, \cdots), \tag{5}$$

and consequently

$$\frac{u_n}{n} = \left(1 + \frac{1}{n}\right) S_n - 2S_{n-1} + \left(1 - \frac{1}{n}\right) S_{n-2} \qquad (n=1, 2, \cdots),$$

whence it follows that

$$\lim_{n=\infty} \left(\frac{u_n}{n}\right) = 0.$$

We can, therefore, take k so that

$$\left|\frac{u_n}{n}\right| < k \qquad (n \geqq 1).$$

Combining this with condition (b) we see that the terms of (1) are, if we omit the first, less in absolute value than the corresponding terms of the series

$$\sum_{n=1}^{\infty} \frac{Nk}{\alpha^{2+\rho} n^{1+\rho}}.$$

Since this last series is convergent, the absolute convergence of (1) and the existence of $F(\alpha)$ for values of $\alpha > 0$ follow at once.

We may by virtue of (5) write

$$F(\alpha) = \sum_{n=0}^{\infty} \{(n+1)S_n - 2nS_{n-1} + (n-1)S_{n-2}\} f_n(\alpha). \tag{6}$$

In the series (6) we have a right to leave out the parentheses, since it is convergent, since each parenthesis contains only three terms, and since by condition (b) the general term of the resulting series has the limit zero. Moreover, in this latter series we have a right to rearrange the terms so that terms involving the same S are grouped together, since in the rearrangement we do not carry any term over more than three terms. Hence

$$F(\alpha) = \sum_{n=0}^{\infty} (n+1) S_n \{f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha)\}. \tag{7}$$

Placing

$$S_n = S + \epsilon_n \tag{8}$$

we have

$$\begin{aligned} F(\alpha) = S \sum_{n=0}^{\infty} (n+1) \{f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha)\} \\ + \sum_{n=0}^{\infty} \epsilon_n (n+1) \{f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha)\}. \end{aligned} \tag{9}$$

But

$$\sum_{n=0}^{n}(n+1)\{f_n(\alpha)-2f_{n+1}(\alpha)+f_{n+2}(\alpha)\}=1-(n+2)f_{n+1}(\alpha)+(n+1)f_{n+2}(\alpha).$$

Whence, by condition (b),

$$\sum_{n=0}^{\infty}(n+1)\{f_n(\alpha)-2f_{n+1}(\alpha)+f_{n+2}(\alpha)\}=1 \qquad (\alpha>0),$$

and (9) reduces to

$$(10) \qquad F(\alpha)-S=\sum_{n=0}^{\infty}\epsilon_n(n+1)\{f_n(\alpha)-2f_{n+1}(\alpha)+f_{n+2}(\alpha)\}.$$

By (c) the series on the right hand side of (10) converges when $\alpha=0$ and has the value 0. Accordingly, if we can show that it is uniformly convergent for values of $\alpha \geqq 0$ our theorem will be proved, since by condition (a) each term is a continuous function of α.

Given a positive quantity δ, arbitrarily small, we wish to show that we can choose q so large that

$$(11) \qquad \left|\sum_{n=\mu}^{n=\nu}\epsilon_n(n+1)\{f_n(\alpha)-2f_{n+1}(\alpha)+f_{n+2}(\alpha)\}\right|<\delta \qquad (\nu\geqq\mu\geqq q,\ \alpha\geqq 0)$$

Take $m>2$ and such that

$$(12) \qquad |\epsilon_n|<\eta=\frac{\rho c^{2+\rho}\delta}{\rho c^{2+\rho}+\rho N+\rho c^2 L_1+2^{1+\rho}c^2 L} \qquad (n\geqq m).$$

We first consider values of α in the interval $0<\alpha<c/2$, and represent by s the greatest integer less than c/α. Let μ and ν be any two integers such that

$$m\leqq\mu\leqq\nu.$$

Three cases must be considered:

(A) $\qquad \mu\leqq s<\nu,$

(B) $\qquad \nu\leqq s,$

(C) $\qquad \mu>s.$

Beginning with case (A) we write

$$(13) \qquad \sum_{n=\mu}^{n=\nu}\epsilon_n(n+1)\{f_n(\alpha)-2f_{n+1}(\alpha)+f_{n+2}(\alpha)\}=\sum_{\mu}^{s}+\sum_{s+1}^{\nu}=R_1+R_2.$$

By virtue of (d)* and (12) we have for R_1

$$(14) \qquad |R_1|<\eta\sum_{0}^{s}(n+1)\{f_n(\alpha)-2f_{n+1}(\alpha)+f_{n+2}(\alpha)\}$$
$$=\eta\{1-f_{s+1}(\alpha)-(s+1)[f_{s+1}(\alpha)-f_{s+2}(\alpha)]\}.$$

* The substitution of (d') for (d) occasions only a slight modification of the proof.

From condition (b) we get

$$(15)\qquad |f_{s+1}(\alpha)| < \frac{N}{[(s+1)\alpha]^{2+\rho}} \leqq \frac{N}{c^{2+\rho}},$$

and from condition (f)

$$(16)\qquad (s+1)|f_{s+1}(\alpha) - f_{s+2}(\alpha)| < \frac{L_1}{[(s+1)\alpha]^{\rho}} \leqq \frac{L_1}{c^{\rho}}.$$

By combining (14), (15), and (16) we obtain

$$(17)\qquad |R_1| < \eta\left\{1 + \frac{N}{c^{2+\rho}} + \frac{L_1}{c^{\rho}}\right\}.$$

According to condition (e) we have for R_2

$$|R_2| < \eta \sum_{s+1}^{\nu}(n+1)|f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha)| < \frac{\eta L}{\alpha^{\rho}} \sum_{s+1}^{\nu} \frac{(n+1)}{n^{2+\rho}}.$$

Furthermore

$$\sum_{s+1}^{\nu} \frac{(n+1)}{n^{2+\rho}} < \int_s^{\nu} \frac{x+1}{x^{2+\rho}}\,dx = \left[-\frac{1}{\rho x^{\rho}} - \frac{1}{(1+\rho)x^{1+\rho}}\right]_s^{\nu} < \left[\frac{1}{\rho s^{\rho}} + \frac{1}{(1+\rho)s^{1+\rho}}\right] < \frac{2}{\rho s^{\rho}}.$$

Hence

$$|R_2| < \eta \frac{2L}{\rho(s\alpha)^{\rho}}.$$

But

$$s \geqq \frac{c}{\alpha} - 1$$

or

$$s\alpha \geqq c - \alpha > c - \frac{c}{2} = \frac{c}{2},$$

so that we have finally for R_2,

$$(18)\qquad |R_2| < \eta \frac{2^{1+\rho} L}{\rho c^{\rho}}.$$

Combining (13), (17), and (18) we get

$$\left|\sum_{n=\mu}^{n=\nu} \epsilon_n (n+1)\{f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha)\}\right|$$
$$< \eta \frac{\rho c^{2+\rho} + \rho N + \rho c^2 L_1 + 2^{1+\rho} c^2 L}{\rho c^{2+\rho}} = \delta.$$

We consider next case (B). We have

$$\left|\sum_{n=\mu}^{n=\nu} \epsilon_n (n+1)\{f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha)\}\right|$$
$$\leqq \eta \sum_{0}^{s} (n+1)\{f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha)\}$$

$$= \eta\{1 - f_{s+1}(\alpha) - (s+1)[f_{s+1}(\alpha) - f_{s+2}(\alpha)]\}$$
$$\leqq \eta\{1 + |f_{s+1}(\alpha)| + (s+1)|f_{s+1}(\alpha) - f_{s+2}(\alpha)|\}$$
$$< \eta\left\{1 + \frac{N}{c^{2+\rho}} + \frac{L_1}{c^\rho}\right\} = \eta\frac{\rho c^{2+\rho} + \rho N + \rho c^2 L_1}{\rho c^{2+\rho}} < \delta.$$

For case (C) we have

$$\left|\sum_{n=\mu}^{n=\nu} \epsilon_n(n+1)\{f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha)\}\right|$$
$$\leqq \eta\sum_{s+1}^{\nu}(n+1)|f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha)|$$
$$< \frac{\eta L}{\alpha^\rho}\sum_{s+1}^{\nu}\frac{n+1}{n^{2+\rho}} < \eta\frac{2^{1+\rho}L}{\rho c^\rho} = \eta\frac{2^{1+\rho}c^2 L}{\rho c^{2+\rho}} < \delta.$$

Thus we have proved (11) in all three cases for $0 < \alpha < c/2$, the value of q being equal to m.

For values of $\alpha \geqq c/2$ we have from condition (e)

$$\left|\sum_{n=\mu}^{n=\nu} \epsilon_n(n+1)\{f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha)\}\right|$$
$$\leqq \sum_{\mu}^{\nu}|\epsilon_n|\frac{\left(\frac{2}{c}\right)^\rho(n+1)L}{n^{2+\rho}} \leqq \left(\frac{2}{c}\right)^\rho LK\sum_{\mu}^{\nu}\frac{(n+1)}{n^{2+\rho}}$$

where K is the greatest numerical value of the quantities $\epsilon_0, \epsilon_1, \epsilon_2, \cdots$. Since the series

$$\sum_{n=1}^{\infty}\frac{n+1}{n^{2+\rho}}$$

is convergent, we can find m_1 such that

$$\left|\sum_{n=\mu}^{n=\nu} \epsilon_n(n+1)\{f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha)\}\right| < \delta \qquad \left(\begin{smallmatrix}\nu \geqq \mu \geqq m_1\\ \alpha \geqq c/2\end{smallmatrix}\right).$$

For $\alpha = 0$ the inequality (11) is obviously satisfied for all values of q since each $f_i(a) = 1$. If, therefore, we choose for q the greater of the two quantities m and m_1, it will hold for all values of $\alpha \geqq 0$, as was to be proved.

Convergence factors of the form $\phi(n\alpha)$ are an important class, and it is interesting to note what restrictions may be placed on $\phi(x)$ in order that the conditions of Theorem I may be satisfied. We will show that if

(i) $$|\phi(x)| < \frac{N}{x^{2+\rho}} \qquad (x > 0),$$

(ii) $$\phi(0) = 1,$$

(iii) $$\phi''(x) \text{ exists and } \phi''(x) \geqq 0\,^* \qquad (0 \leqq x \leqq c),$$

(iv) $$|\phi''(x)| < \frac{L}{x^{2+\rho}} \qquad (x > 0),$$

then the functions $\phi(0) = 1$, $\phi(\alpha)$, $\phi(2\alpha)$, $\cdots$ satisfy the conditions of Theorem I.

Condition (a) follows from the fact that the second derivative of $\phi(x)$ exists for all values of $x \geqq 0$. Conditions (b) and (c) follow at once from (i) and (ii).

We have (cf. MARKOFF, *Differenzenrechnung*, § 8)

$$(19) \quad \phi(n\alpha) - 2\phi[(n+1)\alpha] + \phi[(n+2)\alpha] = \alpha^2 \phi''[(n+\theta)\alpha] \quad (0 < \theta < 2).$$

Hence by virtue of (iii) condition (d) will be fulfilled, if c is there replaced by $c/3$.

Finally, by (19) and (iv)

$$|\phi(n\alpha) - 2\phi[(n+1)\alpha] + \phi[(n+2)\alpha]| < \frac{L}{n^{2+\rho}\alpha^{\rho}} \qquad (\alpha > 0;\ n = 1, 2, \cdots),$$

and (e) is satisfied.

The theorem of FEJÉR mentioned in the introduction, in which the convergence factors are functions of $n\alpha$, is not, in its greatest generality, a special case of Theorem I, but becomes so if we exclude from it all functions whose first derivatives have an infinite number of maxima and minima in the neighborhood of the origin. I shall show this by proving that if we have a function $\phi(x)$ which satisfies the conditions of FEJÉR's theorem and whose first derivative has not an infinite number of maxima and minima in the neighborhood of the origin, then it will satisfy (i)–(iv).

FEJÉR's conditions are

(a) $$|\phi(x)| < \frac{M}{x^{2+\rho}} \qquad (x > 1),$$

(b) $$|\phi''(x)| < \frac{M}{x^{2+\rho}} \qquad (x > 1),$$

(c) $$\phi(0) = 1,$$

(d) $$|\phi''(x)| < \mu\,\dagger \qquad (0 \leqq x \leqq 1),$$

where M, ρ and μ are positive constants.

From (d) it follows that $\phi(x)$ is continuous in the interval $0 \leqq x \leqq 1$. We can therefore choose a positive constant N greater than M and greater than the

* We may substitute for (iii)

(iii′) $$\phi''(x) \leqq 0 \qquad (0 \leqq x \leqq c).$$

† (d) is not explicitly stated as a condition, but is used in the course of the proof.

greatest numerical value of $\phi(x)$ in the interval $0 \leqq x \leqq 1$, so that we have

$$|\phi(x)| < \frac{N}{x^{2+\rho}} \qquad (x > 0),$$

and consequently (i) is satisfied. Condition (ii) is identical with (c). Since $\phi'(x)$ has not an infinite number of maxima and minima in the neighborhood of the origin, either (iii) or (iii′) must be satisfied. By taking for L the greater of the two quantities M and μ we have in view of (b) and (d)

$$|\phi''(x)| < \frac{L}{x^{2+\rho}} \qquad (x > 0),$$

and (iv) is satisfied.

Thus we see that Theorem I, when applied to functions of $n\alpha$, includes all functions satisfying FEJÉR's conditions that are likely to be useful as convergence factors. In addition to this it includes an important class of functions that are not included in FEJÉR's theorem, i. e., functions that satisfy (i)–(iv) but whose second derivative becomes infinite in the neighborhood of the origin. An example of such a function is $e^{-\sqrt{x}}$.

A general theorem which includes FEJÉR's theorem as a special case but does not include functions of the type just mentioned is obtained by replacing condition (d) of Theorem I by the condition

$$|f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha)| < K\alpha^2 \qquad (0 \leqq n\alpha \leqq c).$$

The proof is quite similar to the proof of Theorem I.

§ 2. *Convergence factors in convergent series.*

Theorem I is evidently applicable to convergent series since every convergent series is summable. When dealing with convergent series, however, we do not need to place as much restriction upon the convergence factors as in the general case of a summable series. This is shown by the following theorem which in many respects is more general than Theorem I:

THEOREM II. *If the series*

$$u_0 + u_1 + u_2 + \cdots \tag{20}$$

converges to the value A, then the series

$$u_0 + u_1 f_1(\alpha) + u_2 f_2(\alpha) + \cdots \tag{21}$$

will be absolutely and uniformly convergent for all values of $\alpha \geqq 0$ and therefore will define a continuous function, provided the convergence factors *

* That the series (21) converges more rapidly than (20) appears from condition (b).

$f_1(\alpha), f_2(\alpha), \cdots$ *satisfy the following conditions:*

$$(a) \qquad f_n(\alpha) \text{ is continuous} \qquad (\alpha \geqq 0),$$

$$(b) \qquad |f_n(\alpha)| < \frac{N}{n^{1+\rho}\alpha^{1+\rho}} \qquad (\alpha > 0),$$

$$(c) \qquad f_n(0) = 1,$$

$$(d) \qquad f_n(\alpha) - f_{n+1}(\alpha) \geqq 0\,^* \qquad (n \geqq 1,\ 0 \leqq n\alpha \leqq c),$$

$$(e) \qquad |f_n(\alpha) - f_{n+1}(\alpha)| < \frac{L}{n^{1+\rho}\alpha^{\rho}} \qquad (\alpha > 0),$$

where N, ρ, c and L are positive constants.

We must first derive from the given conditions the following further condition:

$$(f) \qquad |f_n(\alpha)| < K \qquad (\alpha \geqq 0)$$

where K is a positive constant.

We know from condition (d) † that, for a given value of α less than c, the functions $f_n(\alpha)$ form a sequence that never increases as long as $n\alpha$ does not exceed c. For each value of $\alpha < c$ we pick out the greatest integer $r \leqq c/\alpha$. This r satisfies the inequality

$$(22) \qquad \frac{c}{2} < r\alpha \leqq c.$$

We have furthermore

$$(23) \qquad f_1(\alpha) \geqq f_n(\alpha) \geqq f_r(\alpha) \qquad (0 < n\alpha < c).$$

But by condition (a) we can find a positive constant M such that

$$|f_1(\alpha)| < M \qquad (0 < \alpha < c),$$

and from (b) and (22)

$$|f_r(\alpha)| < \frac{N}{(r\alpha)^{1+\rho}} < \frac{2^{1+\rho}N}{c^{1+\rho}},$$

so that if we take as K_1 the greater of the two quantities M and $2^{1+\rho}N/c^{1+\rho}$, we have by (23)

$$|f_n(\alpha)| < K_1 \qquad (0 < n\alpha < c).$$

For $n\alpha \geqq c$ we have from condition (b)

$$|f_n(\alpha)| < \frac{N}{(n\alpha)^{1+\rho}} \leqq \frac{N}{c^{1+\rho}} < K_1 \qquad (n\alpha \geqq c).$$

---

* This is the only condition that is not satisfied by the convergence factors of Theorem I. We may substitute for it the condition

$$(d') \qquad f_n(\alpha) - f_{n+1}(\alpha) \leqq 0 \qquad (0 \leqq n\alpha \leqq c).$$

If (d) (or (d')) holds for all values of n and α, (e) is unnecessary.

† If we use (d') instead of (d), the inequality signs in (22) are reversed, but this does not affect the reasoning.

For $\alpha = 0$ we have from condition (c)

$$f_n(0) = 1.$$

Hence if we choose K greater than either of the two quantities K_1 and 1, we have the desired condition (f). Place now

$$u_0 + u_1 + u_2 + \cdots + u_{n-1} = A + \epsilon_n$$

so that

$$\lim_{n=\infty} \epsilon_n = 0,$$

and let δ be an arbitrarily small, positive quantity. Since by condition (a) each term of (21) is a continuous function of α, our theorem will be proved if we can show that a positive integer q exists for which

$$\left| \sum_{n=\mu}^{n=\nu} u_n f_n(\alpha) \right| < \delta \qquad (\nu \geqq \mu \geqq q\,;\ \alpha \geqq 0). \tag{24}$$

Determine a value of $m > 2$ such that

$$|\epsilon_n| < \eta = \frac{\rho c^\rho \delta}{4\rho c^\rho K + 2^\rho L} \qquad (n \geqq m). \tag{25}$$

We first consider values of α in the interval $0 < \alpha < c/2$. Then

$$\begin{aligned} \sum_{n=\mu}^{n=\nu} u_n f_n(\alpha) &= \sum_{\mu}^{\nu} (\epsilon_{n+1} - \epsilon_n) f_n(\alpha) \\ &= -\epsilon_\mu f_\mu(\alpha) + \sum_{\mu}^{\nu} \epsilon_{n+1} [f_n(\alpha) - f_{n+1}(\alpha)] + \epsilon_{\nu+1} f_{\nu+1}(\alpha). \end{aligned} \tag{26}$$

Let s be the greatest integer less than c/α, and μ and ν any two integers for which

$$m \leqq \mu \leqq \nu.$$

Three cases must be considered:

$$(A) \qquad \mu \leqq s < \nu,$$

$$(B) \qquad \nu \leqq s,$$

$$(C) \qquad \mu > s.$$

Beginning with (A) we put

$$\sum_{n=\mu}^{n=\nu} \epsilon_{n+1} [f_n(\alpha) - f_{n+1}(\alpha)] = \sum_{\mu}^{s} + \sum_{s}^{\nu} = R_1 + R_2. \tag{27}$$

By virtue of (d),* (f), and (25) we have for R_1

$$(28)\qquad \begin{aligned} |R_1| &\leqq \sum_{\mu}^{s} |\epsilon_{n+1}| [f_n(\alpha) - f_{n+1}(\alpha)] \\ &< \eta \sum_{\mu}^{s} [f_n(\alpha) - f_{n+1}(\alpha)] = \eta [f_\mu(\alpha) - f_{s+1}(\alpha)] < 2K\eta. \end{aligned}$$

For R_2 we have by condition (e)

$$\begin{aligned} |R_2| &\leqq \sum_{s+1}^{\nu} |\epsilon_{n+1} [f_n(\alpha) - f_{n+1}(\alpha)]| \\ &< \frac{\eta L}{\alpha^\rho} \sum_{s+1}^{\nu} \frac{1}{n^{1+\rho}} < \frac{\eta L}{\alpha^\rho} \int_s^\nu \frac{dx}{x^{1+\rho}} = \frac{\eta L}{\alpha^\rho} \left[\frac{-1}{\rho x^\rho} \right]_s^\nu < \frac{\eta L}{\rho s^\rho \alpha^\rho}. \end{aligned}$$

But

$$s\alpha \geqq \left(\frac{c}{\alpha} - 1 \right) \alpha = c - \alpha > c - \frac{c}{2} = \frac{c}{2},$$

and hence

$$(29)\qquad |R_2| < \frac{\eta 2^\rho L}{\rho c^\rho}.$$

Combining (26), (27), (28), and (29) we get

$$\begin{aligned} \left| \sum_{\mu}^{\nu} u_n f_n(\alpha) \right| &\leqq |\epsilon_\mu f_\mu(\alpha)| + |R_1| + |R_2| + |\epsilon_{\nu+1} f_{\nu+1}(\alpha)| \\ &< K\eta + 2K\eta + \frac{\eta 2^\rho L}{\rho c^\rho} + K\eta = \eta \left[\frac{4\rho c^\rho K + 2^\rho L}{\rho c^\rho} \right] = \delta. \end{aligned}$$

We consider next case (B). We have

$$\begin{aligned} &\left| \sum_{\mu}^{\nu} \epsilon_{n+1} [f_n(\alpha) - f_{n+1}(\alpha)] \right| \\ &\qquad < \eta \sum_{\mu}^{\nu} [f_n(\alpha) - f_{n+1}(\alpha)] = \eta [f_\mu(\alpha) - f_{\nu+1}(\alpha)] < 2K\eta. \end{aligned}$$

Combining this result with (26) we get

$$\left| \sum_{\mu}^{\nu} u_n f_n(\alpha) \right| < |\epsilon_\mu f_\mu(\alpha)| + 2K\eta + |\epsilon_{\nu+1} f_{\nu+1}(\alpha)| < 4K\eta < \delta.$$

For case (C) we have

$$\left| \sum_{\mu}^{\nu} \epsilon_{n+1} [f_n(\alpha) - f_{n+1}(\alpha)] \right| < \eta \sum_{s+1}^{\nu} |f_n(\alpha) - f_{n+1}(\alpha)| < \frac{\eta L}{\alpha^\rho} \sum_{s+1}^{\nu} \frac{1}{n^{1+\rho}} < \eta \frac{2^\rho L}{\rho c^\rho},$$

which combined with (26) gives

$$\left| \sum_{\mu}^{\nu} u_n f_n(\alpha) \right| < |\epsilon_\mu f_\mu(\alpha)| + \eta \frac{2^\rho L}{\rho c^\rho} + |\epsilon_{\nu+1} f_{\nu+1}(\alpha)| < K\eta + \eta \frac{2^\rho L}{\rho c^\rho} + K\eta < \delta.$$

* The substitution of (d') for (d) occasions only a slight modification of the proof.

Hence the desired inequality (24) is satisfied in all three cases, q being equal to m.

For all values of $\alpha \geqq c/2$ we have

$$|u_n f_n(\alpha)| < \frac{BN}{n^{1+\rho}\alpha^{1+\rho}} \leqq \frac{BN2^{1+\rho}}{n^{1+\rho}c^{1+\rho}}$$

where B is the largest numerical value of the terms $u_0, u_1, u_2, \cdots$. The series $\sum 1/n^{1+\rho}$ converges, and therefore we can choose m_1 such that

$$\left|\sum_{n=\mu}^{n=\nu} u_n f_n(\alpha)\right| < \delta \qquad \left(\nu \geqq \mu \geqq m_1;\ \alpha \geqq \frac{c}{2}\right).$$

When $\alpha = 0$ the series (21) becomes the series (20), and we can so choose m_2 that

$$\left|\sum_{n=\mu}^{n=\nu} u_n\right| < \delta \qquad (\nu \geqq \mu \geqq m_2).$$

Thus, finally, if we take for q the greatest of the three values m, m_1, and m_2, we have the desired inequality (24).

The convergence factors $e^{-n\alpha}$ satisfy the conditions of Theorem II. Hence if the series

$$a_0 + a_1 r + a_2 r^2 + \cdots$$

is convergent, the series

$$a_0 + a_1 r e^{-\alpha} + a_2 r^2 e^{-2\alpha} + \cdots$$

converges uniformly in the interval $\alpha \geqq 0$. If now we set $e^{-\alpha} = x/r$, it follows that

$$a_0 + a_1 r\left(\frac{x}{r}\right) + a_2 r^2\left(\frac{x}{r}\right)^2 + \cdots$$

or

$$a_0 + a_1 x + a_2 x^2 + \cdots$$

converges uniformly in the interval $0 \leqq x \leqq r$. Hence Theorem II includes ABEL's theorem as a special case.

§ 3. *Summable integrals. General facts and lemmas.*

Let $f(x)$ be a function which is integrable in every finite interval lying in the interval $a \leqq x$. Then the integral

$$\int_a^\infty f(x)\,dx$$

shall be said to be summable if the limit

$$\lim_{x=\infty}\left[\frac{1}{x}\int_a^x \int_a^\alpha f(\beta)\,d\beta\,d\alpha\right]$$

exists; and we attach to the integral the value of that limit. The agreement of this definition with the ordinary definition of an integral, is shown by the following theorem:

THEOREM III. *Every convergent integral is summable, and the process of summation gives the value of the integral ordinarily defined.**

Put

$$\int_a^\infty f(x)\,dx = A$$

and

$$\int_a^x f(x)\,dx = A + \epsilon(x),$$

so that

$$\lim_{x=\infty} \epsilon(x) = 0.$$

Then

$$\frac{1}{x}\int_a^x \int_a^\alpha f(\beta)\,d\beta\,d\alpha = \frac{1}{x}\int_a^x [A + \epsilon(\alpha)]\,d\alpha = A - \frac{aA}{x} + \frac{1}{x}\int_a^x \epsilon(\alpha)\,d\alpha.$$

Since the second term approaches zero as x becomes infinite, our theorem will be proved if we can show that the same is true of the third term. Let δ be a positive constant as small as we please, and choose m so that

$$|\epsilon(\alpha)| < \frac{\delta}{2} \qquad (\alpha \geqq m > 0).$$

Then for values of $x \geqq m$ we have

$$\left|\frac{1}{x}\int_a^x \epsilon(\alpha)\,d\alpha\right| \leqq \frac{1}{x}\left|\int_a^m \epsilon(\alpha)\,d\alpha\right| + \frac{1}{x}\left|\int_m^x \epsilon(\alpha)\,d\alpha\right|.$$

The second of these terms reduces to

$$\frac{x-m}{x}|\epsilon(\alpha')| \qquad (m < \alpha' < x)$$

and is therefore less than $\delta/2$. We can also choose p so that when $x > p$ the first term is less than $\delta/2$. Accordingly, since δ is arbitrary,

$$\lim_{x=\infty}\left[\frac{1}{x}\int_a^x \epsilon(\alpha)\,d\alpha\right] = 0,$$

and our theorem is proved.

I shall next establish a series of lemmas that will be needed in the proof of the theorem concerning the introduction of convergence factors into summable integrals.

* The substance of this theorem is stated without proof by HARDY in an article in the Quarterly Journal of Mathematics, vol. 35 (1903–04), p. 54.

LEMMA 1. *If $f(x)$ is uniformly continuous for values of $x \geqq k > 0$, then there exists a positive constant M such that*

$$|f(x)| < Mx \qquad (x \geqq k).$$

Let us choose δ such that

$$|f(x+h) - f(x)| < 1 \qquad (|h| \leqq \delta;\ x \geqq k). \tag{30}$$

Then for every $x \geqq k$ we can find a $y \geqq 0$ such that

$$x = k + y\delta.$$

If n is the greatest integer not exceeding y, we have by (30)

$$|f(k + y\delta) - f(k)| < n + 1 \qquad (y \geqq 0),$$

and therefore

$$\frac{|f(k+y\delta)|}{k+y\delta} < \frac{|f(k)|}{k+y\delta} + \frac{n+1}{k+y\delta}.$$

For values of y in the interval $0 \leqq y < 1$ we have, since $n = 0$,

$$\frac{|f(k+y\delta)|}{k+y\delta} < \frac{|f(k)|}{k} + \frac{1}{k} = K_1.$$

For values of $y \geqq 1$, since $y \geqq n \geqq 1$,

$$\frac{|f(k+y\delta)|}{k+y\delta} < \frac{|f(k)|}{k} + \frac{n+1}{n\delta} \leqq \frac{|f(k)|}{k} + \frac{2}{\delta} = K_2.$$

If then we choose as M the larger of the two constants K_1 and K_2, we have

$$|f(k+y\delta)| < M(k+y\delta) \qquad (y \geqq 0),$$

or

$$|f(x)| < Mx \qquad (x \geqq k).$$

LEMMA 2. *If $f(x)$ is uniformly continuous for values of $x \geqq k > 0$, then $f(x)/x^\rho$ (where ρ is any positive constant) is uniformly continuous in the same interval.*

We can assume without loss of generality that $k \geqq e$, where e is the base of natural logarithms; for, if it is not, it follows that $f(x)/x^\rho$ is uniformly continuous in the closed interval $k \leqq x \leqq e$ since it is continuous in it.

From Lemma 1 it follows that there exists a positive constant M such that

$$\left|\frac{f(x)}{x}\right| < M \qquad (x \geqq k).$$

Given ϵ positive and arbitrarily small, let us choose δ less than $\epsilon/2M$ and such that

$$|f(x+h) - f(x)| < \frac{k^\rho}{2}\epsilon \qquad (x \geqq k;\ 0 < h \leqq \delta).$$

We have

$$(31)\quad \left|\frac{f(x+h)}{(x+h)^\rho}-\frac{f(x)}{x^\rho}\right| \leqq \frac{1}{(x+h)^\rho}\,|f(x+h)-f(x)| + |f(x)|\left|\frac{1}{(x+h)^\rho}-\frac{1}{x^\rho}\right|.$$

For the first term on the right hand side we have

$$(32)\qquad \frac{1}{(x+h)^\rho}\,|f(x+h)-f(x)| < \frac{1}{k^\rho}\,\frac{k^\rho\epsilon}{2} = \frac{\epsilon}{2} \quad (x \geqq k;\ 0 < h \leqq \delta).$$

Also from the Law of the Mean

$$\frac{1}{(x+h)^\rho}-\frac{1}{x^\rho} = \frac{-\rho h}{(x+\theta h)^{\rho+1}} \qquad (0<\theta<1),$$

and hence

$$|f(x)|\left|\frac{1}{(x+h)^\rho}-\frac{1}{x^\rho}\right| < M\frac{x\rho h}{x^{\rho+1}} \leqq M\frac{\rho}{e^\rho}\,h \qquad (x \geqq k;\ 0<h\leqq\delta).$$

But

$$\frac{\rho}{e^\rho} < 1 \qquad (\rho > 0).$$

Consequently

$$(33)\qquad |f(x)|\cdot\left|\frac{1}{(x+h)^\rho}-\frac{1}{x^\rho}\right| < M\delta < \frac{\epsilon}{2} \qquad (x \geqq k;\ 0<h\leqq\delta).$$

Combining (31), (32), and (33) we have

$$\left|\frac{f(x+h)}{(x+h)^\rho}-\frac{f(x)}{x^\rho}\right| < \epsilon \qquad (x \geqq k;\ 0<h\leqq\delta),$$

and the lemma is proved.

LEMMA 3. *If $f(x)$ is uniformly continuous for values of $x \geqq k > 0$, and the integral*

$$\int_a^\infty f(x)\,dx$$

is summable, then

$$\lim_{x=\infty}\left[\frac{f(x)}{x}\right] = 0.$$

We know from Lemma 2 that $f(x)/x$ is uniformly continuous for values of $x \geqq k$. Given ϵ, let us choose a positive $\delta < k$ and such that

$$(34)\qquad \left|\frac{f(x+h)}{x+h}-\frac{f(x)}{x}\right| < \frac{\epsilon}{3} \qquad (x \geqq k;\ 0<h\leqq 2\delta).$$

Let

$$\frac{1}{x}\int_a^x\int_a^\alpha f(\beta)\,d\beta\,d\alpha = S(x).$$

Then

$$f(x) = \frac{d^2[xS(x)]}{dx^2}. \tag{35}$$

Moreover, since $\lim_{x=\infty} S(x)$ exists, we can so determine $x_0 \geqq k$ that

$$|S(x') - S(x)| < \frac{\delta^2 \epsilon}{6} \qquad (x' > x \geqq x_0). \tag{36}$$

But we have *

$$(x+2\delta)S(x+2\delta) - 2(x+\delta)S(x+\delta) + xS(x) = \delta^2 f(x+\theta\delta) \quad (0<\theta<2),$$

whence by (36)

$$\delta^2|f(x+\theta\delta)| < (x+2\delta)|S(x+2\delta) - S(x+\delta)| + x|S(x+\delta) - S(x)| < \frac{2(x+\delta)\delta^2\epsilon}{6} \quad (x \geqq x_0).$$

Consequently, since $x + \delta < 2x$, we get

$$\left|\frac{f(x+\theta\delta)}{x+\theta\delta}\right| < \frac{2(x+\delta)}{(x+\theta\delta)}\frac{\epsilon}{6} < \frac{2(x+\delta)}{x}\frac{\epsilon}{6} < \frac{2\epsilon}{3} \qquad (x \geqq x_0).$$

It follows then from (34) that

$$\left|\frac{f(x)}{x}\right| < \epsilon \qquad (x \geqq x_0),$$

and therefore

$$\lim_{x=\infty}\left[\frac{f(x)}{x}\right] = 0.$$

Before taking up the other lemmas, I wish to introduce the following notation. Let

$$\int f(x)\,dx = \psi(x) + C_1, \qquad \int \psi(x)\,dx = \chi(x) + C_2. \tag{37}$$

Then

$$xS(x) = \int_a^x \int_a^\alpha f(\beta)\,d\beta\,d\alpha = \int_a^x [\psi(\alpha) - \psi(a)]\,d\alpha = \chi(x) - x\psi(a) - \chi(a) + a\psi(a). \tag{38}$$

Lemma 4. *If $f(x)$ is uniformly continuous for values of $x \geqq k > 0$, and the integral*

$$\int_a^\infty f(x)\,dx$$

is summable, then

$$\lim_{x=\infty}\left[\frac{\psi(x)}{x^2}\right] = 0.$$

From Lemma 3 it follows that for a given positive but arbitrarily small ϵ, we

* Cf. Markoff: *Differenzenrechnung*, § 8.

can choose a positive $x_0 > a$ such that

$$\left|\frac{f(x)}{x}\right| < \frac{\epsilon}{2} \qquad (x \geqq x_0).$$

We have also from (37)

$$\psi(x) - \psi(x_0) = (x - x_0) f[x_0 + \theta(x - x_0)] \qquad (0 < \theta < 1),$$

and consequently

$$\left|\frac{\psi(x)}{x^2}\right| \leqq \left|\frac{\psi(x_0)}{x^2}\right| + \frac{x - x_0}{x} \cdot \left|\frac{f[x_0 + \theta(x - x_0)]}{x_0 + \theta(x - x_0)}\right| \cdot \frac{x_0 + \theta(x - x_0)}{x}$$

$$< \left|\frac{\psi(x_0)}{x^2}\right| + \frac{\epsilon}{2} \qquad (x \geqq x_0).$$

Choose $x_1 \geqq x_0$ and such that

$$\left|\frac{\psi(x_0)}{x^2}\right| < \frac{\epsilon}{2} \qquad (x \geqq x_1).$$

Then

$$\left|\frac{\psi(x)}{x^2}\right| < \epsilon \qquad (x \geqq x_1),$$

and accordingly

$$\lim_{x=\infty} \left[\frac{\psi(x)}{x^2}\right] = 0.$$

LEMMA 5. *If we have a function $\phi(x)$ such that*

$$\lim_{x=\infty} [\phi(x)] = C, \qquad \lim_{x=\infty} [\phi''(x)] = 0$$

where C is any constant, then it follows that

$$\lim_{x=\infty} [\phi'(x)] = 0.$$

Given ϵ positive and arbitrarily small, let us choose B such that

$$(39) \qquad |\phi(x') - \phi(x)| < \frac{\epsilon}{2}, \qquad |\phi''(x)| < \frac{\epsilon}{2} \qquad (x' > x \geqq B).$$

From the Law of the Mean

$$\phi(x + 1) - \phi(x) = \phi'(x + \theta) \qquad (0 < \theta < 1),$$

and therefore

$$(40) \qquad |\phi'(x + \theta)| < \frac{\epsilon}{2} \qquad (x \geqq B).$$

Applying again the Law of the Mean, we have

$$\phi'(x + 1) - \phi'(x + \theta) = (1 - \theta)\phi''(x + \theta') \qquad (\theta < \theta' < 1),$$

from which it follows, by (39) and (40), that

$$|\phi'(x + 1)| < |\phi'(x + \theta)| + |\phi''(x + \theta')| < \epsilon \qquad (x \geqq B).$$

Consequently

$$\lim_{x=\infty} [\phi'(x)] = 0.$$

LEMMA 6. *If $f(x)$ is uniformly continuous for values of $x \geqq k > 0$ and the integral*

$$\int_a^\infty f(x)\,dx$$

is summable, then

$$\lim_{x=\infty} [S'(x)] = 0.$$

We have from (38)

$$S''(x) = \frac{2\chi(x)}{x^3} - \frac{2\psi(x)}{x^2} + \frac{f(x)}{x} - \frac{2(\chi(a) - a\psi(a))}{x^3}. \tag{41}$$

From Lemmas 3 and 4 it follows that

$$\lim_{x=\infty}\left[\frac{f(x)}{x}\right] = 0, \qquad \lim_{x=\infty}\left[\frac{\psi(x)}{x^2}\right] = 0,$$

and from equation (38) that

$$\lim_{x=\infty}\left[\frac{\chi(x)}{x^3}\right] = 0.$$

Hence from (41)

$$\lim_{x=\infty} [S''(x)] = 0.$$

Accordingly, since by hypothesis $S(x)$ approaches a limit when $x = \infty$, we have by Lemma 5

$$\lim_{x=\infty} [S'(x)] = 0.$$

LEMMA 7. *If we have a function of two variables $\phi(\alpha, x)$ such that*

(a) $\phi(\alpha, x)$ is continuous in x and in α separately in the region $R(x_2 \geqq x \geqq x_1,\ \alpha_2 \geqq \alpha \geqq \alpha_1)$,

(b) $\phi'_x(\alpha, x)$ and $\phi''_x(\alpha, x)$ exist and

$$|\phi''_x(\alpha, x)| < C$$

in the region R, then $\phi'_x(\alpha, x)$ is continuous in α in the region R.

Take any point (α_0, x_0) that lies in the region R. Choose a constant Δx such that $x_0 + \Delta x$ lies in the interval $x_2 \geqq x \geqq x_1$ and

$$|\Delta x| < \frac{\epsilon}{4C}, \tag{42}$$

where ϵ is an arbitrarily small, positive constant.

Then choose η such that

$$(43)\qquad |\phi(\alpha', x_0+\Delta x)-\phi(\alpha_0, x_0+\Delta x)|<\frac{\epsilon}{4}\Delta x \qquad (|\alpha'-\alpha_0|<\eta),^*$$

$$(44)\qquad |\phi(\alpha', x_0)-\phi(\alpha_0, x_0)|<\frac{\epsilon}{4}\Delta x \qquad (|\alpha-\alpha_0|<\eta).$$

By the Law of the Mean

$$(45)\qquad \frac{\phi(\alpha', x_0+\Delta x)-\phi(\alpha', x_0)}{\Delta x}=\phi_x'(\alpha', x_0+\theta\Delta x) \qquad (0<\theta<1),$$

$$(46)\qquad \frac{\phi(\alpha_0, x_0+\Delta x)-\phi(\alpha_0, x_0)}{\Delta x}=\phi_x'(\alpha_0, x_0+\theta'\Delta x) \qquad (0<\theta'<1).$$

By subtraction of (46) from (45) and with the aid of (43) and (44) we obtain

$$(47)\qquad |\phi_x'(\alpha', x_0+\theta\Delta x)-\phi_x'(\alpha_0, x_0+\theta'\Delta x)|<\frac{\epsilon}{4}+\frac{\epsilon}{4}=\frac{\epsilon}{2} \qquad (|\alpha'-\alpha_0|<\eta).$$

Applying again the Law of the Mean we have

$$\phi_x'(\alpha', x_0+\theta\Delta x)-\phi_x'(\alpha', x_0)=\theta\Delta x\,\phi_x''(\alpha', x_0+\theta_1\theta\Delta x) \qquad (0<\theta_1<1),$$

$$\phi_x'(\alpha_0, x_0+\theta'\Delta x)-\phi_x'(\alpha_0, x_0)=\theta'\Delta x\,\phi_x''(\alpha_0, x_0+\theta_1'\theta'\Delta x) \qquad (0<\theta_1'<1),$$

whence it follows by subtraction that

$$|\phi_x'(\alpha', x_0)-\phi_x'(\alpha_0, x_0)|\leqq|\phi_x'(\alpha', x_0+\theta\Delta x)-\phi_x'(\alpha_0, x_0+\theta'\Delta x|$$
$$+\theta'|\Delta x|\,|\phi_x''(\alpha_0, x_0+\theta_1'\theta'\Delta x)|+\theta|\Delta x|\,|\phi_x''(\alpha', x_0+\theta_1\theta\Delta x)|.$$

In combination with (b), (42), and (47) this gives

$$|\phi_x'(\alpha', x_0)-\phi_x'(\alpha_0, x_0)|<\epsilon \qquad (|\alpha'-\alpha_0|<\eta),$$

and the lemma is proved.

§ 4. *Convergence factors in summable integrals.*

THEOREM IV. *If the function $f(x)$ is uniformly continuous for all values of $x\geqq a$, where a is any positive constant, and the integral*

$$\int_a^\infty f(x)dx$$

is summable and has the value S, then the integral

$$(48)\qquad F(\alpha)=\int_a^\infty f(x)\phi(\alpha, x)\,dx$$

* It is assumed in this and in the following similar inequalities that α' lies in the interval $\alpha_2\geqq\alpha\geqq\alpha_1$.

will be absolutely convergent and continuous for all positive values of α *and will approach* S *as its limit when* $\alpha = +0$, *provided the convergence factor* $\phi(\alpha, x)$ *satisfies the following conditions for* $x \geqq a$:

(*a*) $\phi(\alpha, x)$ *is continuous in* (α, x) $(\alpha \geqq 0)$,

(*b*) $\phi_x''(\alpha, x)$ *exists and is continuous in* (α, x) $(\alpha \geqq 0)$,

(*c*) $$|\phi(\alpha, x)| < \frac{N}{x^{2+\rho}\alpha^{2+\rho}} \qquad (\alpha > 0),$$

(*d*) $$\phi(0, x) = 1,$$

(*e*) $$\phi_x''(\alpha, x) \geqq 0^* \qquad (0 \leqq x\alpha \leqq c),$$

(*f*) $$|\phi_x''(\alpha, x)| < \frac{L}{x^{2+\rho}\alpha^{\rho}} \qquad (\alpha > 0),$$

where N, ρ, c, *and* L *are positive constants.*

We must first derive from the given conditions the following further conditions which hold for $x \geqq a$:

(*g*) $\phi_x'(\alpha, x)$ is continuous in α $(\alpha \geqq 0)$,

(*h*) $$|\phi_x'(\alpha, x)| < \frac{L}{x^{1+\rho}\alpha^{\rho}} \qquad (\alpha > 0).$$

Consider the function $\phi(\alpha, x)$ in the region $R(B \geqq x \geqq a;\ \alpha \geqq 0)$. By condition (a) this function is continuous throughout R. We can also show at once that its second derivative with regard to x remains finite in the region R. For from (b) we know that a positive constant M exists such that

$$|\phi_x''(\alpha, x)| < M \qquad (B \geqq x \geqq a;\ 1 \geqq \alpha \geqq 0),$$

and from condition (f) we have

$$|\phi_x''(\alpha, x)| < \frac{L}{x^{2+\rho}\alpha^{\rho}} < \frac{L}{a^{2+\rho}} \qquad (B \geqq x \geqq a;\ \alpha > 1).$$

Accordingly the function $\phi(\alpha, x)$ satisfies the conditions of Lemma 7. Consequently $\phi_x'(\alpha, x)$ is continuous in α in the region R, and therefore in the region $x \geqq a$, $\alpha \geqq 0$.

By virtue of conditions (c) and (f) we have

$$\lim_{x=\infty} [\phi(\alpha, x)] = 0 \qquad (\alpha > 0),$$

$$\lim_{x=\infty} [\phi_x''(\alpha, x)] = 0 \qquad (\alpha > 0).$$

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* We may substitute for (e) the condition

(*e'*) $$\phi_x''(\alpha, x) \leqq 0 \qquad (0 \leqq x\alpha \leqq c).$$

If (e) (or (e')) holds for all values of x and α, (f) is unnecessary.

Hence by Lemma 5 it follows that

$$\lim_{x=\infty} [\phi'_x(\alpha, x)] = 0 \qquad (\alpha > 0), \tag{49}$$

and consequently

$$\int_x^\infty \phi''_x(\alpha, x)\,dx = -\phi'_x(\alpha, x) \qquad (\alpha > 0).$$

We have then

$$|\phi'_x(\alpha, x)| = \left|\int_x^\infty \phi''_x(\alpha, x)\,dx\right| \leqq \int_x^\infty |\phi''_x(\alpha, x)|\,dx$$

$$< \int_x^\infty \frac{L}{x^{2+\rho}\alpha^\rho}\,dx = \frac{L}{(1+\rho)x^{1+\rho}\alpha^\rho} < \frac{L}{x^{1+\rho}\alpha^\rho} \qquad (\alpha > 0).$$

Having thus established conditions (g) and (h) we can now prove the following inequalities (50) and (51) which hold when $x = a$. By condition (c) we have

$$|\phi(\alpha, a)| < \frac{N}{a^{2+\rho}\alpha^{2+\rho}} < \frac{N}{a^{2+\rho}} \qquad (\alpha > 1),$$

and by condition (a) we can find a positive constant K_1 such that

$$|\phi(\alpha, a)| < K_1 \qquad (0 \leqq \alpha \leqq 1).$$

Denoting by K the greater of the two quantities $N/a^{2+\rho}$ and K_1 we have

$$|\phi(\alpha, a)| < K \qquad (\alpha \geqq 0). \tag{50}$$

Similarly from conditions (h) and (g) we see that a positive constant C can be found such that

$$|\phi'_x(\alpha, a)| < C \qquad (\alpha \geqq 0). \tag{51}$$

We shall next prove that the integral (48) is absolutely convergent for every positive value of α. By Lemma 1 there exists a positive constant M such that in the interval $x \geqq a$ we have

$$|f(x)| < Mx.$$

Consequently

$$|f(x)\phi(\alpha, x)| < \frac{MN}{x^{1+\rho}\alpha^{2+\rho}}.$$

Since also the integral

$$\int_a^\infty \frac{dx}{x^{1+\rho}}$$

is convergent, the absolute convergence of (48) and the existence of $F(\alpha)$ for values of $\alpha > 0$ follow at once.

We have from (35)

$$F(\alpha) = \int_a^\infty f(x)\phi(\alpha, x)\,dx = \int_a^\infty [2S'(x) + xS''(x)]\,\phi(\alpha, x)\,dx.$$

Integrating twice by parts we get

$$
\begin{aligned}
F(\alpha) = {} & [S(x)\phi(\alpha, x)]_a^\infty + [xS'(x)\phi(\alpha, x)]_a^\infty - [S(x)x\phi'_x(\alpha, x)]_a^\infty \\
(52) \qquad & - \int_a^\infty S(x)\phi'_x(\alpha, x)\,dx + \int_a^\infty S(x)\,[\phi'_x(\alpha, x) + x\phi''_x(\alpha, x)]\,dx.
\end{aligned}
$$

Now from the definition of $S(x)$ both $S(a)$ and $S'(a)$ are zero. As x becomes infinite, $S'(x)$ approaches zero by Lemma 6 and $S(x)$ by hypothesis approaches S. From condition (c) we see further that for all positive values of α, $\phi(\alpha, x)$ approaches zero as x becomes infinite.

Combining (49) with conditions (c) and (f) we obtain next

$$\lim_{x=\infty} [x\phi(\alpha, x)] = 0 \qquad (\alpha > 0),$$

$$\lim_{x=\infty} [2\phi'_x(\alpha, x) + x\phi''_x(\alpha, x)] = 0 \qquad (\alpha > 0).$$

Hence by Lemma 5

$$\lim_{x=\infty} [x\phi'_x(\alpha, x) + \phi(\alpha, x)] = 0 \qquad (\alpha > 0),$$

from which it follows that

$$\lim_{x=\infty} [x\phi'_x(\alpha, x)] = 0 \qquad (\alpha > 0).$$

The quantities in brackets in equation (52) therefore disappear, and we have

$$(53) \qquad F(\alpha) = -\int_a^\infty S(x)\phi'_x(\alpha, x)\,dx + \int_a^\infty S(x)[\phi'_x(\alpha, x) + x\phi''_x(\alpha, x)]\,dx.$$

Put

$$S(x) = S + \epsilon(x)$$

so that

$$\lim_{x=\infty} \epsilon(x) = 0$$

and then substitute in (53). We obtain

$$
\begin{aligned}
F(\alpha) = {} & -S\int_a^\infty \phi'_x(\alpha, x)\,dx + S\int_a^\infty [\phi'_x(\alpha, x) + x\phi''_x(\alpha, x)]\,dx \\
(54) \qquad & - \int_a^\infty \epsilon(x)\phi'_x(\alpha, x)\,dx + \int_a^\infty \epsilon(x)[\phi'_x(\alpha, x) + x\phi''_x(\alpha, x)]\,dx.
\end{aligned}
$$

But

$$\int_a^\infty \phi'_x(\alpha, x)\,dx = -\phi(\alpha, a) \qquad (\alpha > 0),$$

$$\int_a^\infty [\phi'_x(\alpha, x) + x\phi''_x(\alpha, x)]\,dx = [x\phi'_x(\alpha, x)]_a^\infty = -a\phi'_x(\alpha, a) \qquad (\alpha > 0).$$

Consequently (54) reduces to

$$(55) \qquad F(\alpha) = \phi(\alpha, a)S - a\phi'_x(\alpha, a)S + \int_a^\infty \epsilon(x)x\phi''_x(\alpha, x)\,dx \qquad (\alpha > 0).$$

By condition (d)

$$\phi'_x(0, x) = 0, \qquad \phi''_x(0, x) = 0,$$

and therefore the right hand member of (55) reduces to S when $\alpha = 0$. Since, by condition (b), $\phi''_x(\alpha, x)$ is a continuous function of α for all values of $\alpha \geqq 0$, our theorem is proved if we can show further that the integral in the right hand member of (55) is uniformly convergent in the interval $\alpha \geqq 0$.

Given an arbitrarily small, positive quantity, δ, let us choose $m > a$ and such that

$$(56) \qquad |\epsilon(x)| < \eta = \frac{\rho c^{2+\rho}\delta}{(\rho+1)c^2 L + \rho c^{2+\rho}(aC + K) + \rho N} \qquad (x \geqq m).$$

Consider first values of α in the interval $0 < \alpha$. Let $s = c/\alpha$ and μ and ν be any two quantities such that

$$m \leqq \mu < \nu.$$

Three cases must be considered

$$(A) \qquad \mu < s < \nu,$$

$$(B) \qquad \nu \leqq s,$$

$$(C) \qquad \mu \geqq s.$$

Beginning with (A) we write

$$(57) \qquad \int_\mu^\nu \epsilon(x) x \phi''_x(\alpha, x)\, dx = \int_\mu^s + \int_s^\nu = R_1 + R_2.$$

By condition (e) we have

$$|R_1| < \eta \int_\mu^s x\phi''_x(\alpha, x)\, dx \leqq \eta \int_a^s x\phi''_x(\alpha, x)\, dx.$$

But

$$\int_a^s x\phi''_x(\alpha, x)\, dx = s\phi'_x(\alpha, s) - a\phi'_x(\alpha, a) - \phi(\alpha, s) + \phi(\alpha, a)$$

and consequently, in view of (h), (c), (50), and (51),

$$\left|\int_a^s x\phi''_x(\alpha, x)\, dx\right| \leqq |s\phi'_x(\alpha, s)| + |a\phi'_x(\alpha, a)| + |\phi(\alpha, s)| + |\phi(\alpha, a)|$$

$$< \frac{L}{s^\rho \alpha^\rho} + aC + \frac{N}{s^{2+\rho}\alpha^{2+\rho}} + K = \frac{L}{c^\rho} + aC + \frac{N}{c^{2+\rho}} + K.$$

Hence

$$(58) \qquad |R_1| < \eta\left[\frac{L}{c^\rho} + aC + \frac{N}{c^{2+\rho}} + K\right].$$

From condition (f) we have for R_2,

$$(59)\qquad \begin{aligned} |R_2| &\leqq \int_s^\nu |\epsilon(x)x\phi_x''(\alpha, x)|\,dx < \eta \int_s^\nu \frac{L}{x^{1+\rho}\alpha^\rho}\,dx \\ &= \eta\left[-\frac{L}{\rho x^\rho \alpha^\rho}\right]_s^\nu < \eta\frac{L}{\rho c^\rho}. \end{aligned}$$

Combining (57), (58), and (59) and using (56) we have

$$\begin{aligned} \left|\int_\mu^\nu \epsilon(x)x\phi_x''(\alpha, x)\,dx\right| &\leqq |R_1| + |R_2| \\ &< \eta\left[\frac{(\rho+1)c^2L + \rho c^{2+\rho}(aC+K) + \rho N}{\rho c^{2+\rho}}\right] = \delta. \end{aligned}$$

For case (B) we have

$$\begin{aligned} \left|\int_\mu^\nu \epsilon(x)x\phi_x''(\alpha, x)\,dx\right| &< \eta\int_\mu^\nu x\phi_x''(\alpha, x) \leqq \eta\int_a^s x\phi_x''(\alpha, x)\,dx \\ &< \eta\left[\frac{L}{c^\rho} + aC + \frac{N}{c^{2+\rho}} + K\right] = \eta\left[\frac{\rho c^2 L + \rho c^{2+\rho}(aC+K) + \rho N}{\rho c^{2+\rho}}\right] < \delta. \end{aligned}$$

We have finally for case (C)

$$\begin{aligned} \left|\int_\mu^\nu \epsilon(x)x\phi_x''(\alpha, x)\,dx\right| &\leqq \int_\mu^\nu |\epsilon(x)x\phi_x''(\alpha, x)|\,dx \\ &< \eta\int_\mu^\nu \frac{L}{x^{1+\rho}\alpha^\rho}\,dx \leqq \eta\int_s^\nu \frac{L}{x^{1+\rho}\alpha^\rho}\,dx < \eta\frac{L}{\rho c^\rho} = \eta\frac{c^2L}{\rho c^{2+\rho}} < \delta. \end{aligned}$$

Hence in all three cases

$$(60)\qquad \left|\int_\mu^\nu \epsilon(x)x\phi_x''(\alpha, x)\,dx\right| < \delta \qquad \left(\nu > \mu \geqq m;\ 0 < \alpha < \frac{c}{m}\right).$$

For $\alpha = 0$ we have

$$\int_\mu^\nu \epsilon(x)x\phi_x''(0, x)\,dx = \int_\mu^\nu \epsilon(x)\,[0]\,dx < \delta \qquad (\mu \geqq a).$$

Finally, by taking for q the greater of the two quantities a and m, and by combining the results just obtained we have

$$\left|\int_\mu^\nu \epsilon(x)x\phi_x''(\alpha, x)\,dx\right| < \delta \qquad (\nu > \mu \geqq q;\ \alpha \geqq 0),$$

and our theorem is proved.

We assumed in Theorem IV, for the sake of simplicity of proof, that q was greater than zero. That this assumption really places no restriction upon the generality of the theorem is shown by the following corollary:

COROLLARY. *If, in Theorem* IV, *a is any constant, and the convergence factor $\phi(\alpha, x)$ satisfies conditions (a), (b), (c), (d), (e), and (f) for $x \geqq b$, where b is any positive constant, and conditions (a) and (d) for $a \leqq x \leqq b$, the theorem holds without change.*

Let

$$\int_a^b f(x)\,dx = A.$$

We have

$$\frac{1}{x}\int_a^x\int_a^\alpha f(\beta)\,d\beta d\alpha = \frac{1}{x}\int_a^x\int_b^\alpha f(\beta)\,d\beta d\alpha + \frac{1}{x}\int_a^x\int_a^b f(\beta)\,d\beta d\alpha$$

$$= \frac{1}{x}\int_b^x\int_b^\alpha f(\beta)\,d\beta d\alpha + \frac{1}{x}\int_a^b\int_b^\alpha f(\beta)\,d\beta d\alpha + \frac{1}{x}\int_a^x A\,d\alpha$$

$$= \frac{1}{x}\int_b^x\int_b^\alpha f(\beta)\,d\beta d\alpha + \frac{1}{x}\int_a^b [\psi(\alpha) - \psi(b)]\,d\alpha + \frac{x-a}{x}A,$$

and consequently

$$S = \lim_{x=\infty}\frac{1}{x}\int_a^x\int_a^\alpha f(\beta)\,d\beta d\alpha = \lim_{x=\infty}\frac{1}{x}\int_b^x\int_b^\alpha f(\beta)\,d\beta d\alpha + A$$

or

$$\lim_{x=\infty}\frac{1}{x}\int_b^x\int_b^\alpha f(\beta)\,d\beta d\alpha = S - A.$$

Hence the integral

$$\int_b^\infty f(x)\,dx$$

is summable and has the value $(S - A)$. Therefore by Theorem IV the integral

$$\int_b^\infty f(x)\phi(\alpha, x)\,dx$$

is absolutely convergent for all positive values of α, its value $G(\alpha)$ is continuous for such values, and

$$\lim_{\alpha=+0} G(\alpha) = S - A.$$

Hence

$$(61)\qquad \int_a^b f(x)\phi(\alpha, x)\,dx + \int_b^\infty f(x)\phi(\alpha, x)\,dx = \int_a^\infty f(x)\phi(\alpha, x)\,dx$$

is absolutely convergent for all positive values of α.

Let

$$\int_a^b f(x)\phi(\alpha, x)\,dx = H(\alpha).$$

Then

$$H(0) = \int_a^b f(x)\phi(0, x)\,dx = \int_a^b f(x)\,dx = A.$$

We have by (61)

$$F(\alpha) = \int_a^\infty f(x)\phi(\alpha, x)\,dx = H(\alpha) + G(\alpha),$$

and since $H(\alpha)$ and $G(\alpha)$ are continuous for all values of $\alpha > 0$, $F(\alpha)$ is continuous for such values. Moreover, since $H(\alpha)$ and $G(\alpha)$ each approach a limit when $\alpha = +0$, so also does $F(\alpha)$ and we have for that limit

$$\lim_{\alpha=+0} F(\alpha) = H(0) + \lim_{\alpha=+0} G(\alpha) = S.$$

§ 5. *Convergence factors in convergent integrals.*

Theorem IV is applicable to convergent integrals since by Theorem III every convergent integral is summable. As in the case of series, however, less restriction can be placed upon the convergence factor, as is shown in the following theorem:

THEOREM V. *If the function $f(x)$ is finite and integrable in every finite interval lying in the interval $x \geqq a$, where a is any positive constant, and the integral*

$$\int_a^\infty f(x)\,dx \tag{62}$$

converges to the value A, then the integral

$$\int_a^\infty f(x)\phi(\alpha, x)\,dx \tag{63}$$

will be absolutely and uniformly convergent for all values of $\alpha \geqq 0$ and therefore will define a continuous function, provided the convergence factor $\phi(\alpha, x)$ satisfies the following conditions for $x \geqq a$:*

(*a*) $\phi(\alpha, x)$ *is continuous in* (α, x) $(\alpha \geqq 0)$,

(*b*) $\phi'_x(\alpha, x)$ *exists* $(\alpha \geqq 0)$,

(*c*) $|\phi(\alpha, x)| < \dfrac{N}{x^{1+\rho}\alpha^{1+\rho}}$ $(\alpha > 0)$,

* That the integral (63) converges more rapidly than (62) can be shown by a method precisely analogous to that suggested for series in a previous footnote (see page 307) since for a given value of α, an m can be determined such that

$$\left|\int_n^\infty f(x)\phi(\alpha, x)\,dx\right| < \left|\int_n^\infty f(x)\,dx\right| \qquad (n \geqq m).$$

$$(d) \qquad \phi(0, x) = 1,$$

$$(e) \qquad \phi'_x(\alpha, x) \leqq 0 \,^* \qquad (0 \leqq x\alpha \leqq c),$$

$$(f) \qquad |\phi'_x(\alpha, x)| < \frac{L}{x^{1+\rho}\alpha^\rho} \qquad (\alpha > 0),$$

where N, ρ, c, and L are positive constants.

We must first derive from the given conditions the following further condition which holds for $x \geqq a$:

$$(g) \qquad |\phi(\alpha, x)| < K \qquad (\alpha \geqq 0),$$

where K is a positive constant.

We know from condition (e) † that $\phi(\alpha, x)$, for a given value of α, either decreases or remains constant with increasing x as long as $x\alpha$ does not exceed c. We have then

$$(64) \qquad \phi(\alpha, a) \geqq \phi(\alpha, x) \geqq \phi\left(\alpha, \frac{c}{\alpha}\right) \qquad (0 < x\alpha < c).$$

But by condition (a) we can find a positive constant M such that

$$|\phi(\alpha, a)| < M \qquad (0 < \alpha < c/a),$$

and from condition (c)

$$\left|\phi\left(\alpha, \frac{c}{\alpha}\right)\right| < \frac{N}{c^{1+\rho}},$$

so that if we take as K_1 the greater of the two quantities M and $N/c^{1+\rho}$ we have by (64)

$$(65) \qquad |\phi(\alpha, x)| < K_1 \qquad (0 < x\alpha < c).$$

For $x\alpha \geqq c$ we have from condition (c)

$$(66) \qquad |\phi(\alpha, x)| \leqq \frac{N}{c^{1+\rho}} \leqq K_1 \qquad (x\alpha \geqq c).$$

For $\alpha = 0$ we have from condition (d)

$$(67) \qquad \phi(0, x) = 1.$$

* This is the only condition that is not satisfied by the convergence factor of Theorem IV. We may substitute for it the condition

$$(e') \qquad \phi'_x(\alpha, x) \geqq 0 \qquad (0 \leqq x\alpha \leqq c).$$

If (e) (or (e')) holds for all values of x and α, (f) is unnecessary.

† If we substitute (e') for (e), the inequality signs in (64) are reversed but this does not affect the reasoning.

Hence if we choose K greater than either of the quantities K_1 and 1, we have from (65), (66), and (67)

$$\phi(\alpha, x) < K \qquad (\alpha \geqq 0).$$

Let δ be an arbitrarily small positive quantity. Since $\phi(\alpha, x)$ by condition (a) is a continuous function of α for all values of $\alpha \geqq 0$, our theorem will be proved if we can show that a positive quantity q exists such that

$$(68) \qquad \left| \int_\mu^\nu f(x)\phi(\alpha, x)\,dx \right| < \delta \qquad (\nu > \mu \geqq q;\ \alpha \geqq 0).$$

Determine $m > a$ and such that

$$(69) \qquad \left| \int_\mu^\nu f(x)\,dx \right| < \eta = \frac{\rho c^\rho \delta}{L + 3K\rho c^\rho} \qquad (\nu > \mu \geqq m).$$

We first consider values of $\alpha > 0$. Integrating by parts we get

$$(70) \qquad \begin{aligned} \int_\mu^\nu f(x)\phi(\alpha, x)dx &= \left[\phi(\alpha, x)\int_\mu^x f(x)\,dx\right]_\mu^\nu - \int_\mu^\nu \left[\int_\mu^x f(x)\,dx\right]\phi'_x(\alpha, x)\,dx \\ &= \phi(\alpha, \nu)\int_\mu^\nu f(x)\,dx - \int_\mu^\nu \left[\int_\mu^x f(x)\,dx\right]\phi'_x(\alpha, x)\,dx. \end{aligned}$$

Let $s = c/\alpha$ and let μ and ν be any two quantities such that

$$m \leqq \mu < \nu.$$

Three cases must be considered:

$$(A) \qquad \mu < s < \nu,$$

$$(B) \qquad \nu \leqq s,$$

$$(C) \qquad \mu \geqq s.$$

Beginning with (A) we write

$$(71) \qquad \int_\mu^\nu \left[\int_\mu^x f(x)\,dx\right]\phi'_x(\alpha, x)\,dx = \int_\mu^s + \int_s^\nu = R_1 + R_2.$$

By virtue of (69) and conditions (e) and (g) we have for R_1

$$(72) \qquad \begin{aligned} |R_1| &\leqq \int_\mu^s \left| \left[\int_\mu^x f(x)\,dx\right]\phi'_x(\alpha, x) \right| dx < \eta \int_\mu^s [-\phi'_x(\alpha, x)]\,dx \\ &= \eta[\phi(\alpha, \mu) - \phi(\alpha, s)] \leqq \eta|\phi(\alpha, \mu)| + \eta|\phi(\alpha, s)| < 2K\eta \\ &\qquad \left(\nu > \mu \geqq m;\ 0 < \alpha < \frac{c}{m}\right). \end{aligned}$$

From condition (f) we have for R_2

$$(73)\qquad \begin{aligned} |R_2| &\leqq \int_s^\nu \left| \left[\int_\mu^x f(x)\,dx \right] \phi'_x(\alpha, x) \right| dx < \eta \int_s^\nu \frac{L}{x^{1+\rho}\alpha^\rho}\,dx \\ &= \frac{\eta L}{\alpha^\rho}\left[-\frac{1}{\rho x^\rho} \right]_s^\nu < \frac{\eta L}{\rho(s\alpha)^\rho} = \frac{\eta L}{\rho c^\rho}. \end{aligned}$$

From (70)–(73) we get

$$\begin{aligned} \left| \int_\mu^\nu f(x)\phi(\alpha, x)\,dx \right| &\leqq |\phi(\alpha, \nu) \int_\mu^\nu f(x)\,dx| + |R_1| + |R_2| \\ &< K\eta + 2K\eta + \frac{\eta L}{\rho c^\rho} = \eta \frac{L + 3K\rho c^\rho}{\rho c^\rho} = \delta. \end{aligned}$$

For case (B) we have

$$\begin{aligned} \left| \int_\mu^\nu \left[\int_\mu^x f(x)\,dx \right] \phi'_x(\alpha, x)\,dx \right| &\leqq \int_\mu^\nu \left| \left[\int_\mu^x f(x)\,dx \right] \phi'_x(\alpha, x) \right| dx \\ &< \eta \int_\mu^s [-\phi'_x(\alpha, x)]\,dx = \eta[\phi(\alpha, \mu) - \phi(\alpha, s)] < 2K\eta, \end{aligned}$$

which combined with (70) gives

$$\left| \int_\mu^\nu f(x)\phi(\alpha, x)\,dx \right| < |\phi(\alpha, \nu) \int_\mu^\nu f(x)\,dx| + 2K\eta < 3K\eta = \eta \frac{3K\rho c^\rho}{\rho c^\rho} < \delta.$$

Finally we have for case (C)

$$\begin{aligned} \left| \int_\mu^\nu \left[\int_\mu^x f(x)\,dx \right] \phi'_x(\alpha, x)\,dx \right| &\leqq \int_\mu^\nu \left| \left[\int_\mu^x f(x)\,dx \right] \phi'_x(\alpha, x) \right| dx \\ &< \eta \int_\mu^\nu |\phi'_x(\alpha, x)|\,dx < \eta \int_\mu^\nu \frac{L}{x^{1+\rho}\alpha^\rho}\,dx \leqq \eta \int_s^\nu \frac{L}{x^{1+\rho}\alpha^\rho}\,dx < \eta \frac{L}{\rho c^\rho}, \end{aligned}$$

which combined with (70) gives

$$\begin{aligned} \left| \int_\mu^\nu f(x)\phi(\alpha, x)\,dx \right| &< |\phi(\alpha, \nu) \int_\mu^\nu f(x)\,dx| + \eta \frac{L}{\rho c^\rho} \\ &< \eta \left[K + \frac{L}{\rho c^\rho} \right] = \eta \frac{L + K\rho c^\rho}{\rho c^\rho} < \delta. \end{aligned}$$

Hence in all three cases we have established the inequality (68), q being taken equal to m.

For $\alpha = 0$ the integral (63) becomes the integral (62), and since the latter is convergent, we can choose m_1 such that

$$\left| \int_\mu^x f(x)\phi(0, x)\,dx \right| = \left| \int_\mu^x f(x)\,dx \right| < \delta \qquad (\nu > \mu \geqq m_1).$$

To establish the inequality (68) we have now only to take q equal to the greater of the two quantities m and m_1.

Theorem V has been proved under the assumption that a is greater than zero. As in the treatment of summable integrals, however, the theorem can easily be extended to the case where a is any constant. We shall show this by proving the following corollary:

COROLLARY. *If, in Theorem* V, *a is any constant and the convergence factor $\phi(\alpha, x)$ satisfies conditions (a), (b), (c), (d), (e), and (f) for $x \geqq b$, where b is any positive constant, and conditions (a) and (d) for $a \leqq x \leqq b$, the theorem holds without change.*

Let

$$\int_a^b f(x)\,dx = B.$$

We have

$$A = \int_a^\infty f(x)\,dx = \int_a^b f(x)\,dx + \int_b^\infty f(x)\,dx$$

and consequently

$$\int_b^\infty f(x)\,dx = A - B.$$

Hence by Theorem V

$$\int_b^\infty f(x)\phi(\alpha, x)\,dx = G(\alpha)$$

is absolutely and uniformly convergent for all values of $\alpha \geqq 0$ and therefore defines a continuous function for such values.

Then

$$\int_a^b f(x)\phi(\alpha, x)\,dx + \int_b^\infty f(x)\phi(\alpha, x)\,dx = \int_a^\infty f(x)\phi(\alpha, x)\,dx$$

is absolutely and uniformly convergent for all values of $\alpha \geqq 0$ and therefore defines a continuous function for such values.

The convergence factor $e^{-x\alpha}$ satisfies the conditions of Theorem V. Hence if the integral

$$\int_a^\infty e^{kx} f(x)\,dx$$

is convergent, the integral

$$\int_a^\infty e^{(k-\alpha)x} f(x)\,dx$$

converges uniformly in the interval $\alpha \geqq 0$. If now we set

$$e^{(k-\alpha)} = m, \qquad e^k = m_1,$$

we have the theorem:

If the integral

$$\int_a^\infty m_1^x f(x)\,dx$$

is convergent, then the integral

$$\int_a^\infty m^x f(x)\,dx$$

converges uniformly in the interval $0 \leqq m \leqq m_1$.

This theorem is due to BONNET (cf. Liouville's Journal, vol. 14 (1849), p. 250).

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GENERAL THEORY OF APPROXIMATION BY FUNCTIONS INVOLVING A GIVEN NUMBER OF ARBITRARY PARAMETERS*

BY

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Introduction.

The following is a special case of the problem to be considered in this paper: Given a function $\phi(x)$ of the real variable x, continuous on a finite interval (a, b); to determine the polynomial $p(x)$ of *given* degree n, which gives the closest approximation to the given function ϕ on the interval (a, b). This problem becomes definite only when the meaning of the phrase "closest approximation" has been precisely stated, and the meaning adopted will depend on the ultimate object in view.

TCHEBYCHEV seems to have been the first to consider this problem.† He regarded that polynomial as giving the best approximation, which rendered the maximum of $|p(x) - \phi(x)|$, as x varied over (a, b), as small as possible. A different point of view would lead one to seek a polynomial of the given degree which rendered as small as possible the expression

$$\int_a^b (p - \phi)^2 dx,$$

or the expression

$$\int_a^b |p - \phi| dx,$$

etc. In all of these cases, and in the more general ones to be referred to presently, the problem consists in the determination of a set of parameters a_i of a function $f(x; a_0, a_1, \cdots, a_n)$ of the real variable x, such that the maximum of $|f(x)|$, as x varies over a given interval, shall be as small as possible.

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† TCHEBYCHEV, *Sur les questions de minima qui se rattachent à la représentation approchée des fonctions*, Bulletin de la Société phys.-math. de l'Académie impériale des Sciences de St. Pétersbourg, vol. 16 (1858), 145–149; Mémoires de l'Academie impériale de Sciences de St. Pétersbourg, vol. 9 (1859), pp. 201–291.

This problem TCHEBYCHEV himself (loc. cit.) seems to have regarded as the *general type of problem connected with the approximate representation of functions*, although he seems to have done little with this general problem beyond stating it. He confines himself to a detailed discussion of the case where the function f takes the form $p(x) - \phi(x)$, or $r(x) - \phi(x)$, where $p(x)$ is as before a polynomial of given degree, and where $r(x)$ is a rational function of which the degrees of the numerator and denominator are prescribed. The parameters a_i referred to above are then, of course, the coefficients of the various powers of x. The fact that the degrees of the polynomials are prescribed, i. e. the number of arbitrary parameters, is essential to this type of problem.

Certain generalizations of the problems treated by TCHEBYCHEV at once suggest themselves. On the one hand, the functions $p(x)$ or $r(x)$ might be replaced by any function of a given *class* $\mathfrak{C}$ of functions $g(x;\ a_0, a_1, \cdots, a_n)$. On the other hand, the form of the function f to be considered might be made more general. In the case actually discussed by TCHEBYCHEV, f was simply $g - \phi$, where ϕ was a given continuous function. Both generalizations are included, if we identify f with a function Ug, where Ug denotes the result of operating on g with some *functional operation* U. If then there exists a set of parameters a_i which renders the maximum of $|Ug|$, as x varies over a certain given finite interval, as small as possible, we will call the resulting function g *a function of approximation in the class* $\mathfrak{C}$ *with reference to* U.

The fundamental theoretical problems that now present themselves are as follows: 1) Given an operation U and a class $\mathfrak{C}$, does a function of approximation in $\mathfrak{C}$ with reference to U exist? 2) What are necessary and sufficient conditions that a function in $\mathfrak{C}$ be a function of approximation in $\mathfrak{C}$ with reference to U?

TCHEBYCHEV has given answers to these problems (loc. cit.) for the special cases already attributed to him, where Ug has the form $g - \phi$, and where g is either a polynomial or a rational fractional function of a specified kind. His methods, which lack the degree of rigor required at this day, have recently been revised by KIRCHBERGER,* and still more recently BOREL † has given elegant proofs of those theorems of TCHEBYCHEV which relate to *polynomials* of approximation.

It is the object of the present paper to give a two-fold generalization of TCHEBYCHEV's theorems; the latter will be found as special cases of our theory (§ 5, *a*). The method followed is largely that of BOREL. In § 1 we state the general problem with the necessary precision and prove the existence of a solution under very general conditions. In §§ 2, 3, 4, restricting ourselves to the

* KIRCHBERGER, *Über Tchebychefsche Annäherungsmethoden*, Dissertation, Göttingen, 1902; in abstract, Mathematische Annalen, vol. 57 (1903), pp. 509–540.

† BOREL, *Leçons sur les fonctions de variables réelles, etc.*, 1905, pp. 82–88.

case in which the parameters enter linearly into the functions g and to the case where we have $Ug = g - \phi$, we prove again the existence of a solution, derive necessary and sufficient conditions which a solution must satisfy, and prove that the solution is unique. The results of these sections are summarized in Theorems 4 and 5. Theorem 4 will be found to express a characteristic property of a general class of "functions of approximation," which seems to have been recognized hitherto only for the case of polynomials and rational functions of approximation. After considering in § 5 certain special cases of the theory developed in the preceding sections, we extend the theory further in § 7 to include the case in which Ug has the form $Vg - \phi$, where V is any one-valued, distributive operation. This makes possible the application of the theory to the approximate representation of functions restricted only to satisfy certain functional equations of a general form; some of these are referred to in § 8, where we obtain as an illustration a theorem concerning the polynomial of approximation of given degree for a linear differential equation with constant coefficients. In § 6 we show how the problem may be formulated analytically.

I. The problem in general.

§ 1. *The general existence theorem.*

Let $f = f(x) = f(x;\ a_0, a_1, a_2, \cdots, a_n)$ be a function of the real variable x and $n+1$ parameters a_i, subject to the following conditions A:

$A1$) $f(x;\ a_0, a_1, a_2, \cdots, a_n)$ shall be defined as a one-valued and continuous function of its arguments for every x of a finite interval (a, b) and for all real finite values of the parameters a_i.*

$A2$) For every positive number M there shall exist an N, such that if the relation

$$|f(x;\ a_0, a_1, a_2, \cdots, a_n)| \leqq M$$

be satisfied for all values of x on (a, b), then the parameters a_i all satisfy the relations

$$|a_i| \leqq N \qquad (i = 0, 1, 2, \cdots, n).$$

Now suppose the parameters a_i to be undetermined but fixed. As x varies over (a, b), $|f|$ will attain its maximum value m at least once. This maximum $m(a_0, a_1, a_2, \cdots, a_n)$ by $A1$) is a continuous function of the a_i. Hence as the a_i vary over all real finite values, m certainly has a lower bound μ which is either positive or zero. Our problem is to ascertain whether there exists a

* Throughout this paper the word interval will always imply that the endpoints are included. This condition $A1$) could, moreover, for the purposes of this section be stated more generally by defining $f(x)$ merely on a set of points E on (a, b), with the restriction that E be *perfect*.

finite set of parameters α_i for which this lower bound is actually attained, i. e., such that we have

$$m(\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_n) = \mu.$$

If such a set exists, we will call it a *minimizing set of parameters for* f. The problem of determining such a set of parameters would seem to arise in connection with most, if not all, problems of approximation by means of functions with a *given* number of arbitrary parameters.

The question concerning the existence of a minimizing set of parameters can be answered in the affirmative. For, if we choose any particular set of parameters, say $a_0', a_1', a_2', \cdots, a_n'$, and denote $m(a_0', a_1', a_2', \cdots, a_n')$ by M, we know that μ is at most equal to M. We can then without loss of generality confine ourselves to functions f such that we have throughout (a, b)

$$|f| \leqq M.$$

But condition $A2$) requires, then, that there exist a number N such that all the parameters of f satisfy the relations

$$(1) \qquad |a_i| \leqq N \qquad (i = 0, 1, 2, \cdots, n).$$

Hence the parameters may be restricted to the finite closed domain defined by the relations (1), and in this domain the function $m(a_0, a_1, a_2, \cdots, a_n)$ attains its lower bound μ for at least one set of values $\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_n$. Hence we have

THEOREM 1. *There exists at least one minimizing set of parameters for any function satisfying conditions* A.

As an immediate corollary of this theorem, we obtain an existence theorem for the very general class of approximation problems referred to in the *Introduction*. Suppose there is given a class of functions, $g(x; b_0, b_1, b_2, \cdots, b_n)$, with the arbitrary parameters b_i, such that a function g is fully determined as soon as the values of the parameters b_i are fixed. Then as we have seen, the problem of determining, on a finite interval, that function g which will give the *best* approximation to a given function (or to a function satisfying a given functional equation) leads to the determination of a minimizing set of parameters for a function Ug, where Ug denotes the function obtained from g by some functional operation U. If then we identify the function Ug with the function f just considered, Theorem 1 gives the desired information concerning the existence of a solution. The parameters b_i of g will in general be the parameters a_i of $f = Ug$. However, it should be noted that in performing the operation U some of the parameters b_i may disappear, and new arbitrary parameters may be introduced. The parameters b_i which actually appear in Ug we will call the *effective* parameters of g with reference to U. Obviously in applying the conditions A to Ug the effective parameters b_i, together with whatever new param-

eters may have been introduced, play the rôle of a_i's. Any function g, the effective parameters of which have the values belonging to a minimizing set for Ug, we call a *function g of approximation with reference to U*. We have, then,

THEOREM 2. *There exists a function g of approximation with reference to the operation U, provided $f = Ug$ satisfies conditions A.*

II. FUNCTIONS OF APPROXIMATION FOR GIVEN FUNCTIONS.

§ 2. *Another form of the existence theorem.*

We now assign to the function f of the preceding section a more explicit but still very general form. We suppose given a set of $n + 1$ functions,

$$s_0, s_1, s_2, \cdots, s_n,$$

of the real variable x. The function s_i we call the elementary function of rank i. By means of some or all of these functions s_j we form a new function,

$$S_k = a_0 s_0 + a_1 s_1 + \cdots + a_k s_k,$$

which we call an S-function of rank k, if s_k is the elementary function of highest rank occurring in S_k. By allowing the parameters a_j to assume all real finite values we obtain the class of S-functions of rank not higher than k, which we will denote by $\mathfrak{S}_k$. It is clear that $\mathfrak{S}_l$ is contained in $\mathfrak{S}_k$, if l is less than k.

We suppose, further, that there is given a function ϕ of x. The function $y = S_n - \phi$ is to play the rôle of the function f of the preceding section. The functions s_j and ϕ are subject to the following conditions B:

$B1$) The functions s_j and ϕ are one-valued and continuous at every point of (a, b).

$B2$) If one parameter a_j is different from zero, the function S_k does not vanish at more than k points of (a, b).

$B3$) For no set of parameters a_j does the function $y = S_n - \phi$ vanish at every point of (a, b).

Condition $B2$) requires a word of explanation. It implies that each point x_1 of (a, b) at which S_k vanishes can be enclosed within a finite interval throughout which (except at x_1) S_k is different from zero. There are then two cases to consider. If S_k changes sign as x passes through x_1, we call x_1 a *simple* zero of S_k; if S_k does not change sign as x passes through x_1, we call x_1 a *double* zero. In applying condition $B2$) *every double zero must be counted twice.**

We restate the discussion in terms of the new symbols as follows. As x varies over (a, b), the function $|y| = |S_n - \phi|$, continuous on (a, b), attains its maximum value m at least once; $m = m(a_0, a_1, a_2, \cdots, a_n)$ is a continuous

* If a or b is a zero of S_k, it counts as simple.

function of the parameters a_j. For we may vary the a_j by a sufficiently small amount to insure that throughout (a, b), where we have

$$(1) \qquad y = a_0 s_0 + a_1 s_1 + a_2 s_2 + \cdots + a_n s_n - \phi,$$

y, and hence m, will change by less than ϵ. We seek to determine a function y in $\mathfrak{S}_n$,

$$\sum_n = \alpha_0 s_0 + \alpha_1 s_1 + \alpha_2 s_2 + \cdots + \alpha_n s_n,$$

such that $m(\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_n)$ shall be equal to the lower bound μ of m; in other words, we seek a minimizing set of parameters for y. If such a minimizing set exists, we call $\sum_n$ a *function of approximation in* $\mathfrak{S}_n$ *for* ϕ.

We have thought it desirable to state these conditions B independently of conditions A, in order that the remainder of the discussion might gain in unity. It is not difficult to show now, however, that conditions B imply A, and hence establish the existence of a solution to our problem. This we shall do by showing that for all functions (1) satisfying a relation $|y| \leqq M$ (M being finite and greater than μ), each of the parameters satisfies a relation

$$|a_j| \leqq N_j,$$

where the N_j are finite numbers.

We note first that a function

$$S_n(x) = \sum_{j=0}^{n} a_j s_j(x)$$

may be determined and is indeed uniquely determined by its values c_h at $n+1$ distinct points ξ_h $(h = 0, 1, \cdots, n)$ on (a, b), i. e., by the equations

$$c_h = \sum_{j=0}^{n} a_j s_j(\xi_h) \qquad (h = 0, 1, \cdots, n).$$

For the determinant

$$|s_j(\xi_h)| \qquad (j, h = 0, 1, \cdots, n),$$

does not vanish; otherwise a function S_n, not identically zero, would exist which vanishes at the $n+1$ points ξ_h, in contradiction to $B2$). Denoting by

$$S_n^{(i)}(x) = \sum_{j=0}^{n} a_j^{(i)} s_j(x)$$

that function in $\mathfrak{S}_n$ which has at the ξ_h the values $c_h^{(i)}$ given by the equations

$$c_h^{(i)} = 0 \qquad (i \neq h),$$
$$c_i^{(i)} = 1,$$

we can express any function S_n having any values c_h at ξ_h by the formula*

$$S_n(x) = \sum_{i=0}^{n} c_i S_n^{(i)}(x) = \sum_{i=0}^{n} \sum_{j=0}^{n} c_i a_j^{(i)} s_j(x).$$

* This is a generalization of LAGRANGE's formula of interpolation.

Now any function S_n satisfying the relation

$$|S_n(x) - \phi(x)| \leqq M$$

satisfies the relation

$$|S_n(x)| \leqq M + \lambda,$$

where λ is the maximum of $|\phi(x)|$ on (a, b). But, by the formula just given, every function S_n can be written in the form

$$S_n(x) = \sum_{i=0}^{n} S_n(\xi_i) S_n^{(i)}(x),$$

or

$$\sum_{j=0}^{n} a_j s_j(x) = \sum_{j=0}^{n} s_j(x) \sum_{i=0}^{n} S_n(\xi_i) a_j^{(i)}.$$

Hence we have

$$a_j = \sum_{i=0}^{n} S_n(\xi_i) a_j^{(i)},$$

and so

$$|a_j| \leqq (M + \lambda) \sum_{i=0}^{n} |a_j^{(i)}| \leqq N_j$$

provided

$$|S_n(x) - \phi(x)| \leqq M.$$

This gives the desired condition. Hence we have the following result:

THEOREM 3. *There exists a function of approximation in $\mathfrak{S}_n$ for ϕ, provided conditions B are satisfied.*

§ 3. *A necessary condition for a function of approximation in $\mathfrak{S}_n$.*

We have just shown (Theorem 3) that there exists at least one function of approximation in $\mathfrak{S}_n$ for ϕ under conditions B. Let this function be $\sum_n$, and as before denote $\sum_n$ by y. Then $|y|$ attains its maximum μ in at least one point of (a, b). Let x' be any point for which $y = \mu$ and x'' any point for which $y = -\mu$. The sets $[x']$ and $[x'']$, consisting of all points x' and x'' respectively, are evidently closed since y is continuous on (a, b). Further, we may determine a number 2δ such that the oscillation of y in any interval of length $\leqq 2\delta$ will be less than a positive ϵ, less than μ. The distance of any point x' from the nearest x'' will then be greater than 2δ. Hence *it is possible to divide the interval (a, b) into a finite number of adjoining subintervals,*

$$L_1, L_2, L_3, \cdots, L_p,$$

such that no L_q contains points of both the sets $[x']$ and $[x'']$, and such that one of two consecutive intervals L_q will contain points of $[x']$ and the other will contain points of $[x'']$. Moreover, the points $\zeta_1, \zeta_2, \zeta_3, \cdots, \zeta_{p-1}$ which

divide (a, b) into the intervals L_q must be chosen distinct from any of the points x' or x'', and this evidently is always possible.

The above discussion will apply to any function in $\mathfrak{S}_n$, and it should be noted that the number p of intervals L_q is fixed for a given S_n. But we can now prove the following condition:

A necessary condition that a given function $\sum_n$ in $\mathfrak{S}_n$ be a function of approximation in $\mathfrak{S}_n$ for ϕ is that the number of intervals L_q exceed $n+1$.

We shall show that under the hypothesis $p \leqq n+1$ we can so construct a function S_n in $\mathfrak{S}_n$ that the maximum of $|S_n - \phi|$ on (a, b) is less than μ. Let

$$T_{p-1} = c_0 s_0 + c_1 s_1 + c_2 s_2 + \cdots + c_{p-1} s_{p-1}$$

be a function in $\mathfrak{S}_{p-1}$ which vanishes at each of the points ζ_i $(i = 1, 2, \cdots, p-1)$. That such a function exists is shown by an argument similar to the one given on p. 336, which makes it clear that only the ratios $c_0 : c_1 : \cdots : c_{p-1}$ are determined by the conditions thus far imposed on T_{p-1}. We may then write

$$T_{p-1} = \eta \cdot T'_{p-1},$$

where T'_{p-1} is fully and uniquely determined, but where η is any arbitrary constant.*

Now, since T_{p-1} vanishes at $p-1$ different points on (a, b) and since by $B2$) it vanishes nowhere else, it follows that T_{p-1}, being continuous, changes sign as x passes from one interval L_q into the next, and only then. We then determine the sign of η so that T_{p-1} is positive within every interval L_q containing a point x' and negative within every interval L_q containing a point x''.

Make every x' the center of an interval I of length δ and every x'' the center of an interval J of length δ, and let K be the set of points on (a, b) not included in any I or J. Then the upper bound μ' of $|\sum_n - \phi|$ on K is certainly less than μ. Now choose $|\eta|$ so small that $|T_{p-1}| = |\eta|\,|T'_{p-1}|$ is less than the smaller of the numbers $\mu - \epsilon$ and $\mu - \mu'$ at every point of (a, b).

In any interval I we now have

$$\mu - \epsilon < \sum_n - \phi \leqq \mu$$

and

$$0 < T_{p-1} < \mu - \epsilon,$$

since no I contains a point ζ_i. Whence throughout any I the relation

$$0 < \sum_n - T_{p-1} - \phi < \mu$$

is satisfied, and similarly in any interval J

$$0 > \sum_n - T_{p-1} - \phi > -\mu.$$

* If $p = 1$, simply take T_{p-1} equal to η.

Finally, at every point of K we have

$$|\Sigma_n - \phi| \leqq \mu' \qquad \text{and} \qquad |T_{p-1}| < \mu - \mu',$$

and therefore

$$|\Sigma_n - T_{p-1} - \phi| < \mu.$$

Hence the function $S_n = \Sigma_n - T_{p-1}$, which is in $\mathfrak{S}_n$ under the hypothesis $p \leqq n+1$, is such that at every point of (a, b) the relation

$$|S_n - \phi| < \mu$$

holds, which is the desired result.

§ 4. *The sufficiency of the condition and the uniqueness of the solution.*

Conversely, let Σ_n be any function in $\mathfrak{S}_n$ for which the necessary condition of § 3 is satisfied. There exist then at least $n+2$ points

$$x_1 < x_2 < x_3 < \cdots < x_{n+2}$$

on (a, b) at which we have

$$|\Sigma_n(x_\kappa) - \phi(x_\kappa)| = \mu \qquad (\kappa = 1, 2, \cdots, n+2),$$

and

$$\Sigma_n(x_\kappa) - \phi(x_\kappa) = -[\Sigma_n(x_{\kappa+1}) - \phi(x_{\kappa+1})] \qquad (\kappa = 1, 2, \cdots, n+1).$$

Now suppose there were another function

$$\Sigma_n' = \alpha_0' s_0 + \alpha_1' s_1 + \cdots + \alpha_n' s_n$$

in $\mathfrak{S}_n$, such that on (a, b) the maximum value of $|\Sigma_n' - \phi|$ were less than, or even merely equal to, μ. Then the function

$$(\Sigma_n - \phi) - (\Sigma_n' - \phi) = \Sigma_n - \Sigma_n'$$

would be alternately positive (or zero) and negative (or zero) at the points x_κ. But this would require the continuous function $\Sigma_n - \Sigma_n'$ to have on (a, b) at least $n+1$ simple zeros (or their equivalent, if double zeros occur), which by $B2$) is impossible, unless all the parameters of

$$\Sigma_n - \Sigma_n' = (\alpha_0 - \alpha_0') s_0 + (\alpha_1 - \alpha_1') s_1 + \cdots + (\alpha_n - \alpha_n') s_n$$

vanish. But this requires Σ_n and Σ_n' to be identical. Hence we reach, in connection with the result of § 3, the following conclusions:

THEOREM 4. *A necessary and sufficient condition that a function Σ_n in $\mathfrak{S}_n$ be a function of approximation in $\mathfrak{S}_n$ for ϕ is that the number of intervals L_q exceed $n+1$.*

THEOREM 5. *There exists one and only one function of approximation in* $\mathfrak{S}_n$ *for a function* ϕ.*

§ 5. *Special cases of the general theory.*

a) The theorems of Tchebychev on approximation by polynomials and rational functions.

If in the preceding theory we place $s_i = x^i$, that is, if we identify the class $\mathfrak{S}_n$ with the class of polynomials of degree not higher than n, we are led at once to the results of TCHEBYCHEV already referred to in the *Introduction.* Theorems 4 and 5 then give:

There exists one and only one polynomial of approximation of degree n for any function ϕ *on any given finite interval throughout which* ϕ *is continuous; and a necessary and sufficient condition that a given polynomial of degree n be a polynomial of approximation of that degree is that the number of intervals* L_q *exceed* $n + 1$.

A similar result was proved originally by TCHEBYCHEV † for the case of a rational function $R_n(x)$, in which the denominator $D(x)$ is fixed and the numerator is of prescribed degree n. We need here only to place $s_i = x^i/D(x)$ to obtain his theorem, which is analogous to the one just given. Conditions B will be satisfied provided no root of $D(x) = 0$ lies on (a, b).

b) Approximation by finite trigonometric series.

If we place $s_i = \cos ix$, our functions S_n assume the form

$$C_n = a_0 + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx.$$

Such a function satisfies the condition $B2$) concerning the number of zeros, provided the length of the interval (a, b) does not exceed π. For every function C_n can be expressed as a polynomial in $\cos x$ of degree not higher than n. If ϕ also satisfies $B3$), *Theorems* 4 *and* 5 *will apply to the approximate representation of a given continuous function by means of functions of the form* C_n. Other types of trigonometric series having similar properties are as easily obtainable.

c) Other special types of functions having similar properties might be discussed, but it hardly seems worth while to go any further into details. Series of *Legendre polynomials* and series of *hyperbolic functions* are perhaps the simplest that suggest themselves, for which the crucial condition $B2$) can be readily satisfied. The possible application of Theorems 4 and 5 appears to be wide, in view of the general character of the available classes $\mathfrak{S}_n$.

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*Since the above was written, a special case of this theorem, concerning the representation of functions by finite trigonometric series, has been announced by M. MAURICE FRÉCHET, *Sur l'approximation des fonctions par des suites trigonométriques limitées*, Comptes Rendus de l'Académie des Sciences de Paris, vol. 144, no. 3 (January 21, 1907), p. 124. Cf. also § 5*b*) above.

† Loc. cit.

§ 6. *Analytic formulation of the problem.*

Conditions B are sufficient to give an analytic formulation to the problem of determining the function of approximation in $\mathfrak{S}_n$ for a given continuous function ϕ, in case the functions S_n and ϕ possess derivatives as to x which are one-valued and continuous everywhere except at a finite number of points $\tau_1, \tau_2, \cdots, \tau_\lambda$ on (a, b), with the property that for every τ_i there exists an e_i such that

$$\lim_{x=\tau_i}(x-\tau_i)^{e_i}\cdot\frac{d}{dx}[S_n(x)-\phi(x)]=0.$$

For Theorem 4 implies the existence of $n+2$ points

$$x_1, x_2, \cdots, x_{n+2}$$

on (a, b) for which we have

$$|\textstyle\sum_n(x_i)-\phi(x_i)|=\mu \qquad (i=1, 2, \cdots, n+2).$$

Since at these points $S_n-\phi$ is either a maximum or a minimum, we have also

$$\frac{d}{dx}[\textstyle\sum_n-\phi]=0 \quad \text{when } x=x_1,$$

except possibly when one of the points x_i coincides with $a, \tau_1, \tau_2, \cdots, \tau_\lambda$, or b.

If then we place

$$g(x)=(x-a)(x-\tau_1)^{e_1}(x-\tau_2)^{e_2}\cdots(x-\tau_\lambda)^{e_\lambda}(x-b),$$

a function of approximation in $\mathfrak{S}_n$ would have to satisfy the following system of equations:

$$(S_n-\phi)^2-\mu^2=0, \qquad g(x)\frac{d}{dx}(S_n-\phi)=0, \quad (x=x_i;\ i=1, 2, \cdots, n+2).$$

These $2n+4$ equations contain just $2n+4$ unknowns, namely, the $n+1$ parameters a_j, the $n+2$ quantities x_i, and the quantity μ. Among the sets of parameters satisfying this system must occur the set α_j giving the function of approximation.

III. Functions of approximation for functional equations.

§ 7. *Extension of the theory.*

The results obtained in the previous paragraphs may be extended readily to the problem of finding functions of approximation for a certain general class of functional equations. If we have given a functional equation $U(y)=0$ and a class of functions $g(x;\ a_0, a_1, \cdots, a_n)$, we define a *function of approximation for $U(y)=0$ in the given class* to be a function g which *renders the maximum of $|U(g)|$ on (a, b) as small as possible.*

Theorems 4 and 5 may be extended to include this type of problem under the following conditions:

We suppose given a set of $m+1$ *elementary* functions,

$$t_0, t_1, t_2, \cdots, t_m,$$

and form, in a way precisely analogous to the one previously used (§ 2), the various classes $\mathfrak{T}_k$ of functions T_k of rank not higher than k,

$$T_k = a_0 t_0 + a_1 t_1 + \cdots + a_k t_k \qquad (k = 0, 1, 2, \cdots, m).$$

We suppose given further a function $\phi(x)$, and a functional operation V which is effective in a single-valued distributive* way on the functions of $\mathfrak{T}_m$, the resulting values $V(T_m)$ being themselves functions of one variable. *Our treatment in* §§ 2, 3, 4 *applies to the case now under consideration in which, in view of the distributive character of* V, *the function* $U(g)$ *has the form*

$$V(T_m) - \phi = a_0 V(t_0) + a_1 V(t_1) + \cdots + a_m V(t_m) - \phi.$$

But this function has the same form as the one already considered in §§ 2, 3, 4, if we identify $V(t_i)$ with s_i, and $V(T_k)$ with S_k. If then we restate conditions B with the substitutions just indicated, and call the new conditions C, we may extend Theorems 4 and 5 to the new type of problem. Before doing so, however, the following remarks should be made regarding the details of the contemplated extension.

If any of the $V(t_i)$ $(i = 0, 1, \cdots, m)$ are zero identically on (a, b), the corresponding parameters a_i are *non-effective* (cf. § 1). Only the *effective* parameters of a T_k play any essential part in the discussion, and hence in stating condition $C2$) (corresponding to $B2$)) the number k must evidently be replaced by $e_k - 1$, where e_k is the number of effective parameters of T_k with respect to V.

If in any function T_k we place equal to zero all the non-effective parameters with respect to V, we call the resulting function *reduced* with respect to V. A reduced function contains just e_k terms and all the parameters occurring in it are effective. In extending our theory we must be careful to specify that the functions T_k with which we operate are to be reduced functions, whenever this is necessary; as, for instance, in the determination of the functions $T_k^{(i)}$ satisfying the relations $V(T_k^{(i)}(\xi_j)) = 1$ or 0, according as we have $i = j$ or $i \neq j$ in the extended existence theorem (cf. p. 336).

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* A functional operation A is said to be *distributive*, if we have $A(\alpha + \beta) = A(\alpha) + A(\beta)$ and $A(c\alpha) = cA(\alpha)$, for every constant value c and for every two functions α, β in the realm of definition of A. Cf. PINCHERLE, *Funktionaloperationen und -gleichungen*, Encyklopädie der Mathematischen Wissenschaften, vol. 2 A 11 (1906), p. 767. In our case we have the notion of an operation which is one-valued and distributive *with reference to a class* of functions, nothing being specified *concerning its behavior or existence* for functions not in the class.

The treatment of this more general case, which evidently reduces to the one first considered when V is the identical operation (i. e., $V(T_m) = T_m$), is throughout essentially identical with the treatment already given in detail for the simpler case. The conditions C may now be stated thus:

C 1) The functions $V(t_i)$ $(i = 0, 1, 2, \cdots, m)$ and ϕ are one valued and continuous at every point of (a, b).

C 2) If one effective parameter a_j is different from zero, the function $V(T_k)$ does not vanish at more than $e_k - 1$ points on (a, b).

C 3) For no set of parameters a_j does the function $y = V(T_m) - \phi$ vanish at every point of (a, b).

A function T_m which renders as small as possible the maximum of $|V(T_m) - \phi|$ on (a, b) we call a function of approximation in $\mathfrak{T}_m$ *with reference to* V *and* ϕ. The extended theorems then are as follows:

THEOREM 6. *A necessary and sufficient condition that a function T_m in $\mathfrak{T}_m$ be a function of approximation in $\mathfrak{T}_m$ with reference to V and ϕ is that the number of intervals M_q exceed the number of effective parameters in T_m.*

Here the intervals M_q are constructed with reference to the function $V(T_m) - \phi$, just as the intervals L_q were constructed with reference to the function $S_m - \phi$.

THEOREM 7. *The effective parameters of a function of approximation in $\mathfrak{T}_m$ with reference to V and ϕ are uniquely determined; i. e., there exists one and only one reduced function of approximation in $\mathfrak{T}_m$ with reference to V and ϕ.*

One or two further details concerning the proof of Theorem 6 require mention. If as in the previous treatment we denote by $\zeta_1, \zeta_2, \cdots, \zeta_{p-1}$ the points which divide (a, b) into the p intervals M_q, it is necessary to construct a function T_l which vanishes at the $p - 1$ points ζ_i and nowhere else on (a, b) (cf. p. 338). This is clearly possible, provided there exists a number $e_l = p$. That such a number does exist is readily seen from the fact that the sequence of numbers e_i $(i = 0, 1, \cdots, m)$ contains each of the integers from 1 to e_m at least once; for we evidently have $e_{i+1} = e_i$ or $e_{i+1} = e_i + 1$ according as a_{i+1} is not or is an effective parameter. The proof, following exactly the lines previously laid down, then leads to the determination of a function T_l satisfying the relation $|V(T_m) - V(T_l) - \phi| < \mu$. But since V is distributive, this implies the relation $|V(T_m - T_l) - \phi| < \mu$, and $T_m - T_l$ under the hypothesis $p \leqq e_m$ is a function in $\mathfrak{T}_m$, which serves to establish the necessity of the condition under consideration. The latter remark applies also to the proof of the sufficiency of the condition. To go through all the details of the proofs of the last theorems appears superfluous.

It is readily seen also that the analytic formulation of the problem given in § 6 may be extended at once to the present case.

8. *Application of the extended theorems.*

In seeking functional equations to which Theorems 6 and 7 apply, we should notice that a particular form of equation is prescribed, viz., $V(y) - \phi = 0$, where y is the unknown function of x and where V must be distributive. Beyond this it is only necessary that conditions C be satisfied. The variety of distributive operations is very large. As examples of interest we may mention those which give rise to linear differential equations with constant or variable coefficients, linear difference equations, linear integral equations, etc.

As a special case, we may consider the following:

Let

$$c_0 \frac{d^\nu y}{dx^\nu} + c_1 \frac{d^{\nu-1} y}{dx^{\nu-1}} + \cdots + c_\nu y - \phi(x) = 0 \tag{1}$$

be a linear differential equation with constant coefficients c_i, $(c_\nu \neq 0)$, and let the class $\mathfrak{T}_m$ be the class of polynomials of degree m. It is clear that conditions C will be satisfied, provided $\phi(x)$ is not identical on (a, b) with any polynomial of degree equal to (or lower than) m. We have then, from Theorems 6 and 7,

THEOREM 8. *There exists one and only one polynomial of approximation of degree m for any linear differential equation* (1) *under the conditions specified, and a necessary and sufficient condition that a given polynomial of degree m be a polynomial of approximation of this degree is that the number of intervals M_q exceed $m + 1$.*

PRINCETON UNIVERSITY,
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ON DERIVATIVES OVER ASSEMBLAGES*

BY

E. R. HEDRICK

§ 1. *Introduction.*

The notion of upper and lower derivatives was introduced by DU BOIS-REYMOND and DINI; † they have been extensively studied by these writers, by SCHEEFFER, ‡ BAIRE, § LEBESGUE ‖ and others. ¶ It is the purpose of this paper to study a concept which will be called the *derivative with respect to an assemblage*, and to extend the known results concerning derivatives, etc.

A proposition which results from this study is that the existence of a *continuous* assemblage derivative insures the existence of the derivative in the usual sense (theorem 4, § 5); this has been made the center of discussion on account of its beauty and applicability. Some further steps are taken, but the evident extensions, some of which are immediate, have been reserved for possible future presentation.

§ 2. *Upper, lower, and assemblage derivatives.*

Given a function $f(x)$ defined ** at the points of an interval $a \leqq x \leqq b$, †† we may extend the notions of continuity, oscillation, approach to limits, and so on, by considering only those values of x which belong to any given assemblage E. Consider in particular the ordinary difference quotient

$$Q(x, h) = \frac{f(x+h) - f(x)}{h}.$$

* Presented to the Society (New York) April 28, 1906. Received for publication January 23, 1907.

† See for example DU BOIS-REYMOND, Mathematische Annalen, vol. 16 (1880), p. 115; DINI, German translation: *Grundlagen für eine Theorie der Functionen einer reellen Grösse*, etc.

‡ Acta Mathematica, vol. 5 (1884), pp. 52, 183, 279.

§ BAIRE, *Thèse, Sur les fonctions de variables réelles*, Annali di Mathematica, ser. 3, vol. 3 (1899); and *Leçons sur les fonctions discontinues*, 1905.

‖ LEBESGUE, *Sur l'intégration et la recherche des fonctions primitives*, 1904.

¶ For example, PASCH, Mathematische Annalen, vol. 30 (1887). I shall refer also to BOREL, *Leçons sur les fonctions de variables réelles*, 1905.

** In this paper it will be understood that the functions used are *single valued* at points at which they are said to be defined.

†† The function may be defined, not for all points of an interval, but for points of any assemblage which includes E.

If E is dense at x_0 both to right and left,* four numbers will be obtained by taking the greatest and least limits of $Q(x_0, h)$ as h approaches zero from the right hand or from the left hand respectively; these will be called the upper [or lower] right hand [or left hand] derived numbers,† with respect to E and will be denoted by

$$D^r_{(E)}f(x_0), \qquad D_{r(E)}f(x_0), \qquad D^l_{(E)}f(x_0), \qquad D_{l(E)}f(x_0).$$

For example, if $M[x_0, d, E, Q(x_0, h)]$ denotes the upper limit of $Q(x_0, h)$ for values of h in the interval $0 < h \leqq d$ for which $x_0 + h$ lies in E, then

$$D^r_{(E)}f(x_0) = \underset{d=0}{\text{lower limit}} \{ M[x_0, d, E, Q(x_0, h)] \}$$

If these four derived numbers are all equal, we shall call their common value *the derivative with respect to* E, and we shall denote it by $D_{(E)}f(x_0)$. If E contains no points to the left [right] of x_0, $D^l_{(E)}$ and $D_{l(E)}$ [$D^r_{(E)}$ and $D_{r(E)}$] are meaningless, in which case they are to be neglected in the preceding definition.

We may now state a fundamental theorem:

THEOREM 1. *If $f(x)$ is defined and continuous for all points of an interval $(x_0 - \epsilon, x_0 + \epsilon)$ about $x = x_0$, there exists an assemblage E_r dense at x_0 on the right, for which the derivative of $f(x)$ at x_0 with respect to E_r exists and has any preassigned value λ between $D^r f(x_0)$ and $D_r f(x_0)$, the upper and lower right-hand derived numbers with respect to the continuum, i. e.:*

$$D_{(E_r)}f(x_0) = \lambda, \qquad D^r f(x_0) \geqq \lambda \geqq D_r f(x_0);$$

[or also

$$D_{(E_l)}f(x_0) = \mu, \qquad D^l f(x_0) \geqq \mu \geqq D_l f(x_0).]$$

The truth of this statement is quite obvious from the *continuity* of $Q(x_0, h)$ since $Q(x_0, h)$ actually takes on for some $h < d$ any value between its upper and its lower limits in the interval $0 < h < d$. The cases $\lambda = D_r$ or $\lambda = D^r$ are included even when $D_r = \pm\infty$, etc.

In any case, if $f(x)$ is merely *defined*, an assemblage E_r exists for which $D_{(E_r)}f(x) = D^r f(x)$.‡ And evidently the theorem holds when $D^r f(x)$, etc. are taken with respect to any assemblage E whatever, instead of with respect to the continuum.

If the derivative of $f(x)$ exists at a point $x = x_0$ with respect to each of a *finite* number of assemblages $E_1, E_2, \cdots, E_n$, and has the same value for each

* See DINI, loc. cit., p. 244; BOREL, loc. cit., p. 28; LEBESGUE, loc. cit., p. 67, etc. These definitions are usually given only for the case in which E is the continuum.

† We shall allow the ideal values $\pm\infty$ for D^r, etc., except when specifically excluded. In most instances the word *limit*, without special mention of such possibilities as $D^r = -\infty$, will be used to avoid long circumlocutions.

‡ But not necessarily any value *between* $D^r f(x_0)$ and $D_r f(x_0)$.

of them, the derivative of $f(x)$ exists for the total assemblage ΣE formed by combining the given assemblages and has the same value; but this is *not* always true if the number of given assemblages is infinite. In fact, given a continuous function $f(x)$, all the values of x in an interval about $x = x_0$ can always be arranged into a sequence of assemblages $E_1, E_2, \cdots, E_n, \cdots$ for each of which the derivative of $f(x)$ exists and has the same value, which may be chosen at random as any number between $D^r f(x_0)$ and $D_r f(x_0)$ [or D^l and D_l]. *

On the other hand, a necessary and sufficient condition that the derivative of a function $f(x)$ defined in any interval exist with respect to the total assemblage ΣE at a point $x = x_0$, is that the derivative of $f(x)$ exist at $x = x_0$ and have the same value with respect to each of the component assemblages E (finite or infinite in number) and *that the corresponding oscillation* $\omega_{(E)}(x_0, d)$ *approach its limit zero uniformly*, i. e., $|\omega_{(E)}(x_0, d)| < \epsilon$ whenever $|d| < \delta$ where δ is independent of the assemblage E under consideration. Or again a necessary and sufficient condition that $f(x)$ have a derivative at $x = x_0$ with respect to any assemblage H is that the derivative of $f(x)$ should exist and have the same value *for every possible sequence in* H.

§ 3. *The law of the mean.*

Various generalizations of the law of the mean have been stated in terms of the four fundamental derived numbers.† All of these are special cases or corollaries of the following somewhat more general statement:

THEOREM 2. *Let $f(x)$ be defined on an assemblage H. If $f(x)$ is at a maximum with respect to H at a point $x = x_0$ of H, i. e., if $f(x_0) \geqq f(x)$ for all x in H, then*

$$D^l_{(E)} f(x_0) \geqq D_{l(E)} f(x_0) \geqq 0 \geqq D^r_{(E)} f(x_0) \geqq D_{r(E)} f(x_0)$$

where E is any subassemblage of H for which x_0 is a limiting point.

The proof is immediate since $M[x_0, d, E, Q(x_0, h)]$, for example, is not positive for sufficiently small positive values of d; hence the lower limit of M, that is $D^r_{(E)} f(x_0)$, is not positive.

This theorem, combined with theorem 1 and with the well-known theorem that any *continuous* function actually assumes its maximum on any *closed* assemblage, leads to many corollaries. Of these corollaries I shall mention only the following:

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* These and the following statements immediately result from theorems on limits for assemblages in general; see for example my paper: *On a Function, etc.*, Annals of Mathematics, 2d series, vol. 7 (1906), p. 177.

† See for example LEBESGUE, loc. cit., p. 70, etc., where a rather complete summary of the usual theorems is given; and BOREL, loc. cit., p. 28.

COROLLARY A. *If $f(x)$ is defined in an interval* $a \leqq x \leqq b$ and if there exists at each point $a \leqq x < b$ an assemblage R_x having points near x to the right and, such that $D_{(R_x)}f(x)$ exists and is positive, then $f(x)$ is at a maximum, if at all, at b.*

It is understood that R_x may be chosen differently at each point x. If $f(x)$ is at a maximum at $x = k$, $a \leqq k < b$, then $D_{(R_x)}f(x) \leqq 0$ by theorem 2.

COROLLARY B. (*Rolle's theorem.*) *If $f(x)$ is defined in the interval† $a \leqq x \leqq b$ and if $f(x)$ is at a maximum or at a minimum at $x = x_0$, $(a < x_0 < b)$, then*

$$L \geqq 0 \geqq l$$

where L and l are respectively the upper and lower limits of $D_{(R_x)}f(x)$ [or of $D_{(L_x)}f(x)$] in the interval $a \leqq x \leqq b$, where R_x [L_x] is a random assemblage at the right [left] of x, for every x for which $a \leqq x < b$ [$a < x \leqq b$].

For if, for example, $D_{(R_x)}f(x) > 0$ (i. e., if $l > 0$), $f(x)$ is at a maximum at $x = b$, by corollary A.

A consideration of the usual function

$$\phi(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

leads to an important corresponding generalization of the Law of the Mean, of which a special case is well known:‡

COROLLARY C. (*Law of the Mean.*) *If $f(x)$ is defined and continuous in the interval $a \leqq x \leqq b$, then*

$$L \geqq \frac{f(b) - f(a)}{b - a} \geqq l$$

where L and l are the upper and lower limits, respectively, of the values of $D_{(R_x)}f(x)$ in the interval, where R_x is an assemblage chosen at random as above, and differently for each value of x.

It is now clear that the upper and lower limits of any fixed set of right-hand assemblage derivatives $D_{(R_x)}f(x)$ in any interval are the same as those for any other such set, or for any left-hand set; in particular they are the same

* Instead of interval we may read *perfect assemblage*, in which case the maximum lies at a right-hand boundary point.

† Or in a perfect assemblage, if x_0 is an interior point.

‡ See LEBESGUE, loc. cit., p. 70. The theorem as stated usually applies only to the four funda mental members. The theorem here stated seems not much more general until it is noticed that R_x may be chosen *and then held fixed*, independently at each x. It should be noticed that these theorems really apply to cases in which $D_{(R_x)}f(x)$ does not exist; if it does not we can choose a new R_x, say R'_x, so that $D_{(R'_x)}f(x)$ exists and has any desired value between $D^r_{(R_x)}$ and $D_{r(R_x)}$.

as the upper and lower limits of any one of the four fundamental numbers in the same interval.* For a subinterval (a_1, b_1) surely exists for which $[f(b_1)-f(a_1)]/[b_1-a_1]$ is as near L (or l) for any preassigned set of $D_{(R_x)}f(x)$ as we please. It follows that the upper and lower limits of any fixed set of $D_{(R_x)}f(x)$ as the interval (a, b) enclosing x_0 is diminished toward zero in any manner, that is the maximum and minimum of any fixed set of $D_{(R_x)}f(x)$ at any point $x = x_0$, are always the same; *hence the oscillation of any fixed set of* $D_{(R_x)}f(x)$ *is the same as that of any other fixed set.*†

§ 4. *Determination of a function by given assemblage derivatives.*

A well known method‡ may now be used to prove the following proposition:

THEOREM 3. *If* $D_{(R_x)}f(x)=0$ *for some right-hand assemblage* R_x *at each point* x *of an interval* $a \leqq x \leqq b$ *in which* $f(x)$ *is defined and continuous,* $f(x)$ *is a constant.*

For let

$$\phi(x)=f(x)+\lambda(x-a), \qquad (\lambda>0),$$

$$\psi(x)=f(x)-\lambda(x-a);$$

then $D_{(R_x)}\phi(x)=\lambda>0$ and $D_{(R_x)}\psi(x)=-\lambda<0$ for $a \leqq x < b$; hence $\phi(x)$ is at a maximum at $x=b$ and $\psi(x)$ is at a minimum at $x=b$; that is,

$$\phi(b) \geqq \phi(x) \geqq f(x) \geqq \psi(x) \geqq \psi(b) \qquad (a \leqq x < b),$$

or

$$f(b)+\lambda(b-a) \geqq f(x) \geqq f(b)-\lambda(b-a) \qquad (a \leqq x < b).$$

It follows that $f(x)$ differs from $f(b)$ by at most the arbitrarily small quantity $\lambda(b-a)$.

Since

$$D_{(E)}[f_1(x)+f_2(x)] = D_{(E)}f_1(x) + D_{(E)}f_2(x)$$

if $D_{(E)}f_1(x)$ and $D_{(E)}f_2(x)$ both exist, we may conclude that if $f_1(x)$ and $f_2(x)$ are each continuous in an interval $a \leqq x \leqq b$ and if $D_{(R_x)}f_1(x)=D_{(R_x)}f_2(x)$ where (R_x) is the same for $f_1(x)$ and $f_2(x)$, but is possibly different for any two values of x, then $f_1(x)=f_2(x)+\text{const.}$ In case the ordinary derivative of one of the functions, say $f_2(x)$, exists, we shall have $D_{(R_x)}f_2(x)=df_2(x)/dx$ for any R_x whatever; hence in this case it is sufficient to know that $D_{(R_x)}f_1(x)=df_2(x)/dx$, where R_x is wholly unrestricted.

* Compare DINI, loc. cit., p. 264, etc.; LEBESGUE, loc. cit., p. 71. The usual statements hold only for the four fundamental numbers.

† The obvious consequences of this remark will not be dwelt upon here. It may be remarked that theorem 4 results at once from this; but the total argument necessary is more complex than that which follows. Again the upper and lower integrals of any fixed set of $D_{(R_x)}f(x)$ are evidently equal to those for any other set, in particular for any of the four fundamental numbers and these various functions are integrable whenever any one set of $D_{(R_x)}f(x)$ is so.

‡ See DINI, loc. cit., p. 113.

These statements may be somewhat generalized by using a slightly different proof, following that of SCHEEFFER or that of LEBESGUE, who have stated similar theorems *with respect to the four fundamental numbers.** These methods lead to the result: if $f(x)$ is defined and continuous in an interval $a \leqq x \leqq b$ and *if* $D_{(R_x)}f(x) = 0$ *for some assemblage* R_x *which has points near* x *to the right, for all* x *in the interval except at the points of a countable*† *assemblage* E, *then* $f(x)$ *is a constant;* or again, E *need only be of measure zero if we also know that the two right-hand fundamental numbers are not both* $+\infty$ *or both* $-\infty$ *at points of* E.‡ The proofs, being similar to those given by SCHEEFFER and LEBESGUE (l. c.) are here omitted. It should be noticed that *a continuous function is determined by a knowledge of* $D_{(R_x)}f(x)$ *for all except the points of* E. A large part of LEBESGUE's work may be slightly generalized —in some cases the generalization being an essential one—by means of these remarks. One such case will be treated in the next section (p. 352).

§ 5. *Continuous assemblage derivatives.*

From what precedes we may easily deduce the result:

THEOREM 4. *If* $f(x)$ *is defined and continuous in an interval* $a \leqq x < b$, *and if some right-hand assemblage* R_x *exists at every point* x *such that* $D_{(R_x)}f(x) = \phi(x)$ *exists and is continuous, then the ordinary derivative* $df(x)/dx$ *of* $f(x)$ *exists and is equal to* $\phi(x)$; *that is, the existence of a continuous right-hand assemblage derivative insures the existence and continuity of the ordinary derivative.*§

For, if $D_{(R_x)}f(x) = \phi(x)$ exists and is continuous, then the integral

$$\psi(x) = \int_a^x \phi(x)dx,$$

exists, is continuous, and has a derivative in the ordinary sense which is

$$\frac{d\psi(x)}{dx} = \phi(x) \qquad (a < x < b),$$

$$\left.\frac{d\psi(x)}{dx}\right|_{(r)} = \phi(a) \qquad (x = a),$$

where (r) indicates the right-hand derivative in the ordinary sense. Hence $f(x)$ and $\psi(x)$ are two continuous functions for which

* SCHEEFFER, loc. cit., p. 282; DINI, loc. cit., p. 274. Both methods are given by LEBESGUE, loc. cit., pp. 76–78. No essential change in method is necessary.

† What is really proved is that the assemblage E may be any assemblage whose power is less than that of the continuum.

‡ This is slightly more general than to say that not both are infinite.

§ For the similar result for the four fundamental numbers, see, e. g., DINI, loc. cit., p. 267.

$$D_{(R_x)}f(x) = D_{(R_x)}\psi(x) = \frac{d\psi(x)}{dx} \qquad (a \leqq x < \xi < b).$$

It follows that

$$f(x) = \psi(x) + \text{const.} \qquad (a \leqq x < b),$$

and hence $f(x)$ has an ordinary derivative.

Several corollaries suggest themselves. One restatement deserving mention is the following: If a continuous curve $y = \phi(x)$ can be drawn which passes between $y_1 = D^r f(x)$ and $y_2 = D_r f(x)$, i. e., $y_1 \leqq y \leqq y_2$, at every point of an interval $a \leqq x \leqq b$ in which $f(x)$ is continuous, then $f(x)$ has an ordinary derivative equal to $\phi(x)$ in that interval. For theorem 1 shows that this statement is equivalent to theorem 4.

As mentioned in the footnote, p. 349, this theorem also results from the fact that the oscillation of any set of $D_{(R_x)}f(x)$ is the same at each point as the oscillation of any of the four fundamental numbers. Indeed the continuity of any set of $D_{(R_x)}f(x) = \phi(x)$ at a single point $x = x_0$ insures the existence of the ordinary derivative at that point, for the oscillation of $D_{(R_x)}f(x)$ at $x = x_0$ and therefore the oscillation of the function $Q(x_0, h)$ is zero at the point $h = 0$.*

Resuming the argument of § 3 we may write

$$L \geqq \frac{f(\beta) - f(\alpha)}{\beta - \alpha} \geqq l \qquad (a \leqq \alpha < \beta \leqq b),$$

where L and l are the upper and lower limits of $D_{(R_x)}f(x)$ for a continuous function $f(x)$ in the interval $a \leqq x \leqq b$. Therefore

$$\Sigma\,[(\beta - \alpha)L] \geqq \Sigma\,[f(\beta) - f(\alpha)] \geqq \Sigma\,[(\beta - \alpha)l]$$

where the sums indicated are extended over any finite set of subintervals in the interval $a \leqq x \leqq b$. Passing to the limit we have

$$\overline{\int_a^x} D_{(R_x)}f(x)\,dx \geqq f(x) - f(a) \geqq \underline{\int_a^x} D_{(R_x)}f(x)\,dx \qquad (a \leqq x \leqq b),$$

whenever the upper and lower integrals $\overline{\int}$ and $\underline{\int}$ exist, which is surely true, for example, if $D_{(R_x)}f(x)$ is limited, i. e., if $f(x)$ is continuous and not both $D^r f(x)$ and $D_r f(x)$ are $+\infty$ or $-\infty$ at any point. If now

$$\overline{\int_a^x} D_{(R_x)}f(x)\,dx = \underline{\int_a^x} D_{(R_x)}f(x)\,dx,$$

i. e., if $D_{(R_x)}f(x)$ is integrable in the RIEMANN-CAUCHY sense, in which case any assemblage derivative is also integrable and has the same integral, the pre-

* However the derivative may still be discontinuous at $x = x_0$ or even not defined near $x = x_0$ in this case.

ceding inequalities become equations; hence we may say that *if $f(x)$ is continuous in an interval $a \leqq x \leqq b$ and if a right-hand assemblage R_x exists $(a \leqq x < b)$ for which $D_{(R_x)}f(x)$ exists and is integrable, then*

$$\int_a^x D_{(R_x)}f(x)\,dx = f(x) - f(a) \qquad (a \leqq x \leqq b);$$

*that is, the indefinite integrals of $D_{(R_x)}f(x)$ coincide with its primitive functions.**

In order that $\int D_{(R_x)}f(x)\,dx$ exist, $D_{(R_x)}f(x)$ must be continuous except at an assemblage of points of measure zero. Since the ordinary derivative of $f(x)$ exists at every point where $D_{(R_x)}f(x)$ is continuous, it follows that *if $D_{(R_x)}f(x)$ is integrable, $f(x)$ has a derivative except at points of an assemblage E of measure zero, and*

$$\int_a^x D_{(R_x)}f(x)\,dx = \int_a^x \frac{df(x)}{dx}\,dx = f(x) + \text{const.} \qquad (a \leqq x \leqq b),$$

it being understood that the second integral is taken neglecting the points of E. It is easily seen that this is the actual state of affairs whenever $D_{(R_x)}f(x)$ exists and is continuous except at points of an assemblage E whose power is less than that of the continuum; or also whenever $D_{(R_x)}f(x)$ is limited everywhere and is continuous except at points of an assemblage E of measure zero.

Finally let us consider a *fixed* assemblage R at each point x of an interval $a \leqq x \leqq b$ in which $f(x)$ is continuous. For example, let R be a *fixed* sequence, i. e., let R be the sequence

$$x + h_1,\ x + h_2,\ x + h_3,\ \cdots,\ x + h_n,\ \cdots \qquad (h_i > 0),$$

where the sequence $(h_1, h_2, h_3, \cdots, h_n, \cdots)$ is the *same* for every value of x. Then the sequence of continuous functions

$$Q(x, h_1),\ Q(x, h_2),\ \cdots,\ Q(x, h_n),\ \cdots$$

has as its limit $D_{(R)}f(x)$ whenever $D_{(R)}f(x)$ exists; and the upper and lower limits of $Q(x, h_i)$ are in any case equal to the upper and lower limits of $D^r f(x)$ or of any other assemblage derivative, in any interval. Suppose that $D_{(R)}f(x)$ exists for the *fixed* assemblage R; then by a theorem due to BAIRE† $D_{(R)}f(x)$, since it is the limit of a sequence of continuous functions, is pointwise discontinuous‡ on every perfect assemblage. But if $D_{(R)}f(x)$ is continuous, the derivative of $f(x)$ exists in the ordinary sense.

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* This generalizes a result given by LEBESGUE, loc. cit., p. 81. See also DUBOIS-REYMOND, loc. cit., p. 115; PASCH, loc. cit., p. 153.

† See BAIRE, *Thèse*, loc. cit., p. 62 or also *Leçons sur les fonctions discontinues*, p. 98. A simpler proof is that given by LEBESGUE; see BOREL, *Leçons*, note II, p. 149.

‡ That is, the function is continuous at least once in every interval.

THEOREM 5. *If there exists any fixed sequence R (or other assemblage) for which $D_{(R)}f(x)$ exists, then that derivative is pointwise discontinuous and moreover the derivative of $f(x)$ exists in the ordinary sense at all points of an assemblage of the second category.* *

For BAIRE has pointed out that a function which is pointwise discontinuous is continuous at points of an assemblage of the second category.

It is not difficult to see that if the above sequence $Q(x, h_i)$ converges to a continuous limit, the convergence is uniform, and the converse is obviously true.† In case it does, the ordinary derivative of $f(x)$ exists, is continuous, and has the same value as $D_{(R)}f(x)$. This convergence is closely allied to what DINI called simple uniform convergence ‡ in the case of any sequence of continuous functions; but the result in this case is much more far-reaching on account of the special nature of sequences which arise from the process of attempted differentiation. For whereas the simple uniform convergence of DINI is sufficient but not necessary for the continuity of the sum of any given sequence, and whereas moreover it is necessary to assume that the given sequence converges, we may announce the result:

THEOREM 6. *A necessary and sufficient condition that the quotient $Q(x, h)$ approach a limit and that that limit be continuous in an interval $a \leqq x \leqq b$ is that $Q(x, h)$ should be simply uniformly convergent* § *in that interval;* i. e., *that there should exist a fixed sequence $(h_1, h_2, \cdots, h_n, \cdots)$ for which $Q(x, h_i)$ converges uniformly.*

In the preceding statements it is evident that the numbers h_i need not be constant; if they merely are continuous functions of x all the statements made still hold.

COLUMBIA, MO.,
January, 1907.

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* An assemblage is of the first category if each point of it belongs to at least one of a *countable* set of non-dense assemblages. Otherwise it is of the second category. BAIRE, *Thèse*, p. 65; *Leçons*, p. 87.

† See my paper in the Annals of Mathematics, 2d series, vol. 7 (1906).

‡ See DINI, loc. cit., p. 147. The statement is equally applicable of course to the convergence of any function $F(x, h)$ as h approaches zero. For if $F(x, h)$ approaches a limit $\phi(x)$ as h approaches zero and if a sequence of values of h can be found for which $F(x, h_i)$ converges uniformly, then $\phi(x)$ is evidently continuous. I shall say that $F(x, h)$ has simple uniform convergence if h_i exist for which $F(x, h_i)$ converges uniformly.

§ This does not here *postulate* that $Q(x, h)$ *itself* approaches any limit whatever.

GEOMETRIE PROIETTIVE DI CONGRUENZA E GEOMETRIE PROIETTIVE FINITE*

DI

BEPPO LEVI

In una nota † presentata a questa Società il 22 Aprile 1905 e pubblicata nelle Transactions nell'Aprile 1906 i sigg. VEBLEN e BUSSEY hanno portata la loro attenzione sopra la possibilità di costruire uno spazio contenente un numero finito di punti ed in cui valga una geometria proiettiva per la quale siano conservate le principali proprietà dell'ordinaria geometria proiettiva lineare. ‡ Essi hanno pure analizzato qualcuno dei caratteri speciali a tali spazi e ne han data una rappresentazione analitica generale, raccogliendo infine, in un ultimo paragrafo, una bibliografia abbastanza vasta di lavori che più o meno hanno attinenza all'argomento trattato.

Pare di qui che agli autori sia sfuggito un cenno assai rapido, ma sufficientemente esplicito, che sopra queste geometrie e sopra altre assai più generali io diedi fin dal 1904 in una Memoria sopra i "*Fondamenti della metrica proiettiva*" §; ond'io chiedo alla Società il permesso di ricordare qui brevemente i punti della mia memoria che si riferiscono alla questione, aggiungendo ancora alcuni sviluppi che non mi paiono privi d'interesse.

La definizione analitica delle geometrie trattate giustificherà pienamente il nome generico che io ho qui attribuito loro di *geometrie proiettive di congruenza.*

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* Presented to the Society October 27, 1906. Received for publication October 21, 1906.

† *Finite projective geometries.*

‡ Cfr. particolarmente, loc. cit., § 4, pp. 245–247.

§ Memorie della R. Accademia delle Scienze di Torino, ser. 2, vol. 54 (1904); v. il no. 21, *La geometria proiettiva*, pp. 42–44.

Solo dopo che anche il presente articolo era già stato scritto ebbi esatta conoscenza della Nota dello HESSENBERG: "*Ueber die projektive Geometrie,*" Sitzungberichte der Berliner mathematischen Gesellschaft (28 Gennaio 1903), la quale è pur ricordata dai sigg. VEBLEN e BUSSEY (loc. cit., § 9, p. 258). Mi è debito perciò di riconoscere allo HESSENBERG larga parte della priorità qui reclamata, poichè in quella Nota egli afferma con precisione l'independenza della geometria projettiva lineare dalla nozione di ordine, e la dimostra per l'appunto colla costruzione di geometrie projettive finite dei tipi che—seguendo i sigg. VEBLEN e BUSSEY—si chiamerebbero (come sarà tosto ricordato) $PG(2, p)$ e $PG(2, p^2)$. Egli rileva pure (come, independentemente, è avvenuto a me) che in geometrie projettive in cui manchi la nozione d'ordine, divengono possibili configurazioni che sono impossibili nell'ordinario spazio projettivo: —così la configurazione d'un quadrangolo completo coi tre punti diagonali allineati, nella $G(2, 2)$ ed anche in piani contenenti infiniti punti (Cfr. il testo, n. II).

Le osservazioni dello HESSENBERG non invadono però per nulla il campo delle ulteriori considerazioni che sono oggetto del presente scritto.

I. Nella citata Memoria, dopo aver stabiliti, in base ai postulati ammessi, i teoremi di Desargues e di Pascal, onde risultò possibile, seguendo il sig. Hilbert, di riferire lo spazio geometrico ad un sistema di coordinate proiettive, io osservava: *

"Mi piace di rilevare come il concetto di ordine degli elementi di una forma di prima specie non abbia avuto fin qui alcuna parte nella istituzione della nostra geometria e non abbia più alcuna ragione di averne in seguito, in tutto lo sviluppo della geometria proiettiva. . . . È essenziale notare che, per questa costruzione della geometria proiettiva, non è necessario che," nella corrispondenza che la rappresentazione per coordinate stabilisce fra i punti degli assi coordinati dello spazio geometrico ed i numeri del campo numerico, "a espressioni aritmeticamente irreduttibili fra loro di questo campo numerico corrispondano punti † diversi dagli assi . . .; bensì è necessario soltanto che a punti diversi corrispondano numeri o espressioni fra loro aritmeticamente irreduttibili. . . ."

Mediante note operazioni geometriche le quali, sugli assi di uno spazio proiettivo, permettono di costruire il punto la cui coordinata relativa all'asse considerato è la somma, la differenza, il prodotto od il quoto delle coordinate relative a due altri punti assegnati ‡ si possono cioè "costruire tutti i punti degli assi corrispondenti agli elementi del campo di razionalità che ha per base un sistema di segni aritmetici (numeri) § corrispondenti a quanti si vogliano punti fissati arbitrariamente sugli assi. Ora si può supporre che due elementi di questo campo, fra loro aritmeticamente irreduttibili, rappresentino tali successioni di operazioni geometriche che conducano allo stesso punto finale. La differenza di questi due elementi aritmetici rappresenterà allora il punto 0" (origine delle coordinate). "L'insieme di tutti gli elementi rappresentanti questo punto 0 costituisce evidentemente un modulo con coefficienti appartenenti al campo di razionalità sopra nominato; e rappresentano uno stesso punto tutte e sole le espressioni di questo campo di razionalità congrue fra loro rispetto a tal modulo. Il prodotto di due punti p, q non sarà mai lo 0 se non è 0 almeno uno di essi." Il modulo nominato sarà cioè primo.

In tutto quanto segue chiamerò questo modulo *il modulo fondamentale della geometria.*

II. I sigg. Veblen e Bussey si occupano precisamente di questi spazi in cui le coordinate omogenee dei punti sono gli elementi di un campo di Galois $G(p^n)$; sono cioè i sistemi di polinomi in una variabile x a coefficienti appar-

* L. c., p. 42–43.

† Nel testo citato in luogo di punti e assi si parla di raggi e piani coordinati; ciò perchè quelle frasi si riferiscono ad una geometria nella stella.

‡ Cf. Hilbert, *Grundlagen der Geometrie*, § 24–26, ovvero la mia Memoria, no. 17.

§ Forse l'uso che io ho fatto di questa parola "numeri" può aver diminuito la veduta della generalità del concetto nella mente del lettore.

tenenti al campo di integrità naturale,* congruenti rispetto al modulo che ha per elementi fondamentali un numero primo p e un polinomio dato in x di grado n, a coefficienti interi (che si possono sempre supporre $< p$) irreduttibile rispetto al modulo p. Anche per essi i polinomi che costituiscono il modulo rappresentano il punto 0. La geometria di un tale spazio di k dimensioni chiamano una $PG(k, p^n)$.

Si vede come il mio punto di vista comprenda quello dei sigg. VEBLEN e BUSSEY come caso particolare, ma sia assai più generale del loro. E scegliendo convenientemente il modulo fondamentale si può costruire uno spazio proiettivo che, conservando tutte le proprietà lineari che provengono dai soli postulati di appartenenza (*Verknüpfung*), goda ancora di proprietà ben differenti dallo spazio ordinario (in particolare vi manchi la nozione di ordine) *pur possedendo infiniti punti.* Però nemmeno è sfuggito a me il caso particolare notevole presentato dallo spazio con un numero finito di punti:†

"In particolare, se il campo di razionalità considerato è il campo naturale generato dal solo elemento 1, mediante le operazioni di addizione, sottrazione, moltiplicazione e divisione, gli elementi rappresentanti lo 0 saranno tutti quelli della forma $(h/k)n$, dove n è un numero primo e k non è divisibile per n." ‡ Sopra ciascun asse, ciascun punto sarà rappresentato dall'insieme dei numeri razionali che differiscono fra loro per numeri della forma $(h/k)n$, ovvero — e ciò avverrà per un solo punto (il punto all' ∞) su ciascun asse — dall'insieme di tutte le frazioni irreduttibili il cui denominatore è multiplo di n.

Data una frazione irreduttibile p/q il cui denominatore non sia multiplo di n, si possono sempre trovare due numeri interi positivi h, l, tali che $ql - nh = p$ e quindi, $p/q = l - (h/q)n$. Secondo la definizione sopra esposta, l'intero l rappresenterà lo stesso punto che la frazione p/q. Fra i simboli numerici rappresentanti un punto determinato, diverso dal punto all' ∞, di un asse coordinato, esistono dunque sempre dei numeri interi; e poichè rappresenteranno ancora lo stesso punto i numeri l e $l + \lambda n$, ove λ è un numero intero qualunque, positivo o negativo, potranno sempre questi numeri supporsi $\geqq 0$ e $< n$. Ne risulta che la retta possiede nella nostra geometria un numero finito $(n + 1)$ di punti. È il caso che i sigg. VEBLEN e BUSSEY rappresentano con $PG(k, n)$. "Se in particolare si fa $n = 2$, si otterrà uno spazio proiettivo . . ." in cui "il quarto armonico dopo tre elementi dati coincide con uno di questi tre.§

* Il fatto che, alla mia considerazione di un campo di razionalità, si sostituisce qui quella di un campo d'integrità dipende da ciò che i sigg. VEBLEN e BUSSEY si riferiscono ad una rappresentazione per coordinate omogenee, oppostamente a quanto avviene nella mia Memoria.

† Spetta però ai sigg. VEBLEN e BUSSEY il merito di essersi posto il problema di definire *tutti* gli spazi proiettivi finiti e di averlo risolto caratterizandoli nei $PG(k, p^n)$ sopra ricordati.

‡ V. la mia *Memoria* citata, p. 43. Nelle linee seguenti della stessa p. 43 si ritrovano le stesse considerazioni che qui seguono, in una esposizione che qui parrebbe meno opportuna al l'intelligenza.

§ Cfr. la Nota dei sigg. VEBLEN e BUSSEY, § 3.

L'esempio offerto dal sig. FANO per dimostrare l'indipendenza della proposizione: *il quarto armonico dopo tre elementi dati è distinto da ciascuno di essi*, dai postulati di appartenenza è una realizzazione geometrica di questa costruzione analitica"; onde questo mostra che, in mancanza di quel postulato "molte proposizioni non diverrebbero erronee, ma illusorie." * E la costruzione analitica esposta mostra che tal singolarità non è per nulla legata al *numero* dei punti ammessi alla retta; essa può presentarsi anche in uno spazio d'infiniti punti.†

III. Le geometrie proiettive di congruenza, mentre permettono di scoprire fatti nuovi, conciliabili colle ipotesi fondamentali della geometria proiettiva, danno pur modo di interpretare in un campo reale, anche costituito d'un numero finito di punti, fatti che negli spazi proiettivi ordinarii possono verificarsi solo nel campo complesso.

Come le considerazioni che seguono mostreranno con maggior precisione, questo è, in larga misura, un riflesso naturale della teoria dei numeri algebrici del KRONECKER, per la quale ogni numero algebrico (reale o complesso) viene ad identificarsi con una congruenza fra elementi razionali.

Un notevole esempio si è presentato nella mia *Memoria* nella considerazione di una geometria piana del tipo ultimamente descritto con $n = 3$: ‡ il piano proiettivo è allora costituito di 13 punti e si compone di 13 configurazioni di 9 punti identiche alla configurazione dei flessi di una cubica, la qual configurazione, come si sa, è impossibile in un ordinario piano reale.

Sopra un fatto dello stesso ordine hanno portato la loro attenzione i sigg. VEBLEN e BUSSEY: si sa che, sulla retta complessa, cade in difetto la proposizione fondamentale di v. STAUDT che una corrispondenza armonica in cui tre punti coincidano coi loro omologhi è necessariamente l'identità; fu questo il punto di partenza per interessanti ricerche del Prof. SEGRE sopra quelle corrispondenze armoniche che egli chiamò *antiproiettività* e che risultan dal prodotto di trasformazioni proiettive e di una trasformazione per coniugio.§ Ora un fatto analogo si riproduce nelle geometrie finite dei sigg. VEBLEN e BUSSEY. Essi osservano infatti || che i punti d'una retta di una loro geometria si riuniscono in *catene* (analoghe alle catene di v. STAUDT), i punti di ciascuna delle quali si deducono razionalmente da tre punti assegnati qualsiansi della catena medesima: seguendo i nominati autori, se più catene esistono su una stessa

* Cfr. la mia *Memoria* pag. 43–44.

† Cfr. la nota (3) a piè della pag. 43 nella *Memoria* citata.

‡ L. c. no. 31—*Un piano metrico di* 9 *punti e un piano proiettivo di* 13—pp. 53–56: questo esempio presenta una particolare importanza nella dimostrazione della compatibilità e delle dipendenze dei postulati della metrica e di quelli della geometria proiettiva.

§ SEGRE, *Un nuovo campo di ricerche geometriche*, Atti della R. Accademia di Torino, 1890.

|| L. c., § 5, pp. 249, 250.

retta, queste si aggruppano in sistemi più vasti (2-*catene*) contenenti ciascuna tutte e sole le catene, combinazioni lineari di due catene del sistema (e queste due catene hanno, in generale, due punti comuni — distinti o coincidenti); se i punti della retta non appartengono tutti alla stessa 2-catena, esistono ancora sulla retta sistemi lineari più vasti di catene (3-*catene*) e così via: tutti i punti di una retta di una $PG(k, p^n)$ costituiscono in tal modo una *n-catena*. Ora i sigg. VEBLEN e BUSSEY mostrano che al teorema di v. STAUDT si può sostituire quest'altro: *Una corrispondenza armonica sopra una retta di una* $PG(k, p^n)$ *non può avere più di* $n+1$ *punti fissi non appartenenti alla stessa* $(n-1)$*-catena senza ridursi all'identità.**

Questa proposizione stabilisce però per il numero dei punti fissi un limite molto superiore al reale. Io mi propongo di dimostrare, nelle pagine seguenti, che essa può esser sostituita con un'altra assai più precisa.

Mi permetterò anzi di svolgere al riguardo alcune considerazioni generali, da cui si vedrà come si apra qui un largo campo di ricerche algebrico-geometriche.

IV. Sia m una retta di un qualsiasi spazio proiettivo: assegnati arbitrariamente sopra la retta tre punti a, b, c, si può sempre — al più mediante una sostituzione lineare i cui coefficienti siano espressioni razionali nelle coordinate dei tre punti a, b, c — assumere la retta m come asse coordinato e fare in modo che le ascisse dei tre punti a, b, c, siano rispettivamente $0, 1, \frac{1}{0}(=\infty)$.† Se allora ξ è l'ascissa di un punto mobile della retta, $\phi(\xi)$ l'ascissa del punto corrispondente per una determinata trasformazione armonica‡ che tenga fissi i punti a, b, c, è noto§ che la $\phi(\xi)$ deve essere una funzione di ξ che soddisfi alle equazioni funzionali

$$(1) \qquad \begin{cases} \phi(\xi+\eta) = \phi(\xi) + \phi(\eta), \\ \phi(\xi^2) = [\phi(\xi)]^2. \end{cases}$$

Reciprocamente ogni funzione ϕ soddisfacente alle equazioni (1) definisce una trasformazione armonica della retta che muta in sè i punti a, b, c. Invero dalla 1ª delle (1) segue

$$\phi(\xi) = \phi(\xi+0) = \phi(\xi) + \phi(0)$$

onde

$$\phi(0) = 0;$$

$$\phi(\tfrac{1}{0}) = \phi(\xi + \tfrac{1}{0}) = \phi(\xi) + \phi(\tfrac{1}{0})$$

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* L. c., § 5, p. 250.

† Dò la preferenza al segno $\frac{1}{0}$ sopra al segno ∞ perchè non nasca alcun dubbio sulla validità dei presenti sviluppi per geometrie finite o, in generale, geometrie nei cui punti non abbiano rappresentazione tutti i numeri reali.

‡ Cioè una trasformazione biunivoca della retta in sè, che muti gruppi armonici in gruppi armonici.

§ DARBOUX, *Sur le théorème fondamental de la géométrie projective*, Mathematische Annalen, vol. 17 (1880), p. 55 (Vedi p. 56-58).

e quindi, poichè non è costantemente $\phi(\xi) = 0$,

$$\phi(\tfrac{1}{0}) = \tfrac{1}{0};$$

infine da

$$\phi(1) = \phi(1^2) = [\phi(1)]^2 \qquad \text{segue} \qquad \phi(1) = 1,$$

e da

$$\phi(0) = \phi(1-1) = \phi(1) + \phi(-1) \qquad \text{segue allora} \qquad \phi(-1) = -1.$$

Si ha inoltre

$$\phi[(\xi+\eta)^2] = \phi(\xi^2 + \eta^2 + 2\xi\eta) = [\phi(\xi)]^2 + [\phi(\eta)]^2 + 2\phi(\xi\eta)$$
$$= [\phi(\xi) + \phi(\eta)]^2 = [\phi(\xi)]^2 + [\phi(\eta)]^2 + 2\phi(\xi)\phi(\eta)$$

onde

$$\phi(\xi\eta) = \phi(\xi)\phi(\eta).$$

Dalle due relazioni

$$\phi(\xi+\eta) = \phi(\xi) + \phi(\eta), \qquad \phi(\xi\eta) = \phi(\xi)\phi(\eta)$$

segue che, se $\xi, \eta, \zeta, \cdots$ sono ascisse di punti qualsivogliano della retta, e $f(\xi, \eta, \zeta \cdots)$ una loro funzione razionale a coefficienti razionali,

$$(2) \qquad \phi[f(\xi, \eta, \zeta \cdots)] = f[\phi(\xi), \phi(\eta), \phi(\zeta), \cdots].$$

In questa relazione (2) sono comprese le (1) come caso particolare: essa può quindi sostituirsi al loro sistema. Se allora ξ, η, ζ, θ sono tali che

$$(\xi\eta\zeta\theta) = -1,$$

la (2) ci dice che

$$\phi(-1) = \phi[(\xi\eta\zeta\theta)] = [\phi(\xi)\phi(\eta)\phi(\zeta)\phi(\theta)],$$

ossia

$$[\phi(\xi)\phi(\eta)\phi(\zeta)\phi(\theta)] = -1,$$

con che si prova che la trasformazione considerata è armonica.

V. Ciò posto si consideri una geometria proiettiva tale che le ascisse dei punti generici della retta m siano funzioni razionali a coefficienti razionali delle ascisse di un certo sistema di punti. Solo per ragion di semplicità e per le applicazioni successive supporremo che questi punti siano in numero finito.* Indicheremo inoltre, d'ora innanzi, i punti della retta m e le loro ascisse cogli

* L'ipotesi del numero finito di punti razionalmente independenti non è per nulla essenziale; tra l'altro potrebbero i coefficienti, anzichè essere razionali, appartenere ad un determinato campo ortoide (König, *Einleitung in die allgemeine Theorie der algebraischen Grössen*, Leipzig, Teubner, 1903, p. 7–8), per esempio, esser semplicemente reali. Però l'ipotesi della razionalità dei coefficienti è necessaria per qualcuna delle considerazioni seguenti; qualche altra chiede poche ovvie modificazioni nell'ipotesi contraria. Inoltre molte delle conseguenze che da quanto segue possono trarsi divengono illusorie se non si ammette almeno che l'aggregato dei punti razionalmente independenti possa essere ben ordinato.

stessi simboli. Si potrà, sopra la nostra retta, determinare una base minima

$$0, \frac{1}{0}, 1, x_1, x_2, \cdots, x_r$$

di punti fra loro razionalmente independenti, tali che ogni altro punto sia un'espressione razionale a coefficienti razionali di essi. La relazione (2), insieme colle condizioni $\phi(0)=0$, $\phi(1)=1$, $\phi(\frac{1}{0})=\frac{1}{0}$, che possono considerarvisi incluse, definirà allora la funzione ϕ per tutti i punti della retta tosto ch'essa sia definita nei punti $x_1, x_2, \cdots, x_r$ per modo che, detti $X_1, X_2, \cdots, X_r$ i punti corrispondenti, — il che si potrà rappresentare colla scrittura

$$X_i = \phi(x_i) \qquad (i=1, 2, \cdots, r) \tag{3}$$

— avvenga che:

1°. Ogni relazione algebrica la quale leghi i punti x_i sia pure soddisfatta se alle x_i si sostituiscano le X_i e reciprocamente ogni relazione algebrica fra le X_i sia soddisfatta pure dalle x_i.

2°. I punti

$$0, \frac{1}{0}, 1, X_1, X_2, \cdots, X_r$$

costituiscano a lor volta una base razionale (che sarà necessariamente minima) pei punti della retta, per modo che ciascuno di questi si esprima razionalmente mediante essi.

Nella condizione 1° è evidentemente inclusa quest'altra, che:

3°. Se un medesimo punto si esprime razionalmente in due modi differenti mediante le x_i (o le X_i) le due espressioni che risultano dopo la sostituzione (3) (o la sua inversa) risultino ancora equivalenti.

Invero, l'equivalenza di due espressioni razionali $f_1(1, x_1, x_2, \cdots, x_r)$, $f_2(1, x_1, x_2, \cdots, x_r)$ si traduce in una relazione algebrica, $f_1 - f_2 = 0$, fra le x_i: allora, a causa della condizione 1°, sarà pure

$$f_1(1, X_1, X_2, \cdots, X_r) - f_2(1, X_1, X_2, \cdots, X_r) = 0.$$

VI. Il supporre che fra gli elementi di una base razionale minima passi una relazione algebrica $F(1, x_1, x_2, \cdots, x_r) = 0$, equivale a considerare una geometria di congruenza; equivale cioè a dire che l'espressione $F(1, x_1, x_2, \cdots, x_r)$ rappresenta lo 0 ed appartiene quindi al modulo fondamentale della geometria.* Inversamente una geometria di congruenza il cui modulo fondamentale sia interamente costituito soltanto da polinomi si può considerare come rappresentante di una geometria avente una base razionale, fra i cui elementi sussistano le relazioni algebriche che si ottengono uguagliando a 0 tali polinomi. La cosa è differente quando del modulo fondamentale faccia parte un numero (necessaria-

* È questo il concetto fondamentale della teoria di KRONECKER dei numeri algebrici.

mente intero e primo); e tal differenza appare ben naturale se appena si osserva che, nel primo caso, l'ammissione di un modulo fondamentale equivale ad ammettere che taluno dei punti razionalmente independenti è però scelto per modo che da questi si può ottenere il punto 0 mediante una determinata successione di operazioni razionali (proiezioni e sezioni), mentre nel secondo caso l'ammissione del modulo fondamentale influisce effettivamente sul concetto di retta (e cioè sul concetto delle operazioni razionali fondamentali: proiezioni e sezioni), per modo che per una determinata successione di tali operazioni, effettuate su punti fissati *a priori*, perchè le dette operazioni abbiano senso determinato (i punti $0, \frac{1}{0}, 1$), riconduce da essi al punto 0.* In ogni caso, basta ritornare sul significato e sulla deduzione della (2) per riconoscere che, nel caso delle geometrie di congruenza, in essa l'uguaglianza va ora sostituita con una congruenza rispetto al modulo fondamentale. Si chiami M questo modulo, e si osservi che ciascun punto X_j sarà rappresentato da una classe di espressioni razionali nelle x_i, congrue fra loro rispetto al modulo M; il problema della determinazione di una trasformazione ϕ equivale allora a *determinare una sostituzione*

$$X_j \equiv \psi_j(x_i) \pmod{M} \qquad (i, j = 1, 2, \cdots, r) \tag{4}$$

tale che

1° *trasformi ogni polinomio del modulo M (in cui alle variabili x_i si pensino sostituite le corrispondenti X_i) in un polinomio del modulo medesimo.*

2° *i punti $0, \frac{1}{0}, 1, X_1, \cdots, X_r$ costituiscano una base razionale dei punti della retta.*

La prima condizione mostra che il modulo fondamentale della geometria, rispetto alla base $0, \frac{1}{0}, 1, X_1, \cdots, X_r$, si otterrà sostituendo semplicemente le lettere X alle x nei polinomi del modulo fondamentale relativo alla base $0, \frac{1}{0}, 1, x_1, \cdots, x_r$; indicando con $\overline{M}$ il risultato di questa sostituzione, la condizione 2°si potrà enunciare chiedendo che *la trasformazione* (4) *abbia una inversa univocamente determinata*

$$x_i \equiv \Psi_i(X_j) \pmod{\overline{M}}. \tag{4'}$$

VII. Le x_i e le X_j erano fin qui simboli rappresentanti punti razionalmente independenti della nostra retta. Si interpretino ora per un istante come variabili. Quando del modulo M non faccia parte un numero, la condizione ch'esso sia primo (n°. I) si traduce nel fatto che appartengano ad esso tutti e soli i polinomii che, uguagliati a 0, determinano, nello spazio di coordinate x_i, iper-

* I casi già ricordati della $G(2, 2)$ in cui esiste una *terna* armonica di punti e della $G(2, 3)$ in cui esiste la configurazione dei 9 flessi d' una cubica, chiariscono in modo evidente questa osservazione. Come in un piano di soli punti razionali (e quindi reali), i punti delle nostre geometrie si ottengono con sole operazioni razionali (projezioni e sezioni) dai punti di coordinate $0, \frac{1}{0}, 1$: ma punti che per tal generazione risulterebbero necessariamente distinti nell' ordinario piano reale, vengono ora a coincidere.

superficie passanti per una determinata varietà irreduttibile (razionalmente).* Segue allora che *la trasformazione* (4), *in cui si sostituisca il segno* $=$ *al* $\equiv$, *rappresenterà una trasformazione algebrica di questa varietà in sè*, la quale, a causa della esistenza di una inversa univoca (4′) *sarà pure birazionale.* Inversamente ogni trasformazione birazionale a coefficienti razionali della varietà rappresentativa del modulo fondamentale in sè definirà una sostituzione (4) la quale muterà una ipersuperficie per essa varietà in una analoga ipersuperficie, e quindi, operata sopra un polinomio qualunque del modulo, lo muterà in un polinomio del modulo. E questa sostituzione ammetterà una inversa (4′): per ottenerla si considerino le espressioni razionali delle x nelle X che, *sulla varietà rappresentativa del modulo* M, definiscono la trasformazione inversa della (4): saranno esse i secondi membri Ψ della sostituzione (4′): invero il prodotto delle due trasformazioni $X_i = \psi_i(x_j)$, $x_j = \Psi_j(X'_i)$ sarà una trasformazione $X_i = \Theta_i(X'_k)$ la quale, quando le X'_k si interpretano come coordinate di un punto generico della varietà rappresentativa del modulo, deve ridursi all'identità, cosicchè deve essere

$$\Theta_i(X'_k) \equiv X'_k \pmod{M'}$$

dove M' è il modulo M scritto nelle variabli X' in luogo delle x.† Quindi *ad ogni trasformazione birazionale a coefficienti razionali della nominata varietà in sè corrisponde una trasformazione armonica della nostra geometria.* Varrà quindi il teorema di v. STAUDT quando la varietà rappresentativa del modulo fondamentale non ammetta trasformazioni birazionali a coefficienti razionali in sè.

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* Invero, non potrebbero tutte queste ipersuperficie avere a comune una varietà riduttibile, altrimenti apparterrebbero al modulo polinomi prodotti di altri non appartenenti ad esso, in quanto rappresentano ipersuperficie passanti per le singole parti soltanto di essa varietà. D'altra parte, per un noto teorema del sig. HILBERT (*Ueber die Theorie der algebraischen Formen*, Mathematische Annalen, vol. 36 (1890), p. 474; KÖNIG, *Einleitung in die allgemeine Theorie der algebraischen Grössen*, p. 366) tutti i primi membri delle equazioni delle ipersuperficie per una varietà determinata costituiscono un modulo. Nè, nel caso nostro, potrebbero le forme del modulo M corrispondere a una parte soltanto delle ipersuperficie per la detta varietà comune, e nemmeno potrebbe tal varietà comune venir a mancare, poichè ancora per un teorema dello HILBERT (*Ueber die vollen Invariantensysteme*, Mathematische Annalen, vol. 42 (1893), p. 320; KÖNIG, l. c, p. 399) ed uno del LASKER (*Zur Theorie der Moduln und Ideale*, Mathematische Annalen, vol. 60 (1905), p. 30; SEVERI, *Su alcune proprietà dei moduli di forme algebriche*, Atti della R. Acc. delle Sc. di Torino, 1906), una conveniente potenza di un polinomio qualunque nel secondo caso, o rappresentante un' ipersuperficie per la varietà considerata nel primo, deve appartenere al modulo: ma il modulo non sarebbe primo (o, più direttamente, non sarebbe soddisfatta la proprietà fondamentale del nostro campo numerico, che un prodotto sia $= 0$ solo se è nullo qualcuno dei fattori) se al modulo potesse appartenere una potenza di un polinomio, e non il polinomio medesimo.

† Invero, posto $\Theta_i(X'_k) = \theta_i(X'_k)/\theta'_i(X'_k)$, il polinomio $\theta_i(X'_k) - X'_i\theta'_i(X'_k)$ dovrà rappresentare una ipersuperficie passante per la varietà considerata e dovrà quindi appartenere al modulo M'; quindi $\theta_i(X'_k) \equiv X'_i\theta'_i(X'_k) \pmod{M'}$ e, poichè certamente $\theta'_i(X'_k)$ non appartiene ad M', $\Theta_i(X'_k) \equiv X'_i \pmod{M'}$.

Una interpretazione delle condizioni 1° e 2° nel caso che il modulo contenga un numero presenta assai maggiori difficoltà.

VIII. Ci volgiamo ora al problema da cui siamo partiti: *Supponendo di considerare una geometria finita, in cui quindi il modulo fondamentale sia della forma*

$$M = [p, F_n(x)]$$

ove p è un numero primo e $F_n(x)$ un polinomio in x a coefficienti interi, di grado n e irreduttibile rispetto al modulo p, qual è il numero massimo di punti $\xi_1, \xi_2, \cdots, \xi_\nu$ tali che il verificarsi della relazione $\phi(\xi_k) = \xi_k$ per ogni $k \leqq i$ non abbia per conseguenza $\phi(\xi_{i+1}) = \xi_{i+1}$, ma il verificarsi dell'uguaglianza*

$$\phi(\xi_k) = \xi_k$$

per tutti questi punti abbia per conseguenza che la trasformazione armonica definita dalla ϕ è l'identità?

In tal modo deve infatti interpretarsi la generalizzazione del teorema di v. Staudt, perchè è ben facile vedere che esistono sulla retta punti tali che ogni trasformazione armonica ϕ che tenga fermi al solito i punti $0, \frac{1}{0}, 1$ e uno di essi è senz'altro l'identità: tale è per esempio il punto x, dove x è la lettera ordinatrice dei polinomi, o, se si vuole, una radice primitiva del campo di Galois; e solo può dubitarsi che possano esistere trasformazioni armoniche ϕ le quali spostino questi punti, tenendo però fermi altri punti della retta, oltre i punti della catena che contiene i punti $0, \frac{1}{0}, 1$. Per risolvere tal questione occorre che ricordiamo anzitutto alcune proprietà dei campi di Galois:

Se ξ_1 è un elemento di un campo di Galois $G(p^n)$ che non sia un numero intero, l'insieme dei polinomi interi in ξ_1, ridotti rispetto al modulo M che definisce il campo, costituisce un campo di Galois $G(p^{n'})$ [che si potrà indicare con $(1, \xi_1)$] contenuto in $G(p^n)$, ed il numero n' è divisore di n. Se poi ξ_2 è un nuovo elemento di $G(p^n)$ non appartenente a $G(p^{n'})$, l'insieme dei polinomi interi in ξ_1, ξ_2, ridotti rispetto al modulo M, costituirà un nuovo campo di Galois $G(p^{n''})$ [che si potrà indicare con $(1, \xi_1, \xi_2)$] contenente $G(p^{n'})$ e contenuto in $G(p^n)$ per modo che n'' sarà divisore di n e n' divisore di n''.†

Si osservi ora che se una trasformazione armonica ϕ trasforma ξ_1 in se

* Sempre quando al modulo appartiene un numero p (necessariamente primo) tutti i coefficienti dei polinomi che costituiscono gli elementi del campo numerico considerato possono supporsi numeri interi per le osservazioni già fatte al No. II.

† Cfr. Dickson, *Linear groups with an exposition of the Galois Field theory*, Leipzig, 1901, pp. 50 e 51. Per riconoscere l'esattezza di queste affermazioni basta osservare che se λ è un elemento non appartenente al campo di Galois $G(p^\nu)$, un campo di Galois contenente λ e $G(p^\nu)$ conterrà almeno i $p^{\nu+1}$ elementi somme degli elementi di $G(p^\nu)$ cogli elementi $h\lambda$ $(0 \leqq h < p)$, e se, oltre queste somme, contiene un altro elemento μ, ne conterrà pure $p^{\nu+2}$ ottenuti combinando in simil modo μ con questi $p^{\nu+1}$ e così via.

stesso, trasformerà in sè tutti gli elementi del campo $(1, \xi_1)$, e se trasforma in sè ancora ξ_2, trasformerà pure in sè tutti gli elementi del campo $(1, \xi_1, \xi_2)$ e così via. Ne risulta che *il numero dei punti ξ_k definiti nel precedente enunciato è minore o uguale al numero dei termini della massima successione di numeri*

$$n', n'', n''', \cdots, n$$

$(>1$ *e* $\leqq n)$, *ciascuno dei quali sia divisore del seguente;* altrimenti detto, *è minore o uguale al numero dei divisori primi di n.*

Ma si può aggiungere che *questo massimo è sempre raggiunto*, cosicchè al "minore od uguale," si potrà sostituire "*uguale*" senz'altro. Si osservi infatti che, nel caso presente in cui gli elementi della nostra geometria analitica appartengono ad un campo di Galois $G(p^n)$, le condizioni 1° e 2° imposte alla trasformazione (4) equivalgono a dire che

$$X \equiv x^{p^\nu} \pmod{M} \qquad (0 \leqq \nu < n). \tag{5}$$

Invero la X, dovendo essere radice della congruenza $F_n(x) \equiv 0 \pmod{M}$, dovrà avere la forma (5)* e d'altronde l'inversa della (5) sarà data evidentemente da $X^{p^{n-\nu}} \equiv x^{p^n} \equiv x \pmod{M}$.† Si corrisponderanno per la trasformazione ϕ gli elementi $x^{\mu p^\nu}$ e X^μ; e sarà

$$X^\mu \equiv x^{\mu p^\nu} \equiv x^\mu$$

quando

$$\mu(p^\nu - 1) \equiv 0 \pmod{p^n - 1}.$$

Ora il massimo comun divisore di $p^\nu - 1$ e $p^n - 1$ è della forma $p^l - 1$ ove $\nu = ll'$, $n = l\lambda$ ed l' e λ sono primi fra loro; ed il più piccolo valore di μ per cui questa congruenza è soddisfatta rende $\mu(p^l - 1) = p^n - 1$. Reciprocamente, scelto arbitrariamente l divisore di n e posto $n = l\lambda$, l' primo con λ, $\nu = ll'$, esiste un $\mu = (p^n - 1)/(p^l - 1)$ e quindi $< p^n - 1$ tale che $\mu(p^\nu - 1) \equiv 0 \pmod{p^n - 1}$ e quindi $X^\mu \equiv x^\mu$. Segue che condizione necessaria e sufficiente affinchè la trasformazione (5) lasci fermo qualche punto ad ascissa non numerica è che ν ed n abbiano divisori comuni: se l è il massimo comun divisore di ν ed n resteranno fissi tutti i punti del campo di Galois $(1, x^\mu)$ ove $\mu = (p^n - 1)/(p^l - 1)$: sarà questo un $G(p^l)$.

Fissata quindi arbitrariamente una qualunque successione

$$n', n'', n''', \cdots, n \qquad (n' > 1)$$

di divisori di n tali che ciascuno di essi sia uguale al precedente moltiplicato per un numero primo, si può ad essa far corrispondere una successione di campi di Galois

$$G(p^{n'}),\ G(p^{n''}),\ G(p^{n'''}) \cdots,\ G(p^n)$$

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* SERRET, *Cours d'algèbre supérieure*, 5me édition, 1885, T. 2, p. 180.

† Cf. DICKSON, l. c., p. 11; SERRET, l. c., No. 346, p. 132.

dei quali l'ultimo sia il campo totale dei punti della nostra retta, e ciascuno dei quali contenga tutti i precedenti; e a ciascuno $G(p^{n^{(k)}})$ *di questi campi si possono far corrispondere trasformazioni* $\phi^{(k)}$ *tali che tengano fermi tutti i punti di* $G(p^{n^{(k)}})$ *(e quindi i punti dei campi precedenti) ma spostino ogni punto dei campi successivi, che non appartenga a questi. Una data* ϕ *sarà una* $\phi^{(k)}$ *tosto che essa tenga fermo un punto di* $G(p^{n^{(k)}})$, *non appartenente a* $G(p^{n^{(k-1)}})$, *e sposti un punto di* $G(p^{n^{(k+1)}})$. *Dato* $n^{(k)}$ *il numero delle* $\phi^{(k)}$ *differenti è il numero dei numeri primi con* $n/n^{(k)}$ *e minori di esso.*

Per assicurare che una ϕ *si riduce all'identità occorre verificare al più che essa tien fissi tanti punti, ciascuno razionalmente independente dai precedentemente considerati, quanti sono i divisori primi del numero* n.

Particolarmente notevole è il caso in cui n sia primo; allora, qualunque sia l'elemento ξ di $G(p^n)$, non numero, il campo $(1, \xi)$ è il campo totale $G(p^n)$; quindi se $\phi(\xi) = \xi$, sarà, per ogni altro elemento η, $\phi(\eta) = \eta$. *Se* n *è primo una trasformazione armonica che, oltre ai tre punti* $0, 1/0, 1$ *tenga pur fermo un punto qualunque non appartenente alla catena di questi tre sarà sempre l'identità.*

TORINO, ITALIA, 13 *Ottobre* 1906.

COLLINEATIONS IN A FINITE PROJECTIVE GEOMETRY*

BY

OSWALD VEBLEN

In the paper by Mr. BUSSEY and myself on *Finite Projective Geometries* it was shown † that in a $PG(k, p^n)$ any transformation of the form

$$(1) \qquad \frac{x_i'}{x_{k+1}'} = \frac{a_{i1}x_1^{p^m} + a_{i2}x_2^{p^m} + \cdots + a_{ik+1}x_{k+1}^{p^m}}{a_{k+11}x_1^{p^m} + a_{k+12}x_2^{p^m} + \cdots + a_{k+1k+1}x_{k+1}^{p^m}} \qquad (i=1, \cdots, k),$$

where m is zero or an integer less than n, is a collineation. As a result of Dr. LEVI's article ‡ it is possible to prove the converse proposition, namely, that *every collineation in* $PG(k, p^n)$ *is of type* (1). The following argument connects this theorem directly with our former article. It will be sufficient to give the argument for the case, $k = 2$.

A projective collineation, § or, in other words, a linear transformation, is determined by the quadrangle into which it transforms the quadrangle $(0\,0\,1)$, $(1\,1\,1)$, $(0\,1\,1)$, $(1\,0\,1)$. It follows that an arbitrary collineation is the product of a projective collineation by a collineation which leaves these four points invariant. The latter type of collineation may be called *antiprojective*, in the language of SEGRE. ‖ Its existence is proved by the existence of transformations of type (1) where $m \neq 0$.

An antiprojectivity, by its definition, leaves invariant the points of the x_1 axis for which x_1/x_3 is 0, 1, or ∞, and, therefore, all points of the chain determined by these three, namely, all points whose coördinates are integers modulo p. From the quadrangle-construction for addition and multiplication and the fact that a collineation transforms a complete quadrangle into a complete quadrangle it follows that if three points of the x_1 axis for which x_1/x_3 is a, b, and c respectively are so related that

* Presented to the Society December 28, 1906. Received for publication November 16, 1906.

† These Transactions, vol. 7 (1906), p. 252.

‡ Page 354 of this number of the Transactions. To our former list of works relating to finite geometries we add references to this paper of LEVI, *Geometrie proiettive di congruenza e geometrie proiettive finite*, and to an earlier paper, *Fondamenti della metrica proiettiva*, Memorie della R. Accademia delle Scienze di Torino, ser. 2, vol. 54 (1904), and also to a book by G. ARNOUX, *Arithmétique Graphique. Les espaces arithmétiques hypermagiques*, Paris, 1894. ARNOUX applied his "arithmetic spaces," which may be identified with our $EG(k, p)$, to construction of magic squares.

§ These Transactions, vol. 7 (1906), p. 253.

‖ Cf. LEVI's remarks on page 357.

$$a + b = c \qquad \text{or} \qquad ab = c,$$

then the points, with coördinates $\phi(a)$, $\phi(b)$, $\phi(c)$, into which they are transformed are so related that

$$\phi(a) + \phi(b) = \phi(c) \qquad \text{or} \qquad \phi(a)\phi(b) = \phi(c).$$

In other words the transformation of the points of the x_1 axis effected by an antiprojectivity is subject to the conditions:*

$$\phi(a) + \phi(b) = \phi(a + b); \tag{2}$$

$$\phi(a) \cdot \phi(b) = \phi(ab); \tag{3}$$

$$\phi(0) = 0, \qquad \phi(1) = 1, \qquad \phi(\infty) = \infty. \tag{4}$$

If e is a primitive root of the $GF[p^n]$ then every mark of the field may be written in the form

$$e^k.$$

But by (3)

$$\phi(e^k) = [\phi(e)]^k.$$

Hence if $\phi(e) = e^m$, then $\phi(e^k) = e^{mk} = (e^k)^m$ and

$$\phi(x) = x^m.$$

If k is a primitive root of the $GF[p]$ then

$$\phi(k) = k.$$

Hence m must be of the form p^l† and therefore, as a transformation of the points of the x_1 axis, an anti-projectivity is given by the equation

$$\frac{x_1'}{x_3'} = \frac{x_1^{p^l}}{x_3^{p^l}},$$

and the number of antiprojectivities, exclusive of the identity, is $n - 1$.

An antiprojectivity is, however, completely determined by the permutation it effects among the points of the x_1-axis. For by projection from the center $(1\ 0\ 1)$ this permutation determines the only possible permutation of points on the fixed line $x_1 = x_2$. But a collineation is completely determined by the

* This derivation of (2) and (3) applies to any desarguesian geometry, whereas the proof of (3) given by LEVI assumes the commutativity of multiplication. This is because LEVI defines an antiprojectivity of a line as a transformation which changes harmonic throws into harmonic throws. The two definitions have not been proved identical in cases where multiplication is non-commutative. DARBOUX's proof of the fundamental theorem of projective geometry amounts to showing that conditions (2), (3), (4) are satisfied in the case of ordinary number only by the identical transformation.

† Similarly if the marks of any $GF[p^\nu]$ included in the $GF[p^n]$ are required to be invariant, $\phi(x) = x^{(p^\nu)^l}$. From this consideration are derived the results of the last section of Dr. LEVI's paper.

permutations of the points of two fixed lines. There are thus $n-1$ antiprojectivities of the plane. Since the following are evidently $n-1$ distinct antiprojectivities they must therefore be all the antiprojectivities of the plane:

$$\frac{x_1'}{x_3'} = \frac{x_1^{p^l}}{x_3^{p^l}}, \qquad \frac{x_2'}{x_3'} = \frac{x_2^{p^l}}{x_3^{p^l}}. \tag{5}$$

The most general collineation is now evidently a product of (5) by the most general linear fractional transformation and that product is of type (1). It follows as a corollary that the order of the collineation group of $PG(k, p^n)$ is the order of the linear fractional group $LF(k+1, p^n)$ multiplied by n.

PRINCETON, N. J.,
November 13, 1906.

GEOMETRY IN WHICH THE SUM OF THE ANGLES OF EVERY TRIANGLE IS TWO RIGHT ANGLES*

BY

R. L. MOORE

In his article *Die Legendre'schen Sätze über die Winkelsumme im Dreieck*,† DEHN has shown what may be stated precisely by the aid of the following abbreviations:

H_{-a} denotes all those theorems of geometry which are logical consequences of HILBERT's axiom-groups I, II, III, IV, viz., his axioms of association, of order, of parallels, and of congruence respectively.

H_{-a-p} denotes all those theorems of geometry which are logical consequences of HILBERT's axiom groups, I, II, IV.

S denotes the assumption that the sum of the angles of every triangle is two right angles.‡

What DEHN has shown is that H_{-a} does not follow from H_{-a-p} and S. I wish to show that, nevertheless, if H_{-a-p} and S are true of a space, then either this space is a H_{-a} space or it is possible so to introduce ideal points that the space thus extended will be a H_{-a} space, these ideal points, moreover, being such that no one of them is between two points of the original space.

One might state this a little more suggestively, if less accurately, as follows: "While the parallel postulate, III, and thus all of that part of Hilbertian Geometry which follows without use of his 'Axiom of ARCHIMEDES' and 'Vollständigkeit Axiom,' can not indeed be proved from his other postulates I, II, IV with III replaced by S, nevertheless it can be shown that a space concerning which these postulates (I, II, IV, S) are valid must be, if not the whole, then at least a part, of a space in which III also is true."§

* Presented to the Society (Chicago) April 22, 1905, as part of a paper entitled *Sets of metrical hypotheses for geometry*. Received in revised form for publication December 26, 1906.

† Mathematische Annalen, vol. 53 (1900), pp. 404–439.

‡ This may be stated precisely as follows: If ABC is a triangle and C is between B and E then, in the angle ACE, there is a point D such that $\measuredangle ACD \equiv \measuredangle BAC$ and $\measuredangle DCE \equiv \measuredangle ABC$.

§ This result has an interesting connection with our spatial experience. Statements have been made to the effect that, since no human instruments, however delicate, can measure exactly enough to decide in every conceivable case whether the sum of the angles of a triangle is equal to two right angles (unless the difference between this sum and two right angles should exceed

Let us first establish a number of preliminary theorems. The theorems of H_{-a-p} are assumed and I shall take for granted, sometimes without giving exact reference, certain of those definitions and theorems of HILBERT's *Festschrift* which are founded on his axiom groups I, II, IV. Reference will also be made to HALSTED's *Rational Geometry* (R. G.) in case of theorems for which demonstrations without use of HILBERT's III are therein indicated.

For simplicity let us confine ourselves to one plane.

THEOREM I. *If two straight lines, a and b, are perpendicular to each other at a point O, then every straight line, c, in the plane ab, has a point in common with either a or b.*

*Dem.** This conclusion of course holds if c coincides with a or b. If it does not coincide with a or b, then it has (cf. HILBERT's Theorem 5 and definition following IV, 3 and Theorem 9) a point C within one of the four right angles into which a and b divide the plane ab. There exists a point A on that ray of this angle which belongs to a and a point B on that ray thereof which belongs to b. There is (cf. R. G., art. 47, theorem) a point D on c such that OD is perpendicular to c. On c there are two points, E and E', such that $DE' \equiv DE \equiv OD$ (cf. HILBERT's IV, 1 and IV, 2). Then (as may be seen with help of S), the angle DOE is one half a right angle and the angle $E'OD$ is one half a right angle. Accordingly the angle $E'OE$ is a right angle. Hence it is easily seen (cf. R. G., arts. 51 and 143) that E and E' can not both be within the right angle AOB. Therefore c, which contains both E and E', has a point within, and a point without, the angle AOB and thus must cut either OA or OB.

THEOREM 2. *If O lies between A and C and also between B and D, and if $\measuredangle AOB$ is a right angle and $\measuredangle ABD \equiv \measuredangle ACD$, then there is a circle passing through A, B, C, and D.*†

Dem. There exist (cf. R. G., 47, 51 and HILBERT's IV, 4) perpendiculars

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a certain minimum amount), it is therefore impossible to settle the question whether our space is Bolyai-Lobachewskian or Euclidean even though it be granted that it is one or the other.

Now the geometry which I wish to show to be a consequence of H_{-a-p} and S is one concerning whose resemblance to Euclidean space one may say even more. One may say (possibly a little roughly): "While, as DEHN has shown, it is indeed true that a space of which H_{-a-p} and S were true would not necessarily be strictly Hilbertian (in the sense of H_{-a}), neverthless no human being confined therein could ever distinguish it from a Hilbertian space even though he were supplied with instruments which could decide for him whether any two segments were exactly equal."

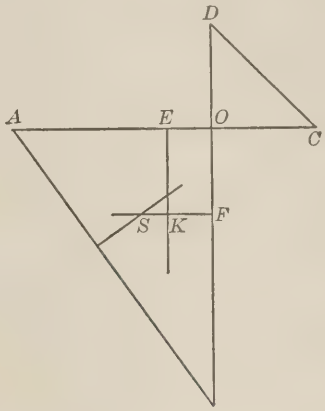
* "*Dem.*" means an *indication* of a demonstration. Likewise "by theorem" does not necessarily mean that this is the only theorem used in the demonstration. Similar statements may be made about the use, in this paper, of the words "therefore," "hence," etc.

Capital letters of the English alphabet designate points.

The sign "$=$" means "is, or are, identical with."

† This theorem gives a sufficient condition that three given points should lie on a circle, also a sufficient condition that 4 given points should lie on a circle.

LE and MF to AC and BD at their middle * points E and F respectively. As LE cannot meet BD (cf. R. G., art. 47), according to Theorem 1 it must meet FM at a point K. Now, by S and Theorem 1, the perpendicular to AB at its middle point must meet either KE or KF. Suppose it meets KF at a



point S. Then $SD \equiv SB \equiv SA$ (by definitions of a right angle and of a perpendicular, HILBERT's Theorem 10, and statement preceding Theorem 13). So there is a circle with S as center passing through A, B, and D. By R. G., art. 57, $\measuredangle SAD \equiv \measuredangle SDA$ and $\measuredangle SAB \equiv \measuredangle SBA$. It may then be seen, even without further use of S, that SAD and SAB are acute and hence SA is not perpendicular to AC and therefore, by R. G., art 138, that the circle which passes through A, B, D must have another point P in common with AC also that $\measuredangle DPA \equiv \measuredangle DBA$ (cf. S and R. G., art. 133); but this, in view of the hypothesis that $\measuredangle DBA \equiv \measuredangle DCA$, is impossible (cf. R. G., 66) unless P is C. The circle therefore through A, B, and D passes also through C. Similarly if the perpendicular to AB at its middle point met EK it would follow that there would be a circle through A, B, and C, and this circle would necessarily pass through D.

Convention. A point O, two straight lines perpendicular to each other at this point, a point I on one of these lines and points Q and S on the other, such that O is between Q and S, are selected once for all and considered as fixed throughout this discussion.

As an aid to the exhibition of an analytic geometry I wish to develop a calculus of those rays (half lines) which start from I towards the O-side of IK where IK is perpendicular to OI. These rays are called "leftward rays." If a leftward ray cuts QS it is called a "cutting ray." Small letters of the English alphabet are used to denote cutting rays.

* See R. G., articles 82, 64 and 66.

If M is a point of QS, $|IM|$ means IT where T is a point of the ray OQ such that $OM \equiv OT$.

The ray IO is designated by the symbol 0 (it is a "zero ray"). If I' is a point on the ray OQ such that $OI' \equiv OI$, then the ray II' is designated by the symbol 1 (it is a "unit ray").

Definition 1 (sum of two cutting rays). If L and M are two points (or the same point) on QS, ray IL + ray IM means ray IN where N is a point of QS such that $ON = OL + OM$ in the vector sense.

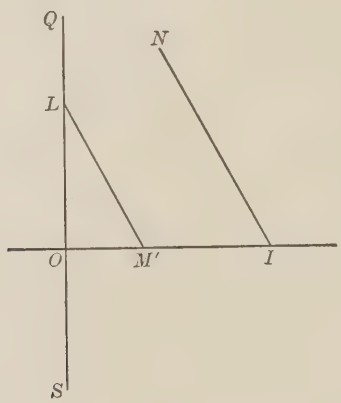
Definition 2. If IM and IN are two leftward rays, then "$IM > IN$" means that IN precedes IM in the sense $IS - IO - IQ$; and "$IM < IN$" means $IN > IM$.

THEOREM 3. *If α and β are two leftward rays, then of the three statements $\alpha > \beta$, $\alpha = \beta$, or $\alpha < \beta$ one and only one is true. If $\alpha < \beta$ and $\beta < \gamma$, then $\alpha < \gamma$.*

THEOREM 4. *The set of all cutting rays is a commutative group with respect to the operation of addition* $(+)$.

Dem. See Definition 1.

Definition 3 (quotient of two cutting rays). If L and M are points of the straight line QS and M is a point of ray OI such that $OM \equiv OM'$ and if IN is a ray such that $\measuredangle OIN \equiv \measuredangle OM'L$, if further IN lies on the Q- or the S-side of OI according as M and L are on the same side, or opposite sides, of O, then $IL/IM = IN$.



Definition 4 (product of two cutting rays), $ab = c$ if and only if $c/a = b$.

THEOREM 5. *If $a/b = c/b$, then $a = c$.*

THEOREM 6. *$b/1 = b$ and $b \times 0 = 0$.*

THEOREM 7. *If $a/b = c/d$, then $b/a = d/c$.*

THEOREM 8. *If A, B, C, D are points on QS and A and B are on the same side or opposite sides of O according as C and D are on the same side*

or on opposite sides of O, then the existence of four points A', B', C', D' on a circle and another point O' such that O' is between B' and C', and between A' and D', angle $A'O'B'$ is a right angle, $OA \equiv O'A'$, $OB \equiv O'B'$, $OC \equiv O'C'$, and $OD \equiv O'D'$, is a necessary and sufficient condition that $IA/IB = IC/ID$.

Dem. Use Definition 3 and Theorem 2.

THEOREM 9. *If $a/b = c/d$, then $a/c = b/d$.*

Dem. Use Theorem 8.

THEOREM 10. *If $a/b = c/d$ and $a/e = f/d$, then $b/e = f/c$.*

*Dem.** Consider three cases —

I. Suppose $|b| < |e|$. Evidently there exist an n such that $b/n = e/d$, and an x such that $b/x = e/c$ whence also $b/e = x/c$ (see Theorem 9). Now $a/f = e/d$ (see hypothesis and Theorem 9), so that $a/f = b/n$. Also $b/e = n/d$ (see Theorem 9), therefore $n/d = x/c$ and thus, by Theorem 9, $x/n = c/d$. But since, by hypothesis, $a/b = c/d$, we have $a/b = x/n$. But from $a/f = b/n$, by Theorem 9, $a/b = f/n$ and thus $x/n = f/n$; therefore, by Theorem 5, $x = f$. But as $b/e = x/c$, it follows finally that $b/e = f/c$.

II. Suppose $e < b$. Use similar reasoning to show that $e/b = c/f$ and thence conclude (by use of Theorem 7) that $b/e = f/c$.

III. Suppose $e = b$. In this case, by Theorem 5, $f = c$ and by Theorem 9 the theorem is true.

THEOREM 11. *If $ab = x$† and $a'b' = ab$, then $a/a' = b'/b$; and conversely, if $ab = x$ and $a/a' = b'/b$, then $a'b' = ab$.*

Dem. I. If $ab = x$ and $a'b' = ab$, then, according to Definition 4, $x/a = b$ and $x/a' = b'$. According to Theorems 6 and 7 $1/b' = a'/x$ and $1/b = a/x$. Hence, by Theorem 10, $b'/b = a/a'$.

II. If $ab = x$ and $a/a' = b'/b$, then, by Definition 4 and Theorem 2, $x/a = b/1$. Hence, by Theorem 7, $a/x = 1/b$ and therefore, by hypothesis and Theorems 10 and 6 and Definition 4, $x = a'b'$.

THEOREM 12. *If $ab = x$, then $ba = ab$.*

Dem. If $ab = x$, then, by Definition 4 and Theorem 6, $x/a = b/1$. Hence, by Theorems 9 and 6, $x/b = a$. Therefore, by Definition 4, $ba = x$. But since, by hypothesis, $ab = x$, $ba = ab$.

THEOREM 13. *If $ab = x$, $(ab)c = y$, and $bc = z$, then $a(bc) = (ab)c$.*

Dem. According to Theorem 11 this proposition will be established if it is proved that $ab/a = bc/c$. Now according to Definition 4 and Theorem 6 $ab/a = b/1$. Likewise $bc/c = b/1$. So $ab/a = bc/c$.

THEOREM 14. *If $ab = x$, $ac = y$, then $a(b + c) = ab + ac$.*

Dem. Use S in connection with 2nd figure on page 34 of HILBERT's *Grundlagen der Geometrie* (2nd edition).

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* The suggestion of this demonstration I am unable at present to trace to its source.

† i. e., ab is a cutting ray.

THEOREM 15. *If* $|a| < 1$ *then for every* b, ab *exists as a cutting ray. For every* c *there exists* x *such that* $|cx| < 1$.

If $|a| < |b|$ and bx exists as a cutting ray, then $|ax| < |bx|$. Given d, e, k, $e \neq 0$, $k \neq 0$, then there exist d', e' such that $|d'| < |k|$, $|e'| < |k|$, and $d/e = d'/e'$.

THEOREM 16. *If* $|a|, |b|, |c|, |d|$ *are all less than* 1, *and if* $a/b = e$ *and* $c/d = f$, *then* $a/b + c/d = (ad + bc)/bd$.

Dem. According to hypothesis and Definition 4, $a = be$, $c = df$; and by hypothesis and Theorems 15, 12, 13, and 14, $ad + bc = (be)d + b(df) = d(be) + b(df) = (db)e + (bd)f = (bd)e + (bd)f = bd(e + f)$. This gives, by the use of Definition 4, $(ad + bc)/bd = e + f = a/b + c/d$.

THEOREM 17. *If* $|a|, |b|, |c|, |d|$ *are all less than* 1, $a/b = e$, $c/d = f$ *and* $ef = y$, *then* $ac/bd = ef$.

Dem. By hypothesis and Definition 4, $c = df$, $a = be$ and, by hypothesis and Theorem 15, $ac = (be)(df)$. Hence, by hypothesis, Theorems 15, 12, and 13, $ac = (bd)(ef)$, and from this, by Definition 4, $ac/bd = ef$.

THEOREM 18. *If* $|a|, |b|, |c|, |d|, |a'|, |b'|, |c'|, |d'|$ *are all less than* 1, *and if* $a/b = a'/b'$, $c/d = c'/d'$, *then* $(ad + bc)/bd = (a'd' + b'c')/b'd'$.

Dem. By hypothesis and Theorems 15 and 11, $ab' = ba'$ and $cd' = dc'$. Hence (see Theorem 15) $(ab')(dd') = (ba')(dd')$ and $(cd')(bb') = (dc')(bb')$. Hence, by Theorems 12, 13, and 14, $(ad)(b'd') + (bc)(b'd') = (bd)(a'd') + (bd)(b'c')$ and thus, by Theorems 12 and 14, $(ad + bc)b'd' = bd(a'd' + b'c')$; therefore by Theorem 11, $(ad + bc)/bd = (a'd' + b'c')/b'd'$.

THEOREM 19. *If* $|a|, |b|, |c|, |d|, |a'|, |b'|, |c'|, |d'|$ *are all less than* 1 *and* $a/b = a'/b'$ *and* $c/d = c'/d'$, *then* $ac/bd = a'c'/b'd'$.

Dem. According to Theorems 15 and 11, $ab' = ba'$, $cd' = dc'$, whence follows (see Theorem 15), $(ab')(cd') = (ba')(dc')$. Hence, by Theorems 15, 12, and 13, $(ac)(b'd') = (bd)(a'c')$ and thus, by Theorem 11, $ac/bd = a'c'/b'd'$.

Definition 5. If either there is no e such that $a/b = e$ or there is no f such that $c/d = f$, then $a/b \times c/d$ means $a'c'/b'd'$ and $a/b + c/d$ means $a'd' + b'c'/b'd'$ where a', b', c', d' are in absolute value less than 1, and $a/b = a'/b'$ and $c/d = c'/d'$.

THEOREM 20. $a/b \times c/d = a'c'/b'd'$ *and* $a/b + c/d = a'd' + b'c'/b'd'$ *where* a', b', c', d' *are all less than* 1 *in absolute value and* $a'/b' = a/b$ *and* $c'/d' = c/d$.

Dem. See Theorems 16 and 17 and Definition 5.

Lemma. If $a'b$, $a'c$ and $(a'b)c$ are cutting rays, then $a'b/a'c = b/c$.

Dem.

$$\begin{aligned} (a'b)c &= (ba')c && \text{by hypothesis and Theorem 12,} \\ &= b(a'c) && \text{by Theorem 13,} \\ &= (a'c)b && \text{by Theorem 12.} \end{aligned}$$

Hence, by Theorem 11, $a'b/a'c = b/c$.

THEOREM 21. *If b and d are different from 0 and a and c are any cutting rays, then there exist x and y $(y \neq 0)$ such that $a/b + x/y = c/d$. If $a/b + x/y = c/d$ and $a/b + w/z = c/d$, then $x/y = w/z$.*

Dem. Use Lemma and Theorems 15, 4, 20, 11.

THEOREM 22. *If a, b, and d are different from 0 and c is any cutting ray, then there exist x and y such that $a/b \times x/y = c/d$. If $a/b \times x/y = c/d$ and $a/b \times w/z = c/d$, then $x/y = w/z$.*

Dem. I. By Theorem 15 there exist a', b', c', d' all less than 1 in absolute value, such that $a/b = a'/b'$, $c/d = c'/d'$. By Theorems 15, 20, 13, 11, and Lemma, $a/b \times b'c'/a'd' = a(b'c')/b(a'd') = (ab')c'/(ba')d' = c'/d'$.

II. Suppose $a/b \times x/y = c/d$ and $a/b \times w/z = c/d$. By Theorem 15 there exist a', b', c', d', x', y', w', z', all less than 1 in absolute value, such that $a/b = a'/b'$, $c/d = c'/d'$, $x/y = x'/y'$, $w/z = w'/z'$. By hypothesis and Theorem 20, $a'x'/b'y' = c'/d'$ and $a'w'/b'z' = c'/d'$. Hence, by Theorems 15, 11, 12, 13, $(a'd')x' = (b'c')y'$ and $(a'd')w' = (b'c')z'$. Hence, by Theorems 12 and 11, $x'/y' = b'c'/a'd' = w'/z'$. Hence $x/y = w/z$.

Convention. Any Greek letter denotes a/b where a and b are cutting rays.

THEOREM 23. *The set of all α's is a number system for which Hilbert's Theorems 1–16 of § 13 of his Grundlagen der Geometrie hold true with respect to $+$, $\times$, and $>$ as defined in this paper.**

Dem. HILBERT's Theorems 1–12 may be proved by use of my Theorems 20, 6, 4, 21, 22, 12, 15, 14, 13, and Lemma to Theorem 22. His Theorems 13–16 may be proved with the use of my Theorems 1–12 in connection with Definitions 1–5.

Definition 6. The length of a segment AB means the ray IT where T is a point on the ray OQ such that $AB \equiv OT$.

The length of AB is denoted by "$e(AB)$."

Lemma I. If in the triangles ABC, $A'B'C'$, $\measuredangle A \equiv \measuredangle A'$, $\measuredangle B \equiv \measuredangle B'$, $\measuredangle C \equiv \measuredangle C'$, then

$$\frac{e(AB)}{e(A'B')} = \frac{e(BC)}{e(B'C')} = \frac{e(CA)}{e(C'A')}.$$

Dem. Use Definitions 6 and 3, Theorem 23, etc., in connection with Theorem 22 of HILBERT's *Grundlagen der Geometrie.*

Lemma II. If BAC is a right angle, then $e^2(AB) + e^2(AC) = e^2(BC)$.

Dem. There is (cf. R. G., Art. 47) a point D on BC such that AD is perpendicular to BC. D is between B and C. Otherwise (as may easily be seen, cf. R. G., Art. 66, etc.), either AD and AC or AD and AB could not meet on the A side of BC. According to S, $\measuredangle ACB \equiv \measuredangle BAD$ and

* See Definitions 1–5.

$\measuredangle CAD \equiv \measuredangle ABC$. It follows by Lemma I, Theorem 23 and Definitions 6 and 1, that

$$e^2(AB) + e^2(AC) = e(BD)e(BC) + e(CD)e(BC)$$
$$= \{e(BD) + e(CD)\}e(BC) = e^2(BC).$$

THEOREM 24. *Given α, β there exists γ such that $\gamma^2 = \alpha^2 + \beta^2$.*

Dem. This theorem is evidently true for the case in which α or $\beta = 0$. If α and β are different from 0, then by Theorems 6 and 15 there exist a, b, c, d, all less than 1 in absolute value, such that $\alpha = a/b$, $\beta = c/d$. By Theorem 23 $\alpha^2 + \beta^2 = [(ad)^2 + (bc)^2]/(bd)^2$. But, by Theorem 15 and Definition 6, ad and bc are lengths of segments. Hence there exists a right-angled triangle BAC such that the length of AB is ad and the length of AC is bc. According to Lemma 2, $e^2(BC) = (ad)^2 + (bc)^2$. Therefore, by Theorem 23, $\alpha^2 + \beta^2 = e^2(BC)/(bd)^2 = [e(BC)/bd]^2$.

Convention. Two straight lines OX and OY perpendicular to each other are selected as axes of coördinates. If P is any point and D and E are the feet of the perpendiculars from P to OX and OY respectively, then x_P means 0, $e(PE)$, or $-e(PE)$ according as P is on OY, on the X-side of OY, or on the non-X-side of OY, and y_P means 0, $e(PD)$, or $-e(PD)$, according as P is on OX, on the Y-side of OX, or on the non-Y-side of OX.

As a result of this convention we have the theorem:

THEOREM 25. *Every point P is represented by one and only one sensed pair of coördinates (x_P, y_P) and every sensed pair of cutting rays represents one and only one point.*

THEOREM 26. *Given a straight line AB, there exist α, β, γ (α, β not both 0) such that the x and y of every point on AB satisfy the equation $\alpha x + \beta y + \gamma = 0$.*

Dem. Use S, Theorems 1 and 23, lemma I, Definitions 6 and 1, also R. G., articles 47 and 49, and HILBERT's axiom IV, 4, Axiom IV, 1, Theorem 11, etc.

THEOREM 27. *If B is between A and C and $x_A \neq x_B$, then $x_A < x_B < x_C$ or $x_A > x_B > x_C$. If B is between A and C and $x_A = x_B$, then $y_A > y_B > y_C$ or $y_A < y_B < y_C$.*

Dem. Use Theorems 1 and 23, R. G., 47, and HILBERT's Theorem 5 et seq.

THEOREM 28. *The length $e(AB)$ equals $\sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}$ and thus the relation $e(AB) = e(A'B')$ is a necessary and sufficient condition for the equality*

$$(x_A - x_B)^2 + (y_A - y_B)^2 = (x_{A'} - x_{B'})^2 + (y_{A'} - y_{B'})^2.$$

Dem. Use lemma 2 in connection with convention preceding Theorem 25.

THEOREM 29. *The existence of α, β, γ, δ satisfying the relations*

$$x_{A'} = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}} x_A - \frac{\beta}{\sqrt{\alpha^2+\beta^2}} y_A + \gamma,$$

$$x_{B'} = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}} x_B - \frac{\beta}{\sqrt{\alpha^2+\beta^2}} y_B + \gamma,$$

$$y_{A'} = \frac{\beta}{\sqrt{\alpha^2+\beta^2}} x_A + \frac{\alpha}{\sqrt{\alpha^2+\beta^2}} y_A + \delta,$$

$$y_{B'} = \frac{\beta}{\sqrt{\alpha^2+\beta^2}} x_B + \frac{\alpha}{\sqrt{\alpha^2+\beta^2}} y_B + \delta,$$

is a necessary and sufficient condition for the congruence $AB \equiv A'B'$.

In view of Theorems 23–27 and Theorem 29 it may be easily seen that one could proceed in a manner similar to that indicated by HILBERT in § 9 of his *Grundlagen der Geometrie* and show that all of his axioms hold true of our geometry, were it not for one obstacle, namely that there is not necessarily a perfect one-to-one correspondence between the set of all points and the set of all sensed pairs of α's. If α is not a cutting ray then (α, β) does not correspond to a point in manner indicated in Theorem 25.

It is natural then to fill up this gap by means of the following definitions:

Definition 7. Every sensed pair (α, β) is called an ideal point and "ideal point" means such a sensed pair; α and β are called its coördinates.

Definition 8.* An ideal straight line means the set of all existent points, real and ideal, whose coördinates satisfy an equation of the form $\alpha x + \beta y + \gamma = 0$.

Definition 9. If of the points A, B, C at least one is ideal then ABC means $x_A > x_B > x_C$ or $x_A < x_B < x_C$ unless $x_A = x_B$, in which case ABC means $y_A > y_B > y_C$ or $y_A < y_B < y_C$.

Definition 10. If one of the points A, A', B, B' is ideal then "$AB \equiv A'B'$" means there exist α, β, γ, δ satisfying the relations

$$x_{A'} = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}} x_A - \frac{\beta}{\sqrt{\alpha^2+\beta^2}} y_A + \gamma,$$

$$x_{B'} = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}} x_B - \frac{\beta}{\sqrt{\alpha^2+\beta^2}} y_B + \gamma,$$

$$y_{A'} = \frac{\beta}{\sqrt{\alpha^2+\beta^2}} x_A + \frac{\alpha}{\sqrt{\alpha^2+\beta^2}} y_A + \delta,$$

$$y_{B'} = \frac{\beta}{\sqrt{\alpha^2+\beta^2}} x_B + \frac{\alpha}{\sqrt{\alpha^2+\beta^2}} y_B + \delta.$$

* In this definition and in what follows, small letters of the English alphabet are not always used, according to earlier convention, to designate cutting rays exclusively.

In view of Theorems 25, 26, 27, 29 and the corresponding Definitions 6, 7, 8, 9 it is evident that our real and ideal points and straight lines form a system which, with respect to order, congruence and association, is related to our set of α's as HILBERT's geometry of § 9 of his *Festschrift* is related to his number system of that paragraph, and moreover in view of Theorems 23 and 24 it is evident that the set of α's satisfy with respect to the operations of addition $(+)$, multiplication $(\times)$, and the relation $>$, sufficient conditions to enable us to proceed according to the method indicated by HILBERT to prove that our geometry satisfies his axioms of groups I–IV. It remains to be shown that no ideal point is between two real points. This may be proved by use of Definitions 7 and 1 and Theorem 23.

For helpful suggestions and criticisms the author desires to tender to Professor VEBLEN his grateful acknowledgment.

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NON-DESARGUESIAN AND NON-PASCALIAN GEOMETRIES*†

BY

O. VEBLEN AND J. H. MACLAGAN-WEDDERBURN

§ 1. The object of this paper is to discuss certain non-pascalian and non-desarguesian geometries. Beside their intrinsic interest, these geometries are useful in forming independence proofs for systems of axioms for geometry.

In a previous paper ‡ it has been shown how to construct all finite projective geometries of three or more dimensions and all two dimensional projective geometries in which the Desargues theorem (cf. third footnote) is valid. The present paper supplements the other by showing the existence of non-desarguesian finite geometries. It is to be noted that, since every finite linear associative algebra in which every number possesses an inverse is commutative, § therefore every non-pascalian finite geometry is necessarily non-desarguesian. Moreover, as has been shown by HESSENBERG,‖ every non-desarguesian geometry is also non-pascalian, whether finite or infinite.

A *projective plane geometry* is a set of elements called points, finite or infinite in number, subject to the following conditions:

1. If A and B are any two points, there is (α) one and (β) only one set of points called a line and containing both A and B.

2. If a and b are any two lines, there is (α) one and (β) only one ¶ point contained in both a and b.

3. Each line contains at least three points.

* Presented to the Society (Chicago), April 22, 1905. Received for publication, December 6, 1906.

† As originally read, the paper called attention to the fact that HILBERT's example of a non-pascalian geometry in his *Festschrift* did not satisfy his order axioms. The error however is not repeated in later editions of his book. This paper also contained a discussion of the non-pascalian geometry obtained by letting the coördinates of a point be quaternions. As such geometries are discussed in the *Abstrakte Geometrie* of K. T. VAHLEN, Leipzig, 1905, this part of the paper is omitted.

‡ VEBLEN and BUSSEY, *Finite projective geometries*, Transactions, vol. 7 (1906), p. 271. The reader is referred to this paper for references to the literature and discussion of the general problem of finite geometries. Condition VII of that paper is the "Pascal Theorem" and the paragraph following VII contains the "Desargues Theorem."

§ J. H. MACLAGAN-WEDDERBURN, *A theorem on finite algebras*, Transactions, vol. 6 (1905), p. 349.

‖ G. HESSENBERG, *Beweis des Desarguesschen Satzes aus dem Pascalschen*, Mathematische Annalen, vol. 61 (1905), p. 161.

¶ The condition 2 (β) is of course logically a consequence of 2 (α) and 1 (β).

The geometry is non-desarguesian or non-pascalian if the theorems of DESARGUES or of PASCAL, respectively, fail to be valid.

If the number of points is finite it is at once evident that every line contains the same number of points and that if $n+1$ is the number of points on a line, the total number of points is n^2+n+1.

Conversely, if the total number of points is n^2+n+1 and the number of points on each line is $n+1$ and every two points lie on one and only one line, it follows that every two lines have one and only one point in common. For, suppose the two lines a, b had no point in common. A point of a would determine with b $n+1$ lines giving thus $n(n+1)+1$ points and, counting in the points of a, we should have at least n^2+2n points instead of n^2+n+1.

By a dual argument, if the number of lines is n^2+n+1 and the number of lines through each point is $n+1$ and every two lines have one and only one point in common, it follows that every two points determine one and only one line.

§ 2. Analytic geometries may be constructed so as to satisfy the given conditions if the coördinates are numbers of a number-system subject only to the following conditions:

(1) The numbers form a commutative group under the operation of addition: that is, for any two numbers a and b there is a unique number c such that $a+b=c$ and a unique number d such that $a+d=b=d+a$, and there is a number 0 such that, for every a, $a+0=a=0+a$; further, addition is associative and commutative,* i. e. $(a+b)+c=a+(b+c)$ and $b+a=a+b$.

(2) For any two numbers a and b there is a unique number c such that $ab=c$ and if $a \neq 0$ unique numbers d, d' such that $da=b$ and $ad'=b$; also $0a=0=a0$ for every a.

(3) Finally there must be some sort of a distributive law. We shall use the following: $c(a+b)=ca+cb$. We shall not use for the present† the other distributive law $(a+b)c=ac+bc$, nor the associative and commutative laws of multiplication, nor shall we assume that there exists an identity element with respect to multiplication.

A *point* is now defined as one of the systems of three coördinates

$$(\alpha) \qquad x,\ y,\ \phi,$$

$$(\beta) \qquad x,\ \phi,\ 0,$$

$$(\gamma) \qquad \phi,\ 0,\ 0,$$

* The commutativity of addition is not an independent postulate. Cf. H. HANKEL, *Theorie der complexen Zahlensysteme* (Leipzig, 1867), page 32; E. V. HUNTINGTON, Annals of Mathematics, vol. 8 (1906), page 25.

† Cf. however the last sentence of this §.

where ϕ is not equal to 0 and is the same for all points. There are $n^2 + n + 1$ points in all if there is a finite number, n, of elements in the number-system.

A *line* is defined as all points which satisfy an equation of one of the forms

$$x\psi + yb + zc = 0, \tag{1}$$

$$y\psi + zc = 0, \tag{2}$$

$$z\psi = 0, \tag{3}$$

where ψ is not equal to 0 and is the same for all lines. There are $n^2 + n + 1$ lines altogether in the finite case. Further there are, in that case, $n + 1$ points on a line. For (3) is satisfied by points of types (β) and (γ): in case $c = 0$ (2) is satisfied by the n points of type (α) in which $y = 0$ and by (γ): in case $c \neq 0$ in (2) if $z = 0$, then $y = 0$ and hence (γ) is one of the points, while if $z = \phi$, y must be equal to the solution of $y\psi = -\phi c$ and x may have any of n values. The argument is similar for an equation of type (1). A similar argument will show that in the finite case the number of lines through a point is $n + 1$.

To show that our points and lines form a projective plane geometry it is necessary to prove that any two lines have in common one and only one point. This requires the consideration of several cases.

(a) Two lines of type (1):

$$x\psi + yb + zc = 0,$$

$$x\psi + yb' + zc' = 0.$$

The coördinates of a point of intersection must satisfy the condition

$$y(b - b') + z(c - c') = 0.$$

In case $c = c'$, b cannot be the same as b' (else the lines would be identical) and hence $y = 0$; the lines then have in common only the point $(\alpha, 0, \phi)$ where α is the unique solution of $x\psi = -\phi c$. In case $c \neq c'$ and $b = b'$, we have $z = 0$, $y = \phi$ and x is the solution of $x\psi = -\phi b$. In case $c \neq c'$, $b \neq b'$ we have $z = \phi$ and hence y is the solution, say α, of $y(b - b') = -\phi(c - c')$ and x is the solution of $x\psi = -(\alpha b + \phi c)$. On account of the commutative law of addition the values of x, y, and z thus found must satisfy the given equations.

(b) Lines of types (1) and (2):

$$x\psi + yb + zc' = 0,$$

$$y\psi + zc' = 0.$$

The coördinate z cannot be zero, else y and x would also be zero. Hence $z = \phi$, y is the solution, α, of $y\psi = -\phi c'$, and x the solution of $x\psi = -(\alpha b + \phi c')$.

(*c*) Lines of types (1) and (3):

$$x\psi + yb + zc = 0,$$

$$z\psi = 0.$$

The coördinate y cannot be zero since then x and z would both be zero. Hence $y = \phi$ and x is the solution of $x\psi = -\phi b$.

(*d*) Two lines of type (2):

$$y\psi + zc = 0,$$

$$y\psi + zc' = 0.$$

Any solution must satisfy the condition $z(c - c') = 0$ or $z = 0$. Hence $y = 0$ and $x = \phi$.

(*e*) Lines of types (2) and (3):

$$y\psi + zc = 0,$$

$$z\psi = 0.$$

The only solution is $(\phi, 0, 0)$.

This, according to the last sentence of § 1, completes the proof that all finite number systems of the type described above lead to finite plane geometries. If the number system is not finite it is necessary to prove algebraically that every two points determine a unique line. If the other distributive law holds also or if multiplication is commutative this can be done by the analysis used above.

§ 3. In order to construct a particular non-desarguesian geometry it is necessary only to adduce a particular number-system. The first class of systems which we shall use is due to L. E. Dickson.* It is such that the number of elements in each system is finite and such that multiplication is associative but not commutative and the distributive law holds only in the form $x(y + z) = xy + xz$. Each system possesses an identity which in the analysis above may be taken as the common value of ϕ and ψ. These systems are discussed on pages 5 to 22 of Dickson's paper. A characteristic system of this type is the following. There are p^{2m} elements, (a, b), where a and b are marks of a field $GF[p^m]$, p being an odd prime. These elements obey the following laws of combination:

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$$

$$(a_1, b_1) \times (a_2, b_2) = (a_1 a_2 + \epsilon\nu b_1 b_2, b_1 a_2 + \epsilon a_1 b_2)$$

where ν is a fixed not-square in the $GF[p^m]$, and $\epsilon = \pm 1$ according as $a_1^2 - \nu b_1^2$ is a square or not-square, i. e., $\epsilon = (a_1^2 - \nu b_1^2)^{(p^m-1)/2}$.

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* L. E. Dickson, *On finite algebras*, Göttinger Nachrichten, 1905, pp. 358–394.

The simplest special case is where $p^m = 3$. The algebra may then be thought of as having two units 1 and j and composed of all numbers of the form $x + yj$, x and y being integers modulo 3 and $j^2 = -1$. The system* is thus composed of nine numbers $0, 1, 2, j, 2j, 1+j, 1+2j, 2+j, 2+2j$.

It is not difficult (as the reader may verify analytically) to find cases which show that the geometry founded on this number-system is non-desarguesian and non-pascalian. Instead of doing this, however, we shall now show how to get another geometry based on the same number-system. The new geometry is so constructed as to be capable of being explicitly exhibited in a compact form.

Since the number-system is associative and distributive [in the sense, $a(b+c) = ab + ac$] the point (x, y, z) may also be represented by $(\kappa x, \kappa y, \kappa z)$, $\kappa \neq 0$. With this understanding a linear transformation changes points into points and, if their coefficients are scalar, two transformations may be combined in the usual way. They do not in general change lines into lines.†

Consider now the following transformation of period 13:

$$A: \qquad x = x' + z', \qquad y = 2x', \qquad z = 2y' + 2z'.$$

Let $A_0, B_0, \cdots, G_0$ denote the points $(2, 0, 1)$, $(2+2j, j, 1)$, $(2+j, 2j, 1)$, $(1+2j, 1+j, 1)$, $(j, 2+2j, 1)$, $(1+j, 1+2j, 1)$, $(2j, 2+j, 1)$ and let $A_k, B_k, \cdots, G_k$ $(k < 13)$ denote the points derived from these by transforming them by the kth power of A. In this notation, seven of the lines of our original geometry are:

$$x + y + z = 0: \; A_0 A_1 A_3 A_9 B_0 C_0 D_0 E_0 F_0 G_0,$$
$$x + yj + z = 0: \; A_0 B_1 B_8 D_3 D_{11} E_2 E_5 E_6 G_7 G_9,$$
$$x + y(2j) + z = 0: \; A_0 C_1 C_8 E_7 E_9 F_3 F_{11} G_2 G_5 G_6,$$

---

* In this special case the distributive law has a very simple form; namely

$$(a + bj)j = -aj - bj^2 = b - aj$$

where neither a nor b is zero. Though the commutative law is not valid it is always true either that $xy = yx$ or that $xy = -yx$.

For the convenience of the reader in verifying computations, we subjoin the multiplication table:

	2	j	$2j$	$1+j$	$1+2j$	$2+j$	$2+2j$
2	1	$2j$	j	$2+2j$	$2+j$	$1+2j$	$1+j$
j	$2j$	2	1	$2+j$	$1+j$	$2+2j$	$1+2j$
$2j$	j	1	2	$1+2j$	$2+2j$	$1+j$	$2+j$
$1+j$	$2+2j$	$1+2j$	$2+j$	2	$2j$	j	1
$1+2j$	$2+j$	$2+2j$	$1+j$	j	2	1	$2j$
$2+j$	$1+2j$	$1+j$	$2+2j$	$2j$	1	2	j
$2+2j$	$1+j$	$2+j$	$1+2j$	1	j	$2j$	2

† For example, the transformation A changes $(1, 0, 0)$, $(0, j, 1)$ $(j, j, 1)$ which lie on $yj + z = 0$ into $(1, 2, 0)$ $(1, 0, 2+2j)$, $(1+j, 2j, 2+2j)$ of which the first two, but not the third, lie on $x(1+j) + y(1+j) + z = 0$.

$$x + y(1+j) + z = 0: \; A_0 B_7 B_9 D_1 D_8 F_2 F_5 F_6 G_3 G_{11},$$
$$x + y(2+2j) + z = 0: \; A_0 B_2 B_5 B_6 C_3 C_{11} E_1 E_8 F_7 F_9,$$
$$x + y(1+2j) + z = 0: \; A_0 C_7 C_9 D_2 D_5 D_6 E_3 E_{11} F_1 F_8,$$
$$x + y(2+j) + z = 0: \; A_0 B_3 B_{11} C_2 C_5 C_6 D_7 D_9 G_1 G_8.$$

A and its powers give 84 other heptads of points which may be obtained by adding successively the integers $1, 2, \cdots, 12$ to the subscripts of the points of each of the above lines, the resultant subscripts being, of course, reduced modulo 13. *The ninety-one heptads so obtained are the lines of a new non-desarguesian and non-pascalian geometry.*

It is not difficult to verify *a posteriori* that this scheme does define a geometry. By trial we see that if two lines intersect in more than one point, then neither of these points can be of the form A_k or B_k. Further, the scheme is invariant under the following transitive substitution group:

$$1, \; (BDG)(CEF), \; (BGD)(CFE),$$
$$(BC)(DF)(EG), \; (BE)(CD)(FG), \; (BF)(CG)(DE).$$

Hence the same is true of the remaining letters. The other axioms may be tested in the same way. That the geometry is non-desarguesian and non-pascalian is proved by examples which are sufficiently indicated in the diagrams below.

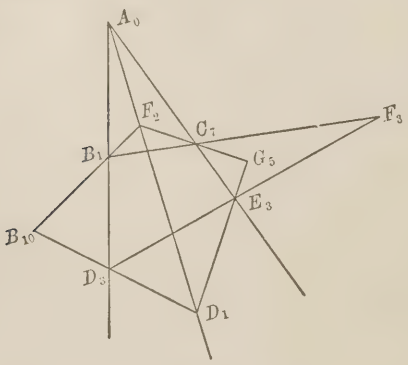

FIG. 1.

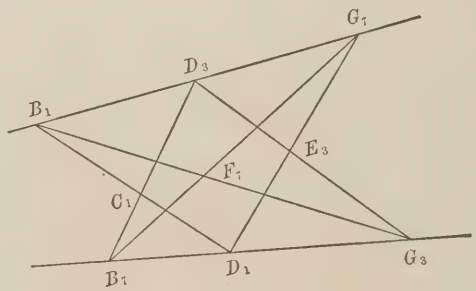

FIG. 2.

More interesting, however, than this tactical verification is the analytical proof that the 91 heptads derived by A and its powers from the lines B are the lines of a plane geometry. As in § 2, it is only necessary to show that any two heptads have one and only one point in common. It is a direct consequence of definition that a heptad is the set of points satisfying the equation,

$$x(a_{11} + a_{31}) + y(a_{12} + a_{32}) + z(a_{13} + a_{33}) + (xa_{21} + ya_{22} + za_{23})\beta = 0,$$

where β is any one of the seven numbers of the system and a_{rs} are the coefficients of any one of the thirteen powers of A. This equation may be abbreviated as follows:

$$(1) \qquad xa + yb + zc + (xa' + yb' + zc')\beta = 0.$$

Consider the intersection of this with

$$(2) \qquad x + y\alpha + z = 0.$$

In (1) and (2), α and β are the only coefficients which are not necessarily scalar. From (1) and (2) we deduce

$$(3) \qquad y(2a\alpha + b) + z(2a + c) + \{y(2a'\alpha + b') + z(2a' + c')\}\beta = 0,$$

and shall now show that there is only one point whose coördinates satisfy (2) and (3) simultaneously. There are three cases.

(i) $2a' + c' = 0$. In this case (3) becomes:

$$(4) \qquad y\{2a\alpha + b + (2a'\alpha + b')\beta\} + z(2a + c) = 0.$$

If the coefficients in (4) do not vanish identically, (2) and (4) have evidently one and only one point in common. On forming the powers* of the transformation A we find that $2a + c$ and $2a' + c'$ cannot vanish simultaneously except for the identical transformation, and in this case the vanishing of the coefficient of y leads to $\alpha = \beta$. Hence except in this trivial case (4) is not an identity.

(ii) $2a + c = 0$. In this case

$$y\{(2a\alpha + b)\beta^{-1} + 2a'\alpha + b'\} + z(2a' + c') = 0.$$

Hence as in (i) there is one and only one point common to (2) and (3).

(iii) $2a' + c' = (2a + c)d \neq 0$, where d is scalar. Then (3) may be written in the form

$$y\{2(a + 2a'd^{-1})\alpha + b + 2b'd^{-1}\} + \{y(2a'\alpha + b')d^{-1} + z(2a + c)\}(1 + d\beta) = 0,$$

or say

$$ye + (yf + zg)\gamma = 0.$$

This is equivalent to

$$y(e\gamma^{-1} + f) + zg = 0.$$

---

* The first twelve powers have the matrices

101	120	201	221	021	122	212	110	001	022	111	020
200	202	210	102	112	012	211	121	220	002	011	222
022	111	020	100	101	120	201	221	021	122	212	110

and the thirteenth power is the identical transformation.

Hence, since $g \neq 0$, (1) and (3) and therefore also (1) and (2) have one and only one point in common.

The point transformation A transforms lines of the new geometry into lines. Hence, if any two lines intersect in more than one point, the lines obtained by transforming these simultaneously by any power of A will also intersect in more than one point. But any pair of lines can, by transformation by a suitable power of A, be transformed into such a pair as has been considered above. Hence any two lines intersect in one and only one point.

§ 4. On pages 381–394, loc. cit.,* L. E. DICKSON has given another set of algebras which are available for geometrical purposes. The following multiplication table (cf. (68), loc. cit., p. 394) defines an algebra which satisfies the conditions of § 2 together with the commutative law of multiplication and has an identity element. It therefore leads to a geometry.

	i	j
i	j	$b+\beta i$
j	$b+\beta i$	$-\beta^2-8bi-2\beta j$

Any number of the algebra is of the form $a+di+ej$, where a, d and e are any marks of any field F, not having the modulus 2, in which $x^3=b+\beta x$ is irreducible. In such an algebra an equation of the form $ax=b$ has a unique solution if $a \neq 0$.

If $b=2$, $\beta=0$ and F is the field of rational numbers this table becomes

	i	j
i	j	2
j	2	$-16i$.

Points and lines are defined as above, but as multiplication is commutative the equation of a line is more conveniently written in the form $ax+by+cz=0$.

That the geometry is non-desarguesian is shown by the following example. Consider the six points $A=(0,0,1)$, $A'=(0,i,1)$, $B=(i,i,1)$, $B'=(i,j,1)$, $C=(j,0,1)$, $C'=(j,i,1)$. These points lie in pairs on the lines $x=0$, $x-iz=0$, and $x-jz=0$, which meet in the point $(0,1,0)$. The equations of the sides of the triangles ABC and $A'B'C'$ are as follows:

$$AC\colon\ y=0, \qquad A'C'\colon\ y-iz=0,$$

$$AB\colon\ x-y=0, \qquad A'B'\colon\ x+(1+i+j)y/17-(2+i+j)z/17=0,$$

$$BC\colon\ x-(1-i)y-jz=0, \qquad B'C'\colon\ x+y-(i+j)z=0.$$

*See also L. E. DICKSON, *Linear algebras in which division is always uniquely possible*, these Transactions, vol. 7 (1906), p. 371.

If D, E and F are the intersections of AC with $A'C'$, AB with $A'B'$, and BC with $B'C'$, the coördinates of these points are: $D=(1,0,0)$, $E=(x=y=8(17/8+2i+j)/177, 1)$ and $F=[(-1+i+2j)/3, (1+2i+j)/3, 1]$. The equation of the line DE is $y-8(17/8+2i+j)z/177=0$ and evidently does not contain F. Hence D, E and F are not collinear and the geometry is non-desarguesian.

In the same way by taking the six points $A'=(1,0,1)$, $B'=(i,0,1)$, $C'=(i+j,0,1)$, $A=(1,1,0)$, $B=(i,1,0)$, $C=(j,1,0)$, it can be shown that the geometry is non-pascalian.

The *finite* number-system of the present type with the smallest number of elements is obtained by letting F be the $GF[3]$ and letting $b=1$ and $\beta=1$. This however leads to a geometry with 28 points on a line and $(27)^2+27+1=757$ points in all and has not been exhibited explicitly.

§ 5. The number systems discussed above all have identity elements. The presence of an identity is not necessary for the analysis of § 2, and examples of number systems that lead to geometries, though lacking the identity, are not hard to find. Two simple cases are constructed as follows:

Let e_1 and e_2 be two independent units. The systems under consideration consist of all numbers of the form ae_1+be_2 where a and b are integers reduced modulo 2. Addition is defined by the equation

$$(ae_1+be_2)+(ce_1+de_2)=(a+c)e_1+(b+d)e_2.$$

The two multiplication tables are

(i)

	e_1	e_2	e_3
e_1	e_3	e_2	e_1
e_2	e_2	e_1	e_3
e_3	e_1	e_3	e_2

(ii)

	e_1	e_2	e_3
e_1	e_1	e_2	e_3
e_2	e_3	e_1	e_2
e_3	e_2	e_3	e_1

where $e_3=e_1+e_2$. In (i) multiplication is commutative and in both systems the two-sided distributive law holds. Each system gives rise to a geometry with five points on a line. Both geometries however turn out to be identical with $PG(2,2^2)$ which is based on the Galois field, $GF(2^2)$. This suggests the following theorem: *There is only one projective plane geometry with five points on a line.*

To prove this consider the following Pappus configuration in which O, A, B, C, A_1, B_1, C_1 are distinct. Consider the lines DE, EF, FD. These lines obviously cannot intersect the line OA in any of the points A, B, C; hence as there are only two other points on OA, D, E, and F must be collinear. The geometry is therefore pascalian. DESARGUES's theorem may be proved in the

same way or it may be deduced from PASCAL's theorem. Now there is only one pascalian geometry for a given finite number of points on a line,* therefore there is only one geometry with five points on a line.

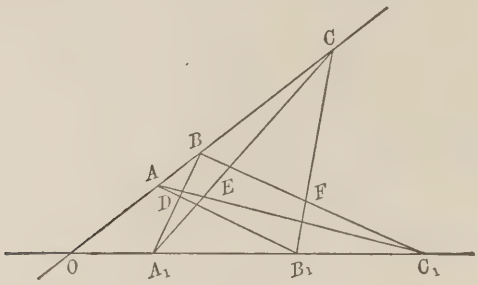

FIG. 3.

Thus a non-desarguesian geometry must have at least six points to the line. A number system which furnishes a geometry with this number of points is the following. Denote the elements by 0, 1, 2, 3, 4, and let addition be identical with ordinary addition modulo 5. Let multiplication be defined by the table:

	1	2	3	4
1	1	2	3	4
2	4	3	2	1
3	3	1	4	2
4	2	4	1	3

The element, 1, is a left-hand identity. All the conditions of § 2 are satisfied and we may conveniently take $\phi = \psi = 1$. The resulting geometry however turns out to be $PG(2, 5)$. The problem of determining a non-desarguesian geometry of minimum number of points to the line remains unsolved.

* Cf. these transactions, vol. 7 (1906), p. 247 and § 1 above.

[Note added April 22, 1907. Dr. C. R. MACINNES has found by a tactical investigation that the only plane projective geometry of six points to a line is the $PG(2, 5)$ and that there is none at all with seven points to a line. The latter result has also been found by SAFFORD, American Mathematical Monthly, vol. 14 (1907), p. 84.]

MODULAR THEORY OF GROUP-MATRICES*

BY

LEONARD EUGENE DICKSON

1. The importance of the concept group-matrix in the theory of finite groups was recognized by DEDEKIND as early as 1880. The development of a general theory of group-matrices is due, however, to FROBENIUS, (Berliner Sitzungsberichte, from 1896 to the present). In particular, FROBENIUS has applied the theory to the representation of a finite group as a (non-modular) linear group. Since linear congruence groups are of importance in the theory of groups, particularly in questions of isomorphisms, and play a fundamental rôle in the applications of groups to the theory of functions and to geometry, the study of the representations of a finite group as a linear congruence group is of decided importance.

It is here proved that, if p^π is the highest power of the prime p dividing the order g of a group G, the group-matrix of G can be transformed, by a matrix whose elements are integers taken modulo p, into a compound matrix in which the submatrices to the right of the main diagonal have zero elements throughout, while the p^π submatrices in the diagonal are identical. Let D denote one of the diagonal submatrices, so that D is a square matrix of order g/p^π. Then the group-determinant Δ of G is congruent to $|D|^{p^\pi}$ modulo p. This result is in marked contrast to the non-modular theory, in which each algebraically irreducible factor of Δ enters to a power exactly equal to its degree.

It is shown in § 8 that the group-matrices of all groups of order p^π can be reduced to their canonical form modulo p by one and the same transformation.

An interesting theorem on group-characters is obtained in § 11.

Just as simplicity is attained in the algebraic theory only when certain algebraic irrationalities are introduced to permit of the complete factoring of the group-determinant into algebraically irreducible factors, so corresponding simplicity in the modular theory can be attained only by the use of Galois imaginaries (roots of irreducible congruences) in order to normalize completely the diagonal matrices D (§ 10). It is therefore proposed to take as the field of reference the field F_p defined as the aggregate of the Galois fields $GF[p^n]$,

* To be presented to the Society, September 5, 1907. Received for publication May 10, 1907.

$n = 1, 2, 3, \cdots$. Every equation with coefficients in F_p is completely solvable within F_p.

When the modulus p does not divide the order of G, the fundamental theorems of the algebraic theory are also true in the field F_p, as was first pointed out in the writer's paper in these Transactions, vol. 3 (1902), pp. 285–301. For the same case, the exposition of the algebraic theory by SCHUR (Berliner Sitzungsberichte, 1905, pp. 406–432) is valid in the field F_p.

When the modulus divides the order of G, the problem presents marked contrasts to the algebraic theory. It is to this outstanding case that the present paper is directed, as also the companion paper, "Modular theory of group-characters" (Bulletin of the American Mathematical Society, July, 1907).

2. Let G be a group of finite order g; let H be a subgroup,

$$H: \qquad s_1 = I, s_2, \cdots, s_h,$$

of order h and index $q = g/h$. Let $e_1 = I, e_2, \cdots, e_q$ be right-hand extenders of H to G, so that

$$G = H + He_2 + \cdots + He_q.$$

Form the left-hand multiplication-table of G with the operations

$$I, e_2, \cdots, e_q;\ s_2, s_2e_2, \cdots, s_2e_q;\ s_3, s_3e_2, \cdots, s_3e_q;\ \cdots$$

in the first row, and their inverses, in this order, in the first column. The body of the table is a compound matrix M with h^2 matrices $M_{i,j}$ as its elements. The matrix in the ith row of matrices and jth column of matrices is

$$(1) \qquad M_{i,j} = M_k, \qquad \text{if} \qquad s_i^{-1}s_j = s_k.$$

We have

$$(2) \qquad M_k = (e_r^{-1}s_ke_r) = \begin{pmatrix} s_k & s_ke_2 & \cdots & s_ke_q \\ e_2^{-1}s_k & e_2^{-1}s_ke_2 & \cdots & e_2^{-1}s_ke_q \\ \cdot & \cdot & \cdot & \cdot \\ e_q^{-1}s_k & e_q^{-1}s_ke_2 & \cdots & e_q^{-1}s_ke_q \end{pmatrix}.$$

If H is a subgroup of order h of a group G, the body of the multiplication-table of G may be exhibited as a compound matrix whose h^2 submatrices M_k have the same relative arrangement as the elements s_k in the multiplication-table $(s_i^{-1}s_j)$ of H.

To the h elements s_i of a group H we make correspond h independent variables x_{s_i}. Then the group-matrix of H is

$$(3) \qquad (x_{s_i^{-1}s_j}) \qquad (i, j = 1, \cdots, h).$$

3. Let p^π be the highest power of p dividing the order $p^\pi q$ of a group G, and let H be a Sylow subgroup of order p^π.

It will be shown in § 5 that there exists a matrix (b_{ij}) which transforms the group-matrix (3) of H into a matrix whose diagonal elements are congruent, modulo p, to $\sum x_{s_i}$, and whose elements to the right of the diagonal are congruent to zero. Then, if I denotes the unit matrix of order q, the compound matrix $(b_{ij}I)$ transforms the group-matrix of G, arranged as in § 2, into a compound matrix, each of the p^π submatrices in the main diagonal being congruent to

$$\left(\sum_{k=1}^{p^\pi} x_{e_r^{-1}s_k e_c}\right) \qquad (r, c = 1, \cdots, q), \tag{4}$$

while each submatrix to the right of the main diagonal has all its elements congruent to zero modulo p. Hence we may state the following

THEOREM. *If p^π is the highest power of p dividing the order of a group G, the group-determinant of G is congruent modulo p to D^{p^π}, where D is the determinant of the matrix* (4).

Consider the linear transformation T on ξ_i $(i = 0, 1, \cdots, p^\pi - 1)$ whose matrix is the group matrix of H_{p^π} with a convenient order for the elements x_{s_i} in the first row (§ 5). When T is expressed in terms of the new variables

$$\eta_i = \sum_{j=i}^{p^\pi - 1} \binom{j}{i} \xi_j \qquad (i = 0, 1, \cdots, p^\pi - 1), \tag{5}$$

the coefficients being binomial coefficients, it takes the canonical form (modulo p) given above. The corresponding normalization of the transformation whose matrix is the group-matrix of G is accomplished by the introduction of the new variables

$$\eta_{iq+s} = \sum_{j=i}^{p^\pi - 1} \binom{j}{i} \xi_{jq+s} \qquad (i = 0, 1, \cdots, p^\pi - 1;\ s = 1, \cdots, q). \tag{6}$$

4. Matrix (4) has the important property that, in the terminology of FROBENIUS, it is *a matrix belonging to the group G*. Removing the restriction that H is a Sylow subgroup, we obtain the matrix

$$X = \left(\sum_{k=1}^{h} x_{e_r^{-1}s_k e_c}\right) \qquad (r, c = 1, \cdots, q). \tag{7}$$

We prove that matrix X belongs to the group G, viz., that

$$XY = Z \quad \text{if} \quad \sum_R x_R y_{R^{-1}A} = z_A \qquad (R \text{ ranging over } G). \tag{8}$$

The element in the rth row and cth column of XY is

$$\sum_{j=1}^{q}\left(\sum_{m=1}^{h} x_{e_r^{-1}s_m e_j}\right)\left(\sum_{l=1}^{h} y_{e_j^{-1}s_l e_c}\right) = \sum_{k=1}^{h}\sum_R x_R y_{R^{-1}e_r^{-1}s_k e_c} = \sum_{k=1}^{h} z_{e_r^{-1}s_k e_c},$$

where $R = e_r^{-1}s_m e_j$, $s_k = s_m s_l$.

A second proof that matrix X belongs to the group G results incidentally from the following discussion which gives a (partial) normalization of the group-matrix of G by an algebraic transformation, in contrast to the (more complete) normalization in § 3 by a modular transformation. The transformation on $\xi_1, \cdots, \xi_{qh}$ whose matrix is the group-matrix of G (§ 2) is

$$\xi'_{(i-1)q+r} = \sum_{j,t} x_{e_r^{-1}s_i^{-1}s_j e_t} \xi_{(j-1)q+t} \qquad (i, j = 1, \cdots, h;\ r, t = 1, \cdots, q). \tag{9}$$

Consider the functions analogous to (6) with $i = 0$:

$$\eta_s = \sum_{j=1}^{h} \xi_{(j-1)q+s} \qquad (s = 1, \cdots, q).$$

Then

$$\eta'_r = \sum_{j,t} \left(\sum_{i=1}^{h} x_{e_r^{-1}s_i^{-1}s_j e_t} \right) \xi_{(j-1)q+t} = \sum_t \left(\sum_{k=1}^{h} x_{e_r^{-1}s_k e_t} \right) \xi_{(j-1)q+t} = \sum_{t=1}^{q} \left(\sum_{k=1}^{h} x_{e_r^{-1}s_k e_t} \right) \eta_t.$$

Hence the variables $\eta_1, \cdots, \eta_q$ undergo a transformation whose matrix is (7). In particular, $|X|$ is an algebraic factor of the group determinant of G.

5. THEOREM. *If G is a group of order p^π, p a prime, with the operations g_i, the group-matrix of G can be transformed* into one whose diagonal elements are congruent to $\sum x_{g_i}$ modulo p, while the elements to the right of the main diagonal are congruent to zero.*

The proof is by induction from $\pi - 1$ to π. It proved in § 7 for $\pi = 1$.

To apply § 2, let H be a subgroup of order $p^{\pi-1}$. Then $q = p$, and we may set $e_i = e^{i-1}$ $(i = 2, \cdots, p)$. By the hypothesis for the induction, the group matrix (x_{s_k}) of H can be transformed into a matrix whose diagonal elements are congruent to $\sum x_{s_k}$ modulo p, while those to the right of the main diagonal are congruent to zero. Let the transforming matrix be (a_{ij}), $i, j = 1, \cdots, p^{\pi-1}$. Then, if I is the unit matrix of order p, the compound matrix $(a_{ij} I)$ will transform the group-matrix (x_{g_i}) of G_{p^π} into a compound matrix, each of whose diagonal matrices D_i is (7) for $h = p^{\pi-1}$, while the elements of the matrices to the right of the diagonal are zero. Now $H_{p^{\pi-1}}$ is invariant in G_{p^π}. Hence $e^{-1}s_k e$ ranges over the operations of H where s_k does, so that

$$\sum_{k=1}^{p^{\pi-1}} x_{e_r^{-1}s_k e_c} = \sum x_{e^{-(r-1)}s_k e^{c-1}} = \sum x_{s_k e^{c-r}}.$$

We introduce the abbreviation

$$\sigma_t = \sum_{k=1}^{p^{\pi-1}} x_{s_k t}. \tag{10}$$

Since e^p belongs to H, we have

$$\sigma_{te^p} = \sigma_t. \tag{11}$$

* The transforming matrix is given in § 8.

Hence each diagonal matrix D_i is the cyclic matrix

$$(12)\qquad (\sigma_{e^{c-r}}) = \begin{pmatrix} \sigma_I & \sigma_e & \sigma_{e^2} & \cdots & \sigma_{e^{p-1}} \\ \sigma_{e^{p-1}} & \sigma_I & \sigma_e & \cdots & \sigma_{e^{p-2}} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \sigma_e & \sigma_{e^2} & \sigma_{e^3} & \cdots & \sigma_I \end{pmatrix}.$$

By § 7, this matrix of the cyclic group of order p can be transformed into a matrix whose diagonal elements are congruent to

$$\sum_{i=0}^{p-1} \sigma_{e^i} = \sum_{j=1}^{p^\pi} x_{g_j},$$

modulo p, while the elements to right of the diagonal are $\equiv 0$.

6. It remains to normalize the group-matrix of the cyclic group of order p by a transformation modulo p. Without increasing the difficulty of the problem, we treat the cyclic group of order p^π. We shall need two algebraic lemmas.

The transformation, with binomial coefficients,

$$(13)\qquad \eta_i = \sum_{j=i}^{g-1} \binom{j}{i} \xi_j \qquad (i = 0, 1, \cdots, g-1),$$

has, algebraically, the following inverse:

$$(14)\qquad \xi_i = \sum_{j=i}^{g-1} (-1)^{i+j} \binom{j}{i} \eta_j \qquad (i = 0, 1, \cdots, g-1).$$

This will be the case if, and only if,

$$(15)\qquad \sum_{j=i}^{l} (-1)^{l+j} \binom{l}{j} \binom{j}{i} = \delta_{il} \qquad (\delta_{ii} = 1,\ \delta_{il} = 0 \text{ if } i \neq l).$$

This well known formula (cf. NETTO, *Combinatorik*, p. 255, formula 43) is a special case of the following one, which is needed later:

$$(16)\qquad E_{mrs} \equiv \sum_{i=0}^{r} (-1)^i \binom{r}{i} \binom{i+m}{s} = \begin{cases} 0 & (r > s) \\ (-1)^r \binom{m}{s-r} & (r \leqq s). \end{cases}$$

To evaluate E, we apply

$$\binom{r}{i} = \binom{r-1}{i} + \binom{r-1}{i-1}$$

and obtain two sums P and Q. Making a similar replacement for $\binom{i+m}{s}$ in Q, we get

$$Q = -\sum_{i-1=0}^{r-1} (-1)^{i-1} \binom{r-1}{i-1} \binom{i-1+m}{s} - \sum_{i-1=0}^{r-1} (-1)^{i-1} \binom{r-1}{i-1} \binom{i-1+m}{s-1}$$
$$= -P - E_{m,\, r-1,\, s-1}.$$

Hence we have the recursion formula

$$E_{mrs} = -E_{m,\,r-1,\,s-1}.$$

Thus for $r \leqq s$, $E_{mrs} = (-1)^r E_{m0s-r}$ and (16) follows. For $r - s = \rho > 0$,

$$E_{mrs} = (-1)^s E_{m\rho 0} = (-1)^s \sum_{i=0}^{\rho} (-1)^i \binom{\rho}{i} = (-1)^s (1-1)^\rho = 0.$$

For the case $m \leqq s$, formula (16) may be obtained from formula (41), (NETTO, loc. cit., p. 255), by replacing m, n, q, s by r, s, $s-m$, $i-s+m$, respectively.

7. THEOREM. *The group-matrix of the cyclic group of order p^π can be transformed into a matrix, given by* (22), *having congruent elements, modulo p, in every parallel to the main diagonal, the elements to the right of the latter being congruent to zero.*

The group-matrix of the cyclic group of order g is

$$A = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{g-2} & \alpha_{g-1} \\ \alpha_{g-1} & \alpha_0 & \alpha_1 & \cdots & \alpha_{g-3} & \alpha_{g-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{g-1} & \alpha_0 \end{pmatrix}. \tag{17}$$

Let $\alpha_{-k} = \alpha_{g-k}$. Then A is the matrix of the transformation

$$\xi_i' = \sum_{j=0}^{g-1} \alpha_{j-i}\, \xi_j \qquad (i = 0, 1, \cdots, g-1). \tag{18}$$

Introducing the variables η by means of (13) and (14), we get

$$\eta_s' = \sum_{r=0}^{g-1} \gamma_{sr}\eta_r, \qquad \gamma_{sr} = (-1)^r \sum_{i=0}^{r} \sum_{j=s}^{g-1} (-1)^i \binom{r}{i}\binom{j}{s} \alpha_{i-j}. \tag{19}$$

The value of γ_{sr} will be determined modulo p, for the case $g = p^\pi$.

For $0 \leqq m \leqq g-1$, the coefficient of α_{-m} in $(-1)^r\gamma_{sr}$ is $c + c'$, where c is the sum of the terms with $j = i + m$, c' the sum with $j = i + m - g$.

Let first $j = i + m$. The minimum value of i is the greater of 0 and $s - m$. But, for $m < s$ and $i < s - m$, $\binom{j}{s} = 0$. Hence in c we may allow i to begin with the value zero. The maximum value of i is the lesser of r and $g - 1 - m$. Hence if $m < g - r$, c is given by (16). Next, let $m \geqq g - r$, so that $i \leqq g - 1 - m$. Now c will be unaltered modulo p if we allow i to take the additional values $g - m + t$ $(t = 0, 1, \cdots, s-1)$, since

$$\binom{p^\pi + t}{s} \equiv 0 \pmod{p} \qquad (t < s < p^\pi).$$

Indeed, there is no term z^s in

$$(1 + z^{p^\pi})(1+z)^t \equiv (1+z)^{p^\pi + t} \pmod{p}. \tag{20}$$

Thus, if $g-r \leqq m < g-r+s$, then $g-m+s-1 \geqq r$ and c is congruent to (16). Finally, for $m \geqq g-r+s$, $c+c'$ is congruent to (16) in view of (21).

Let next $j = i+m-g$. Then i ranges from $g-m+s$ to r, so that terms c' occur only for $m \geqq g-r+s$. For $s < p^\pi = g$, $t < g$,

$$\binom{t}{s} \equiv \binom{g+t}{s}, \qquad \binom{i+m-g}{s} \equiv \binom{i+m}{s} \pmod{p},$$

by (20). Hence $c' = 0$ if $m < g-r+s$; while for $m \geqq g-r+s$,

$$c' \equiv \sum_{i=g-m+s}^{r} (-1)^i \binom{r}{i}\binom{i+m}{s} \pmod{p}. \tag{21}$$

Therefore in every case the coefficient of α_{-m} in $(-1)^r \gamma_{sr}$ is congruent modulo p to the sum (16). Thus

$$\gamma_{rs} \equiv 0 \text{ (if } r > s), \qquad \gamma_{sr} \equiv \sum_{m=0}^{g-1} \binom{m}{s-r} \alpha_{-m} \text{ (if } r \leqq s).$$

Hence $\gamma_{sr} \equiv \gamma_{s-r0}$ $(r \leqq s)$. Set $a_t = \gamma_{t0}$. Then (19) becomes

$$\eta_s' = \sum_{t=0}^{s} a_t \eta_{s-t}, \qquad a_t \equiv \sum_{m=0}^{g-1} \binom{m}{t} \alpha_{-m} \pmod{p}. \tag{22}$$

8. Theorem. *The group-matrices of all groups of order p^π, with a suitable order for the elements of the first row, can be transformed simultaneously into their canonical forms modulo p by the same transforming matrix.*

Employing the notations of § 5, we prove that the transformation T on the variables ξ_i, whose matrix is the group-matrix (y) of a given group G of order p^π, can be reduced to its canonical form modulo p by the introduction of the new variables (5), viz., that transformation (5) transforms T into its canonical form. Then the matrix of (14), i. e., the inverse of the matrix of (5), will transform the group-matrix (y) into its canonical form modulo p. The proof is by induction from $\pi-1$ to π. We therefore assume that the transformation whose matrix is the group-matrix (x) of the subgroup H of order $p^{\pi-1}$ can be reduced to its canonical form by the introduction of the new variables

$$\eta_i = \sum_{j=i}^{a} \binom{j}{i} \xi_j \qquad (i=0, 1, \cdots, a = p^{\pi-1}-1). \tag{23}$$

The matrix (a_{ij}) of § 5 is thus the inverse of the matrix of (23). Hence $(a_{ij} I)$ is the inverse of the matrix of

$$\eta_{ip+s} = \sum_{j=i}^{a} \binom{j}{i} \xi_{jp+s} \qquad (i=0, 1, \cdots, a;\ s=0, 1, \cdots, p-1). \tag{24}$$

It remains to normalize the diagonal matrices D_k $(k = 0, 1, \cdots, a)$, each of

which is of the form (12). Hence by § 6 we introduce the new variables

$$\zeta_{kp+i} = \sum_{j=1}^{p-1} \binom{j}{i} \eta_{kp+j} \quad (k=0, 1, \cdots, a;\ i=0, 1, \cdots, p-1). \tag{25}$$

Eliminating the η's from (24) and (25), we get

$$\zeta_{lp+r} = \sum_{s=r}^{p-1} \sum_{m=l}^{a} \binom{s}{r} \binom{m}{l} \xi_{mp+s} \quad (l=0, 1, \cdots, a;\ r=0, 1, \cdots, p-1). \tag{26}$$

To prove that (26) is equivalent modulo p to transformation (5), set

$$i = lp + r, \qquad j = mp + s.$$

In view of the limits of the summation indices in (26), we have

$$0 \leqq r \leqq s \leqq p-1, \qquad 0 \leqq l \leqq m \leqq a \qquad (a = p^{\pi-1} - 1).$$

It therefore follows from the writer's Dissertation (Annals of Mathematics, ser. 1, vol. 11 (1896–7), pp. 75, 76), that

$$\binom{j}{i} \equiv \binom{s}{r} \binom{m}{l} \pmod{p}. \tag{27}$$

The induction is therefore complete.

9. We have now established the lemmas employed in the proof of the general theorem of § 3. The problem of the ultimate canonical form modulo p of the group-matrix of a given group G of order $p^\pi q$ is therefore reduced to the problem of normalizing the diagonal matrices (4). One step of this normalization is readily effected in § 10, the resulting matrix (34) having the desirable property that the elements of the non-diagonal submatrices are all zero. In particular, this normalization is complete if a Sylow subgroup of order p^π is invariant in G.

10. Let H_{p^π} be invariant in $K_{p^\pi m}$, but in no larger subgroup of the given group $G_{p^\pi q}$. Set

$$K = H + He_2 + \cdots + He_m, \qquad G = K + Kf_2 + \cdots + Kf_n \quad (n = q/m). \tag{28}$$

Then (4) may be exhibited as a compound matrix (E_{ij}) whose n^2 elements E_{ij} are square matrices of order m:

$$E_{ij} = \left(\sum_{k=1}^{p^\pi} x_{f_i^{-1} e_r^{-1} s_k e_c f_j} \right) \qquad (r, c = 1, \cdots, m).$$

For r fixed, $e_r^{-1} s_k$ and $s_k e_r^{-1}$ range over the same set of elements. Set

$$\sigma_t^{ij} = \sum_{k=1}^{p^\pi} x_{f_i^{-1} s_k t f} \qquad (t = I, e_2, \cdots, e_m), \tag{29}$$

the m functions with i and j fixed being independent. Hence

$$E_{ij} = (\sigma^{ij}_{e_r^{-1}e_c}) \qquad (r, c = 1, \cdots, m). \tag{30}$$

Now $e_r^{-1} e_c$ may be given the form se_v with s in H. Then

$$\sigma^{ij}_{e_r^{-1}e_c} = \sigma^{ij}_{e_v}, \qquad (He_r)^{-1}(He_c) = He_v. \tag{31}$$

Hence, *for each i and j, E_{ij} is the group-matrix of the quotient-group K/H.* The order, m, of the latter is relatively prime to the modulus p. Hence by § 1 there exists a matrix μ, of order m and with elements in the field F_p, which transforms E_{11} into its ultimate canonical form C_{11}. Then the compound matrix

$$\begin{pmatrix} \mu & 0 & 0 & \cdots \\ 0 & \mu & 0 & \cdots \\ . & . & . & . \end{pmatrix} \tag{32}$$

transforms the compound matrix (E_{ij}) into $(\mu^{-1} E_{ij} \mu) = (C_{ij})$. Here the C_i have simultaneously their canonical forms:

$$C_{ij} = \begin{pmatrix} c_1^{ij} & 0 & 0 & \cdots \\ 0 & c_2^{ij} & 0 & \cdots \\ . & . & . & . \end{pmatrix}, \tag{33}$$

where c_s^{ij} is a square matrix, whose determinant is irreducible in F_p, and whose elements are linear functions of $\sigma_1^{ij}, \cdots, \sigma_m^{ij}$, with coefficients independent of i, j. Applying to (C_{ij}) a certain transformation which merely permutes the variables, we get

$$\begin{pmatrix} C_1 & 0 & 0 & \cdots \\ 0 & C_2 & 0 & \cdots \\ . & . & . & . \end{pmatrix}, \qquad C_s = (c_s^{ij}) \qquad (i, j = 1, \cdots, n). \tag{34}$$

The problem has therefore been reduced to the normalization of certain matrices C_s ("belonging" to the group G) of order n, the index of H_{p^π} in the largest subgroup (of the given group G) in which H_{p^π} is invariant.

If the Sylow subgroup H_{p^π} is invariant in G, then $n = 1$ and the normalization is complete.

11. From the theorem in § 5 we readily derive an interesting result on group-characters. Let H be a group with the h elements s_i and group-determinant Δ, defined as the determinant of matrix (3). Let G_{p^π} be any Sylow subgroup with the elements g_i and group-determinant D. If we set

$$x_s = 0 \text{ for every element } s \text{ of } H \text{ not in } G_{p^\pi}, \tag{35}$$

Δ becomes equal to an exact power of D. By § 5,

$$D \equiv L^{p^\pi} \pmod{p}, \qquad L = \sum_{i=0}^{p^\pi - 1} x_{g_i}.$$

In our field F_p, let Φ be an irreducible factor of degree f of Δ. Hence, under the assumption (35), Φ becomes a function congruent to L^f modulo p. The coefficient of

$$x_I^{f-1} x_{g_i} \qquad (g_i \neq I) \tag{36}$$

in L^f is f. By definition the coefficient of (36) in Φ is the character $\chi(g_i)$, while $\chi(I) = f$. We may therefore state the

THEOREM. *If χ is the group-character defined by a factor, of degree f and irreducible in the field F_p, of the group-determinant of a finite group H, then*

$$\chi(g) \equiv f \pmod{p} \tag{37}$$

for every element g of period a power of p in H.

The definition of characters is employed also for reducible factors. But

$$\Phi = \Phi'\Phi'' \text{ implies } \chi(g) = \chi'(g) + \chi''(g).$$

Hence (37) *is true also of algebraic group-characters.*

THE UNIVERSITY OF CHICAGO,
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EXISTENCE PROOF FOR A FIELD OF EXTREMALS TANGENT TO A GIVEN CURVE*

BY

OSKAR BOLZA

In a recent paper,† Professor BLISS has given sufficient conditions for a minimum of the integral

$$J = \int_{t_0}^{t_1} F(x, y, x', y')\,dt \tag{1}$$

with respect to one-sided variations. His proof is based upon the construction of a *field of extremals tangent to a given curve.* He establishes the existence of such a field first for the special case where all curves considered are representable in the form $y = f(x)$, and then reduces the general case of parameter representation to the former by a point-transformation of the plane.

The object of the following note is to give a *direct proof for the existence of these fields* which play an important part also in other investigations of the calculus of variations.‡

§ 1. *The set of extremals tangent to a given curve.*

The terminology and assumptions concerning the function F being the same as in § 24 of my *Lectures on the Calculus of Variations*, we consider a curve *of class* C''

$$\tilde{\mathfrak{C}}: \qquad x = \tilde{x}(a), \qquad y = \tilde{y}(a), \qquad A_1 \leqq a \leqq A_2,$$

without multiple points, which lies *in the interior of the region of the* x, y-*plane in which the function* F *is supposed to be of class* C''' for every $(x', y') \neq (0, 0)$, and *satisfies the inequality*

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† Transactions of the American Mathematical Society, vol. 5 (1904), p. 477.

‡ Compare LINDEBERG, Mathematische Annalen, vol. 59 (1904), p. 321.

$$F_1[\tilde{x}(a),\ \tilde{y}(a),\ \tilde{x}'(a),\ \tilde{y}'(a)] > 0 \text{ in } (A_1 A_2), \tag{2}$$

where $\tilde{x}' = d\tilde{x}/da$, $\tilde{y}' = d\tilde{y}/da$.

For simplicity, we suppose that the parameter a is the arc of the curve $\tilde{\mathfrak{C}}$ measured from some fixed initial point.

Under these conditions it follows from the general existence theorems* for differential equations applied to the differential equation of the extremals† for the integral (1), that through every point $P(a)$ of the curve $\tilde{\mathfrak{C}}$ one and but one extremal $\mathfrak{E}_a$ can be drawn which is tangent to $\tilde{\mathfrak{C}}$ at P in such a manner that the positive tangents of the two curves coincide. For the parameter t on the extremal $\mathfrak{E}_a$ we may choose the arc of the extremal measured from the point P so that for every value of a the point P corresponds on $\mathfrak{E}_a$ to the value $t = 0$.

If we vary a, we thus obtain a set of extremals

$$x = \phi(t, a), \qquad y = \psi(t, a), \tag{3}$$

for which the functions ϕ, ψ have the following properties:

1) The functions

$$\phi,\ \phi_t,\ \phi_{tt};\ \psi,\ \psi_t,\ \psi_{tt}$$

are as functions of t and a of class C' in the domain

$$0 \leqq t \leqq l, \qquad A_1 \leqq a \leqq A_2, \tag{4}$$

where l is a sufficiently small positive quantity *independent of* a.‡

2) The functions ϕ, ψ satisfy the following initial conditions:

$$\begin{aligned} \phi(0, a) &= \tilde{x}(a), & \psi(0, a) &= \tilde{y}(a), \\ \phi_t(0, a) &= \tilde{x}'(a), & \psi_t(0, a) &= \tilde{y}'(a). \end{aligned} \tag{5}$$

From (5) we obtain by differentiation

$$\begin{aligned} \phi_a(0, a) &= \tilde{x}'(a), & \psi_a(0, a) &= \tilde{y}'(a), \\ \phi_{ta}(0, a) &= \tilde{x}''(a), & \psi_{ta}(0, a) &= \tilde{y}''(a). \end{aligned} \tag{6}$$

From these equations we derive for the Jacobian

$$\Delta(t, a) = \frac{\partial(\phi, \psi)}{\partial(t, a)}$$

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* Compare BLISS, Annals of Mathematics, ser. 2, vol. 6 (1905), pp. 49–67.

† Compare KNESER, *Lehrbuch der Variationsrechnung*, §§ 27, 29 and BOLZA, *Lectures on the Calculus of Variations*, § 25, *b*).

‡ Compare the corollary given by BLISS in the article on differential equations just referred to, p. 53, at the end of section 1.

the result:

$$\Delta(0, a) = 0, \qquad \Delta_t(0, a) = \frac{1}{r} - \frac{1}{\tilde{r}}, \tag{7}$$

if we denote by $1/r$ and $1/\tilde{r}$ the curvature at the point P of the curve $\mathfrak{C}_a$ and of the curve $\tilde{\mathfrak{C}}$ respectively.

We make the *further assumption* that

$$\frac{1}{r} - \frac{1}{\tilde{r}} \neq 0$$

along $\tilde{\mathfrak{C}}$, and in order to fix the ideas we suppose * that

$$\frac{1}{r} - \frac{1}{\tilde{r}} > 0. \tag{8}$$

From this additional assumption it follows that two positive quantities $l_0 \leqq l$ and m can be determined so that

$$\Delta(t, a) \geqq tm \tag{9}$$

in the domain

$$0 \leqq t \leqq l_0, \qquad A_1 \leqq a \leqq A_2. \tag{10}$$

For if we define the function $\chi(t, a)$ for the domain (4) by the equations

$$\chi(t, a) = \begin{cases} \dfrac{\Delta(t, a)}{t}, & \text{when } t \neq 0, \\ \Delta_t(0, a), & \text{when } t = 0, \end{cases}$$

it is easily seen that $\chi(t, a)$ is continuous in the domain (4), and since moreover $\chi(0, a) > 0$ in $(A_1 A_2)$, it follows that a positive quantity $l_0 \leqq l$ can be assigned such that $\chi(t, a) > 0$ in the domain (10). If we denote by m the minimum of $\chi(t, a)$ in the domain (10), we obtain (9).

§ 2. *Proof that the set of extremals* (3) *furnishes a field.*

We now choose two quantities a_1, a_2 so that

$$A_1 < a_1 < a_2 < A_2$$

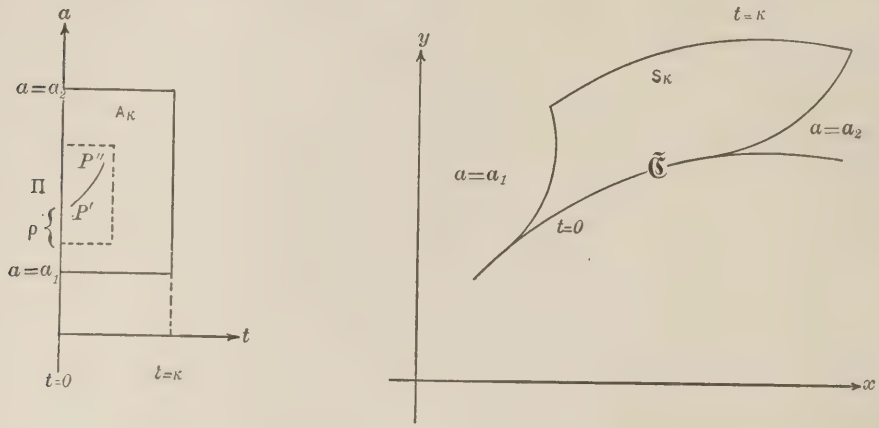
and propose to prove that under the assumptions enumerated in § 1, *a positive quantity $k \leqq l_0$ can be assigned such that the equations* (3) *define a one-to-one correspondence between the rectangle*

* In order that the curve $\tilde{\mathfrak{C}}$ may furnish a minimum for the integral (1) with respect to one-sided variations on the left of $\tilde{\mathfrak{C}}$, it is necessary that $1/r - 1/\tilde{r} \geqq 0$; compare BOLZA, *Lectures*, p. 194.

$$\mathbf{A}_\kappa: \qquad 0 \leqq t \leqq \kappa, \qquad a_1 \leqq a \leqq a_2$$

in the t, a-plane and its image $\mathbf{S}_\kappa$ in the x, y-plane.

We suppose that it were not so; that is we suppose that, however small κ may be taken, there always exists in $\mathbf{A}_\kappa$ at least one pair of distinct points whose images in the x, y-plane coincide. Reasoning then exactly as in the proof for the exist-



ence of a field which I have given in § 34 of my *Lectures* for the case where $\Delta(t, a) \neq 0$, we reach the result that under this hypothesis there would exist a point $\Pi(t = 0, a = \alpha)$ in the rectangle $\mathbf{A}_\kappa$ such that every vicinity of Π contains at least one pair of distinct points of $\mathbf{A}_\kappa$ whose images in the x, y-plane coincide.

We are going to prove that this leads to a contradiction with the inequality (9).

For this purpose we notice that our assumptions concerning the curve $\tilde{\mathfrak{C}}$ imply* that $\tilde{x}'(\alpha)$, $\tilde{y}'(\alpha)$ are not both zero; let $\tilde{x}'(\alpha) \neq 0$, or as we may write on account of (6),

$$\phi_a(0, \alpha) \neq 0.$$

We may then apply Dini's theorem on implicit functions to the function $\phi(t, a)$ and the point $t = 0$, $a = \alpha$. From this theorem it follows† that below any

* Compare the definition of "curve of class C''" on p. 116 of my *Lectures*.

† Choose $d > 0$ so that $\phi(t, a)$ is of class C' and $\phi_a(t, a) \neq 0$ for $|t| \leqq d$, $|a - \alpha| \leqq d$. Let A be the maximum of $|\phi_t(t, a)|$, B the minimum of $|\phi_a(t, a)|$ in this domain. Choose $0 < d_0 < d$ and

$$\sigma < d_1 - d_0, \ \rho < d_0, \ \frac{\sigma}{2}, \ \frac{B\sigma}{2A}.$$

Compare Peano-Genocchi, *Differentialrechnung und Grundzüge der Integralrechnung*, pp. 138–141.

preassigned positive quantity δ two positive quantities ρ and σ can be determined having the following properties: If $P'(t', a')$ and $P''(t'', a'')$ be any two distinct points of the vicinity (ρ) of the point $\Pi(0, \alpha)$ for which

$$\phi(t', a') = \phi(t'', a''), \tag{11}$$

then in the first place $t'' \neq t'$ (say $t' < t''$) and in the second place the two points P', P'' can be joined by a curve representable in the form

$$a = \mathfrak{a}(t), \qquad t' \leqq t \leqq t'',$$

such that

$$\phi[t, \mathfrak{a}(t)] = \phi(t', a') \text{ for } t' \leqq t \leqq t''. \tag{12}$$

The function $\mathfrak{a}(t)$ is of C', and satisfies the inequality

$$|\mathfrak{a}(t) - a')| < \sigma \text{ for } t' \leqq t \leqq t''$$

and the initial conditions

$$\mathfrak{a}(t') = a', \qquad \mathfrak{a}(t'') = a''. \tag{13}$$

Differentiating (12) we obtain

$$\phi_t[t, \mathfrak{a}(t)] + \phi_a[t, \mathfrak{a}(t)]\,\mathfrak{a}'(t) = 0. \tag{14}$$

On the other hand, it follows from the characteristic property of the point Π that there exists at least one pair of distinct points P', P'' in the domain

$$0 \leqq t < \rho, \qquad |a - \alpha| < \rho$$

for which not only (11) holds but at the same time

$$\psi(t', a') = \psi(t'', a''). \tag{15}$$

For such a pair of points the function $\psi[t, \mathfrak{a}(t)]$ is of class C' in $(t't'')$ and takes, according to (13) and (15), the same value for $t = t'$ and $t = t''$. Hence its derivative must vanish at least for one value $t = \tau$ between t' and t'':

$$\psi_t(\tau, \mathfrak{a}(\tau)] + \psi_a[\tau, \mathfrak{a}(\tau)]\,\mathfrak{a}'(\tau) = 0.$$

Combining this equation with the equation derived from (14) by putting $t = \tau$, we obtain the result:

$$\Delta[\tau, \mathfrak{a}(\tau)] = 0.$$

But if we take ρ and σ sufficiently small, the point $t = \tau$, $a = \mathfrak{a}(\tau)$ lies in the domain (10); moreover, τ is positive since $0 \leqq t' < \tau < t''$. Hence we have indeed reached a contradiction with the inequality (9), and therefore the statement enunciated at the beginning of this section is proved.

The image $\mathfrak{L}$ of the boundary of the rectangle $\mathbf{A}_\kappa$ is a continuous closed curve without multiple points. According to a theorem due to SCHÖNFLIES,* the point-set $\mathbf{S}_\kappa$ is therefore identical with the interior of $\mathfrak{L}$ together with the curve $\mathfrak{L}$ itself.

THE UNIVERSITY OF CHICAGO,
February 27, 1907.

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*Göttinger Nachrichten, 1899, p. 282; compare also OSGOOD, ibid., 1900, p. 94; and BERNSTEIN, ibid., 1900, p. 98.

A NEW FORM OF THE SIMPLEST PROBLEM OF THE CALCULUS OF VARIATIONS*

BY

GILBERT AMES BLISS

The simplest problem of the calculus of variations in the plane has been studied in two different forms. In the earlier one the curves considered are represented by equations

$$y = y(x),$$

where $y(x)$ is a single-valued function, and the integral to be minimized is

$$I = \int f(x, y, y')\,dx. \tag{1}$$

In the later form, which seems first to have been studied in detail by WEIERSTRASS, the curves are taken in parametric representation

$$C: \qquad x = \phi(t), \qquad y = \psi(t), \tag{2}$$

and the integral in the form

$$I = \int F(x, y, x_t, y_t)\,dt, \tag{3}$$

where the function F is positively homogeneous in the derivatives x_t and y_t. Both methods are open to objections, the former because it is inconvenient to apply to curves for which the function $y(x)$ is not single valued or for which the slope becomes infinite. In fact, no complete discussion has been attempted for the problem of minimizing the integral (1) without restricting the functions $y(x)$ to be single-valued and to have a finite derivative. The second form of the problem is complicated by the fact that in all the calculations involved the condition of homogeneity

$$F(x, y, \kappa x_t, \kappa y_t) = \kappa F(x, y, x_t, y_t) \qquad (\kappa > 0),$$

must be kept in mind and frequently applied.

In a paper entitled *A generalization of the notion of angle* † the writer has introduced a third form of the problem, in which the integral involves only invariants of the curve (2) under change of parameter-representation. The integral there given is

* Presented to the Society April 27, 1907. Received for publication April 8, 1907.

† Transactions of the American Mathematical Society, vol. 7 (1906), p. 184.

$$I = \int f(x, y, \tau)\, ds, \tag{4}$$

where s is the length of arc along the curve (2), and τ is the angle made by the curve with the x-axis, which is defined by the equations

$$\cos\tau = \frac{x_t}{\sqrt{x_t^2 + y_t^2}}, \qquad \sin\tau = \frac{y_t}{\sqrt{x_t^2 + y_t^2}}. \tag{5}$$

The integral (3) may be easily put into the form (4), for on account of the homogeneity of F,

$$F(x, y, x_t, y_t) = F(x, y, \cos\tau, \sin\tau)\sqrt{x_t^2 + y_t^2}.$$

But in (4) the function $f(x, y, \tau)$ need not be periodic as is the case with the right member of the last equation. Besides this generalization, the advantages of having the integral in the form (4) are that one can use the parametric representation for curves and at the same time avoid the troublesome homogeneity condition connected with the Weierstrassian form of the problem. It must be admitted that the derivation of the important formulas is not in general as simple as is the case for the integral (1), but otherwise the advantages of both of the older methods seem to be retained. The new form of the problem is especially advantageous in connection with the inverse problem of the calculus of variations and its application to geometry as made by HAMEL,* and in determining the invariants of the problem under point transformations.

In the paper by the writer referred to above only the Euler equation and the equations defining transversality, which were necessary in finding the generalization of angle, were developed. It is proposed here to complete the theory of the problem connected with the integral (4) by deriving the forms of the first and second variations and the functions which are commonly used in proving the necessary conditions in the calculus of variations. The derivation of the conditions themselves can be made as is usual in the books on the subject.†

§ 1. *The first variation.*

Suppose then that the integral to be studied is

$$I = \int f(x, y, \tau)\sqrt{x_t^2 + y_t^2}\, dt,$$

where

$$\sqrt{x_t^2 + y_t^2} = \frac{ds}{dt}$$

* *Über die Geometricen in denen die Geraden die Kürzesten sind*, Mathematische Annalen, vol. 57 (1903), p. 231.

† See especially BOLZA, *Lectures on the Calculus of Variations.*

and where the function $f(x, y, \tau)$ is of class C^{iv}* in a region R of (x, y, τ)-points having an interior. The curves considered will all be represented in the parametric representation (2) where ϕ and ψ are functions of class C^{iv} and define values (x, y, τ) lying interior to the region R for all values of t in the interval $t_0 \leqq t \leqq t_1$. As the value of τ is to be well-defined for every value of t, it is evident that the expression $x_t^2 + y_t^2$ must be everywhere different from zero, and the curve will therefore be supposed to have no singular points.

If the integral I is taken along one of the family of curves

$$(6) \qquad C_\alpha: \qquad x = \phi(t, \alpha), \qquad y = \psi(t, \alpha)$$

where $\phi(t, \alpha)$ and $\psi(t, \alpha)$ are of class C^{iv} in the region

$$t_0 \leqq t \leqq t_1, \qquad |\alpha| \leqq k \qquad (k > 0),$$

and for $\alpha = 0$ define the curve (2), then the value of I becomes a function $I(\alpha)$ which for $\alpha = 0$ reduces to the value I taken along C. The first derivative is

$$\frac{dI}{d\alpha} = \int_{t_0}^{t_1} \left\{ f_x x_\alpha + f_y y_\alpha + f_\tau \tau_\alpha + f \frac{x_t x_{\alpha t} + y_t y_{\alpha t}}{x_t^2 + y_t^2} \right\} \sqrt{x_t^2 + y_t^2}\, dt.$$

In this and following formulas literal subscripts denote partial differentiation, except the subscript s which means that there has first been a partial differentiation with respect to t and then a division by $s_t = \sqrt{x_t^2 + y_t^2}$.

After substitution of the value of τ_α calculated from equations (5) and (6), the last equation becomes

$$\frac{dI}{d\alpha} = \int_{s_0}^{s_1} \{ f_x x_\alpha + f_y y_\alpha + (f \cos\tau - f_\tau \sin\tau) x_{\alpha s} + (f \sin\tau + f_\tau \cos\tau) y_{\alpha s} \}\, ds,$$

or by the usual partial integration of the calculus of variations,

$$\frac{dI}{d\alpha} = \int_{s_0}^{s_1} \{ P x_\alpha + Q y_\alpha \}\, ds + [(f \cos\tau - f_\tau \sin\tau) x_\alpha + (f \sin\tau + f_\tau \cos\tau) y_\alpha]_{s_0}^{s_1},$$

where

$$P = f_x - \frac{d}{ds}(f \cos\tau - f_\tau \sin\tau), \qquad Q = f_y - \frac{d}{ds}(f \sin\tau + f_\tau \cos\tau).$$

By an easy calculation P and Q are found to satisfy the relation

$$P \cos\tau + Q \sin\tau = 0,$$

so that a new expression T can be introduced by means of the equations

$$P = T \sin\tau, \qquad Q = -T \cos\tau.$$

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* A function of class $C^{(n)}$ is by definition one which is itself continuous and has continuous derivatives up to and including those of the nth order. The assumptions on $f(x, y, \tau)$ and the curve (2) insure the existence of all the derivatives used in the following developments.

The final form of the first derivative is then

$$(7)\qquad \frac{dI}{d\alpha}=\int_{s_0}^{s_1} wTds+[(f\cos\tau-f_\tau\sin\tau)x_\alpha+(f\sin\tau+f_\tau\cos\tau)y_\alpha]_{s_1}^{s_0},$$

where

$$(8)\qquad w=x_\alpha\sin\tau-y_\alpha\cos\tau,$$

$$(9)\qquad T(x,y,\tau,\tau_s)=f_x\sin\tau-f_y\cos\tau+f_{x\tau}\cos\tau+f_{y\tau}\sin\tau+(f+f_{\tau\tau})\tau_s.$$

In case the curves (6) all pass through the two end-points 0 and 1 of the arc C, one sees readily by putting $t=t_0$ or $t=t_1$ in equations (6) and differentiating for α, that x_α, y_α both vanish at the points 0 and 1, so that *for a set of curves* (6) *all passing through two fixed points* 0 *and* 1 *the derivative* $dI/d\alpha$ *has the value*

$$\frac{dI}{d\alpha}=\int_{s_0}^{s_1} wTds,$$

where w *and* T *are the expressions in* (8) *and* (9) *above.*

The function w depends upon the family of curves (6). But if w is given in advance, a family can always be found including the arc C for $\alpha=0$ and having the given value w. It is only necessary to take curves in the form

$$(10)\qquad x=\phi(t)+\alpha\xi,\qquad y=\psi(t)+a\eta,$$

where ξ and η are to be determined by the equations

$$\xi\sin\tau-\eta\cos\tau=w,$$

$$\xi\cos\tau+\eta\sin\tau=0.$$

In particular, if w vanishes for $t=t_0$ and $t=t_1$, the curves (10) will all pass through two fixed points.

By methods well-known in the calculus of variations it follows then that *any curve which joins two given fixed points and minimizes the integral* I *must be a solution of Euler's differential equation*

$$(11)\qquad T(x,y,\tau,\tau_s)=0.$$

§ 2. *The second variation.*

For the case when the end-points are fixed the second derivative $d^2I/d\alpha^2$ can be calculated from the expression

$$\frac{dI}{d\alpha}=\int_{t_0}^{t_1} T\cdot w\sqrt{x_t^2+y_t^2}\,dt.$$

It is seen readily that the derivative of the integrand is

$$(12)\qquad w\{T_xx_\alpha+T_yy_\alpha+T_\tau\tau_\alpha+(f+f_{\tau\tau})\tau_{s\alpha}\}\sqrt{x_t^2+y_t^2}+T\frac{\partial}{\partial\alpha}(w\sqrt{x_t^2+y_t^2}).$$

The value of this second derivative is desired for $\alpha = 0$, that is, along the curve C. If C satisfies the first necessary condition for a minimum, $T = 0$, the term with the factor T disappears. Furthermore from (9) and (5)

$$(13) \qquad T_\tau = f_s + f_{\tau\tau s}, \qquad \tau_\alpha = (x_\alpha \cos\tau + y_\alpha \sin\tau)\tau_s - w_s,$$

$$\tau_{s\alpha} = \frac{\partial}{\partial\alpha}\frac{\tau_t}{\sqrt{x_t^2 + y_t^2}} = (x_\alpha \cos\tau + y_\alpha \sin\tau)\tau_{ss} - w_{ss} - \tau_s^2 w,$$

so that for $\alpha = 0$ the expression (12) becomes

$$(14) \quad w\left\{\left(T_x + \cos\tau\frac{d}{ds}f_1\tau_s\right)x_\alpha + \left(T_y + \sin\tau\frac{d}{ds}f_1\tau_s\right)y_\alpha - f_1\tau_s^2 w - \frac{d}{ds}f_1 w_s\right\}$$

where

$$f_1 = f + f_{\tau\tau}.$$

But along the curve C

$$\left(T_x + \cos\tau\frac{d}{ds}f_1\tau_s\right)\cos\tau + \left(T_y + \sin\tau\frac{d}{ds}f_1\tau_s\right)\sin\tau = T_s = 0.$$

When this equation and

$$\left(T_x + \cos\tau\frac{d}{ds}f_1\tau_s\right)\sin\tau - \left(T_y + \sin\tau\frac{d}{ds}f_1\tau_s\right)\cos\tau = T_x\sin\tau - T_y\cos\tau$$

are solved for the parentheses and the results substituted in (14), one finds without difficulty

$$(15) \qquad \left.\frac{d^2 I}{d\alpha^2}\right|^{\alpha=0} = \int_{s_0}^{s_1} w\left(f_2 w - \frac{d}{ds}f_1 w_s\right)ds = \int_{s_0}^{s_1}\{f_1 w_s^2 + f_2 w^2\}\,ds,$$

where

$$f_2(x, y, \tau, \tau_s) = T_x\sin\tau - T_y\cos\tau - f_1\tau_s^2.$$

The latter part of (15) is derived from the former by a simple partial integration, the terms outside the integral disappearing because for fixed end-points w vanishes at $s = s_0$ and $s = s_1$. These, however, are the well-known forms for the second variation, a factor $\alpha^2/2$ only being omitted, from which one derives the two following conditions.

Legendre's necessary condition. *The function $f_1 = f + f_{\tau\tau}$ must be $\geqq 0$ for a minimum, or $\leqq 0$ for a maximum, at every point of a minimizing arc C.*

Jacobi's necessary condition. *If the arc C satisfies Euler's equation and if furthermore $f_1 \neq 0$ along C, then a further necessary condition for a minimum or maximum is that no integral w of Jacobi's equation,*

$$f_2 w - \frac{d}{ds}f_1 w_s = 0,$$

exists which vanishes more than once in the interval $s_0 \leqq s < s_1$, the arguments of $f_2(x, y, \tau, \tau_s)$ and $f_1(x, y, \tau)$ in this equation being those corresponding to the arc C.

§ 3. *Hilbert's invariant integral and the E-function of Weierstrass.*

As soon as it is known that the arc C is a solution of Euler's equation along which $f_1 \neq 0$ and which satisfies Jacobi's condition, the existence theorems for differential equations enable one to imbed C in a field of solutions of Euler's equation. In fact, if a point 2 is taken near enough to 0 on C and having the order 201 with the points 0 and 1, then the solutions of (11) which pass through the point 2 will be a one-parameter family of curves which simply cover a portion of the plane in the neighborhood of the arc C.* Their equations can be found in the form

$$x = \phi(t, \alpha), \qquad y = \psi(t, \alpha) \qquad (0 \leqq t \leqq T,\ |\alpha| \leqq k), \tag{16}$$

where the value $t = 0$ corresponds to the point 2 on each curve, and where for $\alpha = 0$ the arc C itself is represented. Under the assumptions originally made upon the function $f(x, y, \tau)$ the functions ϕ and ψ will be of class C^{iv}.

In the field surrounding C the equations (16) are uniquely solvable for t and α in terms of x and y,

$$t = t(x, y), \qquad \alpha = \alpha(x, y), \tag{17}$$

and by implicit function theory these functions will be of class C' at least.

HILBERT's invariant integral may be derived from the integral

$$I = \int_0^t f(x, y, \tau)\, ds$$

taken along any one of the solutions (16) from the point $2\,(t = 0)$ to a point corresponding to the value t. This integral is a function of t and α, or by means of equations (17), a function of x and y in the field. If one takes the integral

$$I^* = \int_{u_3}^{u_4} \left\{ \frac{\partial I}{\partial x} x_u + \frac{\partial I}{\partial y} y_u \right\} du \tag{18}$$

over any path

$$\bar{C}: \qquad x = x(u), \qquad y = y(u), \tag{19}$$

in the field joining two given points 3, 4, it is evident that its value will always be the same however the path is chosen. The integral (18) is indeed HILBERT's invariant integral. It is desirable, however, to find more definitely the expressions for $\partial I/\partial x$ and $\partial I/\partial y$. By steps like those of § 1 it is seen that

$$\frac{\partial I}{\partial t} = f(x, y, \tau)\sqrt{x_t^2 + y_t^2},$$

$$\frac{\partial I}{\partial \alpha} = (f \cos \tau - f_\tau \sin \tau) x_\alpha + (f \sin \tau + f_\tau \cos \tau) y_\alpha.$$

* In the case when the function $f(x, y, \tau)$ is periodic with the period 2π in τ the family of solutions through the point 0 has the same property. See BLISS, *The construction of a field of extremals about a given point*, Bulletin of the American Mathematical Society, vol. 13 (1907), p. 321.

Since the curves (16) are all solutions of Euler's equation the expression T vanishes identically, while x_α and y_α vanish at the point 0, so that in equation (17) only the terms outside the integral sign at the point t are left. It follows therefore that

$$\frac{\partial I}{\partial x} = f(x, y, \tau)\sqrt{x_t^2 + y_t^2}\frac{\partial t}{\partial x} + [(f\cos\tau - f_\tau \sin\tau)x_\alpha + (f\sin\tau + f_\tau\cos\tau)y_\alpha]\frac{\partial \alpha}{\partial x},$$

with a similar equation for the y-derivative. The integrand of HILBERT's invariant integral (18) is evidently

$$f(x, y, \tau)\sqrt{x_t^2 + y_t^2}\frac{dt}{du} + [(f\cos\tau - f_\tau\sin\tau)x_\alpha + (f\sin\tau + f_\tau\cos\tau)y_\alpha]\frac{d\alpha}{du},$$

which can be easily changed into the form

$$(f\cos\tau - f_\tau\sin\tau)x_u + (f\sin\tau + f_\tau\cos\tau)y_u$$

since

$$\sqrt{x_t^2 + y_t^2} = x_t\cos\tau + y_t\sin\tau,$$

$$x_t\frac{dt}{du} + x_\alpha\frac{d\alpha}{du} = x_u, \qquad y_t\frac{dt}{du} + y_\alpha\frac{d\alpha}{du} = y_u.$$

If an angle σ is defined by the equations

$$\cos\sigma = \frac{x_u}{\sqrt{x_u^2 + y_u^2}}, \qquad \sin\sigma = \frac{y_u}{\sqrt{x_u^2 + y_u^2}},$$

Hilbert's integral may be written in the form

$$I^* = \int \{f\cos(\sigma - \tau) + f_\tau\sin(\sigma - \tau)\}\sqrt{x_u^2 + y_u^2}\,du,$$

where σ and τ are the direction-angles of the arbitrarily chosen curve (19) *and one of the solutions* (16) *of Euler's equation, respectively, at their point of intersection in the field. This integral has the fundamental properties that its value taken between any two points of the field is independent of the path over which it is taken, and on one of the solutions* (16) *of Euler's equation it has the same value as the original integral I.*

The latter property follows at once, since along an extremal $\sigma = \tau$.

The WEIERSTRASS E-function is the function

$$E(x, y, \sigma, \tau) = f\cos(\sigma - \tau) + f_\tau\sin(\sigma - \tau) - f(x, y, \sigma)$$

occurring in the integrand of the difference

$$I(\bar{C}) - I^*(\bar{C}) = \int E\sqrt{x_u^2 + y_u^2}\,du, \tag{20}$$

taken along any curve $\bar{C}$ of the field. If the ends of the arc over which the integration is taken lie on the arc C, this difference has the value

$$I(\bar{C}) - I^*(\bar{C}) = I(\bar{C}) - I(C) \tag{21}$$

on account of the properties of I^* above mentioned.

By means of special variations WEIERSTRASS derived the following fourth necessary condition involving the E-function.*

WEIERSTRASS'S NECESSARY CONDITION. *If the arc C is a solution of Euler's equation which minimizes the integral I with respect to any curve lying in a properly chosen neighborhood of C and joining its end-points, then the inequality*

$$E(x, y, \sigma, \tau) \geqq 0$$

must hold for every set of values (x, y, τ) corresponding to a point on C and for all values of σ. In case there is a maximum E must be $\leqq 0$.

It is well known that the four necessary conditions given above do not at the same time insure a minimum or a maximum. By strengthening the inequalities involved in them, however, new sets of conditions may be derived in various ways which with the help of a field and the equations (20), (21) can be proved sufficient. For the sake of completeness one of the theorems which define sufficient conditions is given here.

If the arc C is a solution of Euler's equation along which Legendre's and Weierstrass's conditions hold in the stronger forms $f_1 > 0$ and $E > 0$, and if furthermore Jacobi's condition is satisfied not only in the interval $s_0 \leqq s < s_1$ but also in $s_0 \leqq s \leqq s_1$, then a neighborhood of C can always be found in which this arc actually minimizes the integral I.

§ 4. *Variable endpoints.*

In the preceding sections the formulæ derived were for the case of fixed end-points. When a minimizing arc C is sought, however, which joins a fixed point 0 with a fixed curve D, a new condition on the directions of C and D at their point of intersection must be satisfied and the JACOBI condition takes on a somewhat different form. For the purpose of deriving these conditions let the representation of the family of curves (6) be chosen, as is usually done, so that their intersections with the curve D all occur at the fixed parameter-value t_1. Then the curve D is represented by the equations

$$x = \phi(t_1, \alpha), \qquad y = \psi(t_1, \alpha),$$

in which α is to be regarded as the parameter. The integral I, taken over a curve of this set and regarded as a function of α, has the first derivative

* See BOLZA, *Lectures on the calculus of variations*, p. 138.

$$(22)\qquad \frac{dI}{d\alpha}=\int_{s_0}^{s_1} wTds+\left[\sqrt{x_\alpha^2+y_\alpha^2}\,\{f\cos(\sigma-\tau)+f_\tau\sin(\sigma-\tau)\}\right]^{t=t_1},$$

a formula derived from equation (7), in which σ denotes the direction of the curve D derived from the equations

$$\cos\sigma=\frac{x_\alpha}{\sqrt{x_\alpha^2+y_\alpha^2}},\qquad \sin\sigma=\frac{y_\alpha}{\sqrt{x_\alpha^2+y_\alpha^2}},$$

for $t=t_1$. The terms outside the integral in (7) vanish at the point 0 exactly as before, but not necessarily at $t=t_1$. If the arc C is to minimize I with respect to all curves joining the point 0 with the curve D, it must certainly minimize I with respect to curves which join 0 with the intersection of C and D, and consequently must be a solution of Euler's equation as in § 1. In that case the integral part of the expression (22) vanishes for $\alpha=0$. But the rest must also vanish for $\alpha=0$, and so the following condition must be satisfied.

Condition of transversality. *If the arc C is a solution of Euler's equation which minimizes the integral I with respect to curves joining a fixed point 0 with a fixed curve D, then at its intersection with D the equation*

$$(23)\qquad f\cos(\sigma-\tau)+f_\tau\sin(\sigma-\tau)=0$$

*must also be satisfied, where σ denotes the direction of D and τ the direction of C.**

For the second derivative $d^2I/d\alpha^2$, the integral part has the form given first in equation (15), and it remains only to find the derivative of the terms in (22) which are outside the integral sign. For $\alpha=0$ the expression

$$[f\cos(\sigma-\tau)+f_\tau\sin(\sigma-\tau)]\frac{\partial}{\partial\alpha}\sqrt{x_\alpha^2+y_\alpha^2}$$

vanishes on account of equation (23). The rest of the derivative, omitting a factor $\sqrt{x_\alpha^2+y_\alpha^2}$, is

$$(24)\qquad \begin{aligned}&[-f\sin(\sigma-\tau)+f_\tau\cos(\sigma-\tau)](\sigma_\alpha-\tau_\alpha)\\&\quad+(f_xx_\alpha+f_yy_\alpha+f_\tau\tau_\alpha)\cos(\sigma-\tau)+(f_{\tau x}x_\alpha+f_{\tau y}y_\alpha+f_{\tau\tau}\tau_\alpha)\sin(\sigma-\tau).\end{aligned}$$

In this expression, from (13) and the theory of plane curves,

$$(25)\qquad \tau_\alpha=\sqrt{x_\alpha^2+y_\alpha^2}\cos(\sigma-\tau)\cdot\tau_s-w_s,\qquad \sigma_\alpha=\frac{\sqrt{x_\alpha^2+y_\alpha^2}}{r},$$

* The equation (23) is satisfied if

$$\cos\sigma=\frac{f\sin\tau+f_\tau\cos\tau}{\sqrt{f^2+f_\tau^2}},\qquad \sin\sigma=\frac{-f\cos\tau+f_\tau\sin\tau}{\sqrt{f^2+f_\tau^2}},$$

as in the writer's earlier paper referred to above.

where r is the radius of curvature of the curve D. With the help of (24) and (25) the value of $d^2I/d\alpha^2$ is found to be

$$(26)\qquad \frac{d^2I}{d\alpha^2} = \int_{s_0}^{s_1} w\left(f_2 w - \frac{d}{ds} f_1 w_s\right) ds + \left[(x_\alpha^2 + y_\alpha^2)\left(\frac{P_1}{r} + P_2 + Q\frac{w_s}{w}\right)\right]^{t=t_1},$$

where

$$P_1 = -f\sin(\sigma - \tau) + f_\tau \cos(\sigma - \tau),$$

$$P_2 = (f_x \cos\sigma + f_y \sin\sigma)\cos(\sigma - \tau) + (f_{\tau x}\cos\sigma + f_{\tau y}\sin\sigma)\sin(\sigma - \tau) + f_1 \sin(\sigma - \tau)\cos(\sigma - \tau)\tau_s,$$

$$Q = -f_1 \sin^2(\sigma - \tau).$$

But by means of the expression (26), the Jacobi condition can be proved* in the following form.

JACOBI'S NECESSARY CONDITION. *If the arc C is a solution of Euler's equation joining the fixed point 0 with the fixed curve D, and along which $f_1 \neq 0$, then a further necessary condition for a minimum or a maximum is that no solution w of Jacobi's equation*

$$f_2 w - \frac{d}{ds} f_1 w_s = 0$$

exists which satisfies the condition

$$\frac{P_1}{r} + P_2 + Q\frac{w_2}{w} = 0$$

at the intersection of C and D and vanishes in the interval $s_0 < s \leqq s_1$.

The alterations which must be made in the necessary conditions for the case of variable end-points in order to get sufficient conditions are so similar to the ones given in § 3 that the corresponding theorem need not be stated here.

PRINCETON, *March*, 1907.

*See BLISS, *The second variation of a definite integral when one end-point is variable*, Transactions of the American Mathematical Society, vol. 3 (1902), p. 132.

ON CERTAIN ISOTHERMIC SURFACES*

BY

ARCHER EVERETT YOUNG

I. *Introduction.*

§ 1. *Isothermic Surfaces.* CHRISTOFFEL proposed the problem of determining all pairs of surfaces, S and S_1, for which a one-to-one correspondence of their points can be established, such that the tangent planes at corresponding points will be parallel and the angles between corresponding lines will be equal. His discussion † of the problem shows that, aside from minimal and certain imaginary surfaces, the only solutions are surfaces which, when referred to lines of curvature, have as the expressions for their linear elements, respectively,

$$ds^2 = \lambda(du^2 + dv^2), \qquad ds_1^2 = \frac{1}{\lambda}(du^2 + dv^2),$$

where λ is a function of u and v. CHRISTOFFEL's problem, then, reduces to that of finding all possible isothermic surfaces.

No great progress has been made in the investigation of these surfaces, although WEINGARTEN ‡ and DARBOUX § have shown that they can be defined by a non-linear partial differential equation of the fourth order. The only solutions carried to completion have been obtained by imposing one or more additional conditions; the resulting surfaces have thus been very particular solutions of the general problem.||

§ 2. *Problem discussed in this paper.* The characteristic form for the linear element of many of the isothermic surfaces, when referred to lines of curvature, is well known.

Those of the quadrics and BONNET surfaces ¶ are respectively

$$ds^2 = (u+v)\left(\frac{du^2}{U} + \frac{dv^2}{V}\right), \qquad ds'^2 = \frac{1}{(u+v)^2}\left(\frac{du^2}{U_1} + \frac{dv^2}{V_1}\right),$$

* Presented to the Society (Chicago), December 29, 1905. Received for publication, August 1, 1906.

† Crelle, vol. 57 (1867), pp. 218–228.

‡ Sitzungsberichte der Akademie der Wissenschaften zu Berlin, vol. 2 (1883), p. 1163.

§ *Théorie des surfaces*, vol. 2, p. 248.

|| DARBOUX, loc. cit., vol. 2, p. 246.

¶ BONNET, Journal de L'Ecole Polytechnique, vol. 42, pp. 121 and 132–151.

where U and U_1 are certain functions of u, and V and V_1 of v. It is clear from CHRISTOFFEL's results that there must be other surfaces corresponding respectively to the expressions

$$ds_1^2 = \frac{1}{(u+v)}\left(\frac{du^2}{U} + \frac{dv^2}{V}\right), \qquad ds'^2_1 = (u+v)^2\left(\frac{du^2}{U_1} + \frac{dv^2}{V_1}\right).$$

Evidently the four preceding expressions are included under the general form

$$(1) \qquad ds^2 = (u+v)^K\left(\frac{du^2}{U} + \frac{dv^2}{V}\right),$$

where K is an arbitrary constant, and U and V arbitrary functions of u and v, respectively.

We have undertaken in this paper to determine the surfaces whose linear elements can be written in this form, the lines of reference being lines of curvature. The remainder of this first section is given up to the analytical discussion of the equations involved. It appears that the only surfaces, with the exception of the plane and cylindrical surfaces, that satisfy the conditions of the problem, are those associated with the BONNET surface, the quadrics, or the sphere referred to confocal spherical ellipses.

The associates of the BONNET surfaces are discussed in section II, and the others in section III of this paper. In the latter section, also, attention is called to an apparent oversight of BONNET, occurring in his discussion of the characteristic form for the linear element of the quadrics.

§ 3. *Problem reduced to the solution of a single equation.* The problem of determining all surfaces having their linear element in the form (1), when the lines of reference are lines of curvature, reduces to that of finding all solutions of the following system of equations: *

$$(2) \qquad \frac{\partial M}{\partial v} - \frac{\partial N}{\partial u} = PQ, \qquad \frac{\partial P}{\partial v} + MQ = 0, \qquad \frac{\partial Q}{\partial u} - NP = 0,$$

where P and Q are functions of u and v to be determined, and M and N, for the linear element (1), have the values:

$$M = -\frac{K\sqrt{\frac{V}{U}}}{2(u+v)}, \qquad N = \frac{K\sqrt{\frac{U}{V}}}{2(u+v)}.$$

From (2) it follows easily that

$$\frac{\partial(P^2+M^2)}{\partial v} = 2M\frac{\partial N}{\partial u}, \qquad \frac{\partial(Q^2+N^2)}{\partial u} = 2N\frac{\partial M}{\partial v}.$$

* BONNET, loc. cit., p. 133.

Substituting these values of M and N and integrating, we have

$$P^2 = \frac{K^2}{4}\left[\frac{U'}{(u+v)} - \frac{U+V}{(u+v)^2} + U_1\right], \quad Q^2 = \frac{K^2}{4}\left[\frac{V'}{u+v} - \frac{(U+V)}{(u+v)^2} + V_1\right],$$

where the primes denote derivatives, and U_1 and V_1 are new functions of u and v respectively.

The first equation of (2) with these values substituted becomes

$$(3) \qquad \begin{aligned} K^2\left[U' - \frac{U+V}{u+v} + (u+v)U_1\right]\left[V' - \frac{U+V}{u+v} + (u+v)V_1\right] \\ = \left[U' + V' - 2\frac{U+V}{u+v}\right]^2, \end{aligned}$$

and the system (2) is solved as soon as all possible sets of values of K, U, U_1, V, V_1, which satisfy this equation, are found.

§ 4. *Discussion of equation* (3)*. In discussing equation (3) we shall assume that for some value, α, all the functions $U(\alpha)$, $U_1(\alpha)$, $V(-\alpha)$, and $V_1(-\alpha)$ are analytic.†

Multiplying both sides of (3) by $(u+v)^2$, and then replacing u by α and v by $-\alpha$, we have

$$(K^2 - 4)\{U(\alpha) + V(-\alpha)\} = 0.$$

Hence

$$1^\circ \quad K = \pm 2, \quad \text{or} \quad 2^\circ \quad U(\alpha) = -V(-\alpha).$$

Case 1°, when $K = -2$, has been discussed by BONNET.‡ The functions, as determined by him, which satisfy (3) when K has this value, will satisfy (3) evidently when K has the other value. We have § then, for $K = \pm 2$,

$$U = \alpha u^2 + \alpha' u + \alpha'', \qquad U_1 = -\tfrac{1}{2}U'',$$

$$V = \text{arbitrary function of } v, \qquad V_1 = \text{function of } v \text{ and } V.$$

Case 2° needs further consideration. Writing $U = f(u)$, we have

$$\begin{aligned} U(\alpha) &= f(\alpha), & V(-\alpha) &= -f(-\alpha), \\ U'(\alpha) &= f'(\alpha), & V'(-\alpha) &= f'(-\alpha), \\ U''(\alpha) &= f''(\alpha), & V''(-\alpha) &= -f''(-\alpha). \end{aligned}$$

* The following discussion of equation (3) was suggested by Professor J. E. WRIGHT.

† The question as to whether this assumption limits our results we shall not attempt to settle in this paper. The same point arises in connection with several other equations which the writer is studying and may be discussed by itself in another article.

‡ Loc. cit., p. 133–151.

§ This solution is not based upon the assumption at the beginning of this article with regard to the functions, but is the most general solution of (3) when $K^2 = 4$.

If u be replaced by α and v by $(-\alpha+h)$, then (3) may be written

$$(4)\qquad \begin{aligned} K^2[\tfrac{1}{2}f''h + U_1 h - \tfrac{1}{6}f'''h^2 + \cdots][-\tfrac{1}{2}f''h + \tfrac{1}{2}f'''h^2 + \cdots \\ - \tfrac{1}{6}f'''h^2 + \cdots + hV_1] = [\tfrac{1}{2}f'''h^2 - \tfrac{1}{3}f'''h^2 + \cdots]^2, \end{aligned}$$

where α is the argument of f, and the terms omitted contain cubes and higher powers of h.

Equating coefficients of h^2, we find

$$[\tfrac{1}{2}f''(\alpha) + U_1(\alpha)][-\tfrac{1}{2}f''(\alpha) + V_1(-\alpha)] = 0.$$

Assuming, then, that $U_1(\alpha) = -\tfrac{1}{2}f''(\alpha)$, and equating coefficients of h^3, we have

$$f'''(\alpha)[-\tfrac{1}{2}f''(\alpha) + V_1(-\alpha)] = 0,$$

so that if $f'''(\alpha) \neq 0$, then $V_1(-\alpha) = \tfrac{1}{2}f''(\alpha)$, which is the condition we passed over in the preceding equation. Finally, the coefficients of h^4 in (4) give

$$(K^2 - 1)f'''^2(\alpha) = 0.$$

Hence two cases are left for consideration:

$$\text{(i)}\qquad f'''(\alpha) = 0,$$

and therefore

$$U = \alpha u^2 + \alpha' u + \alpha'', \qquad V = -\alpha v^2 + \alpha' v - \alpha'',$$

$$\text{(ii)}\qquad K = \pm 1, \qquad U_1 = -\tfrac{1}{2}U'', \qquad V_1 = -\tfrac{1}{2}V''.$$

Case (i) will be considered later.

Case (ii). In order to determine the forms of the functions U and V we shall replace U_1 and V_1 by the values given in (ii), and then write (3) in the form

$$(5)\qquad \begin{aligned} [(U+V) - U'(u+v) + \tfrac{1}{2}U''(u+v)^2][(U+V) - V'(u+v) + \tfrac{1}{2}V''(u+v)^2] \\ = [(U'+V')(u+v) - 2(U+V)]^2. \end{aligned}$$

Giving v some constant value, for which V is analytic, and putting

$$(U+V) - V'(u+v) + \tfrac{1}{2}V''(u+v)^2 = F,$$

we may write (5) in the form

$$F[F - (u+v)F' + \tfrac{1}{2}F''(u+v)^2] = [F'(u+v) - 2F]^2;$$

which, by the substitution

$$F = \phi \cdot (u+v)^2,$$

becomes

$$\tfrac{1}{2}\phi\phi'' + \frac{\phi\phi'}{(u+v)} = \phi'^2,$$

where the primes denote derivatives with regard to u. Integrating, we find

$$\frac{1}{\phi} = \frac{\psi(v)}{(u+v)} + \chi(v),$$

where ψ and χ are arbitrary functions of v, and hence

$$F = (U+V) - V'(u+v) + \tfrac{1}{2}V''(u+v)^2 = \frac{(u+v)^3}{Au+B},$$

where A and B are functions of v. Finally, solving for U, we have

$$U = a + bu + cu^2 + \frac{(u+d)^3}{hu+k},$$

where a, b, c, etc., are constants, as U must be independent of v. This result is based on the assumption that the function ψ does not vanish.

If $\psi = 0$, then $f''' = 0$, a case already considered. If $\psi \neq 0$, the constants h and k cannot both be zero at the same time.

If $h \neq 0$, with a new set of variables $(u - k/h)$ and $(v + k/h)$, the new function can be written in the form

$$U = \alpha + \alpha' u + \alpha'' u^2 + \frac{\alpha'''}{u}.$$

If $h = 0$ and $k \neq 0$, the expression for the function may be written

$$U = \alpha + \alpha' u + \alpha'' u^2 + \alpha''' u^3.$$

The solutions of equation (3), which have just been found, are as follows:

1°. $K =$ arbitrary constant, $U = \alpha u^2 + \alpha' u + \alpha''$, $V = -\alpha v^2 + \alpha' v - \alpha''$.

2°. $K = \pm 2$, $U = \alpha u^2 + \alpha' u + \alpha''$, $V =$ arbitrary function of v.

3° (a). $K = \pm 1$, $U = \alpha u^3 + \alpha' u^2 + \alpha'' u + \alpha'''$, $V = \alpha v^3 - \alpha' v^2 + \alpha'' v - \alpha'''$.

3° (b). $K = \pm 1$, $U = \alpha u^2 + \alpha' u + \alpha'' + \dfrac{\alpha'''}{u}$, $V = -\alpha v^2 + \alpha' v - \alpha'' + \dfrac{\alpha'''}{v}$.

II. *Associates of the* Bonnet *surfaces.*

§ 6. *Solution* 1° *of equation* (3). The surfaces arising from the first solution of equation (3), § 4, are the plane and cylindrical surfaces.*

The plane corresponds to the linear element.

$$ds^2 = (u+v)^K \left(\frac{du^2}{\alpha u^2 + \alpha' u + \alpha''} + \frac{dv^2}{-\alpha v^2 + \alpha' v - \alpha''} \right)$$

for any value of K, the form of the lines of reference depending of course upon the value of this constant.

* The proof of these statements follows from equations (2) and (3) and the values of the direction cosines of the normals to the surface found in the usual way.

For $K=2$, one set is composed of trochoids, and the other of logarithmic curves, which include the catenary as a particular case.* Cylindrical surfaces correspond to the linear element above when $K=0$.

§ 7. *Solution* 2° *of equation* (3). In discussing the surfaces arising from the solution of equation (3) when $K=-2$, BONNET found that two cases should be considered, depending upon whether the constant α which appears in the expression for U vanishes or not.

The expressions † for the linear elements and the cartesian coördinates of the surfaces in the two cases, as found by him, are respectively

$$(6)\qquad ds^2=\frac{1}{(u+v)^2}\left[\frac{du^2}{\alpha' u-u^2}+\frac{dv^2}{V}\right],$$

$$x=\frac{2\sqrt{\alpha' u-u^2}}{\alpha'(u+v)},\qquad z=-\frac{2\sqrt{v^2+\alpha' v}}{\alpha'(u+v)}\sin V_3,$$

$$y=-\frac{2\sqrt{v^2+\alpha' v}}{\alpha'(u+v)}\cos V_3,\qquad V_3=\int\frac{4\sqrt{v^2+\alpha' v-V}}{2(v^2+\alpha' v)\sqrt{V}}dv;$$

$$(7)\qquad ds^2=\frac{1}{(u+v)^2}\left(\frac{du^2}{d'u}+\frac{dv^2}{V}\right),$$

$$x=2\sqrt{\frac{u}{\alpha'}}\frac{1}{(u+v)},\qquad z=-2\sqrt{\frac{v}{\alpha'}}\frac{\sin V_3}{(u+v)},$$

$$y=-2\sqrt{\frac{v}{\alpha'}}\frac{\cos V_3}{(u+v)},\qquad V_3=\int\frac{\sqrt{\alpha' v-V}}{2v\sqrt{V}}dv.$$

BONNET shows that the surfaces defined by (6) and (7) are characterized by having lines of curvature whose geodesic curvature is constant.‡ According to him, the surfaces defined by (6) are the envelopes of a variable sphere, whose center describes a plane curve, while its surface always passes through two fixed points, real or imaginary, which lie symmetrically with regard to the plane of the centers. In the case of the surfaces defined by (7), the two fixed points are coincident and lie in the plane of centers.

The cartesian coördinates (x_1, y_1, z_1) of the surfaces associated with (6) and (7) can be obtained by solving the system of equations,§

* See § 8.

† BONNET, loc. cit., p. 150. He gives only the expressions for the first case, but the others are easily derived from them.

‡ RIBAUCOUR has shown that the only surfaces having this property are cones, cylinders, and surfaces of revolution, together with surfaces obtained from these by inversion. (See DARBOUX, loc. cit., vol. 3, p. 122.) The surfaces (6) are the inverse of cones and surfaces of revolution, and (7) the inverse of cylindrical surfaces.

§ DARBOUX, loc. cit., vol. 2, p. 244.

$$\lambda \frac{\partial \theta_1}{\partial u} = \frac{\partial \theta}{\partial u}, \qquad \lambda \frac{\partial \theta_1}{\partial v} = -\frac{\partial \theta}{\partial v},$$

where λ, which has its usual meaning, is obtained in the one case, from the expression for the linear element in (6), and in the other (7), and θ and θ_1 are to be replaced by x, x_1, y, y_1 and z, z_1 successively, the values of x, y and z being taken from (6) or (7), as the case may be.

We have, then, for the surfaces associated with (6),

$$ds_1^2 = (u+v)^2 \left[\frac{du^2}{\alpha' u + u^2} + \frac{dv^2}{V} \right],$$

and

$$x_1 = \frac{2}{\alpha'} \sqrt{\alpha' u - u^2} \left(v + \frac{\alpha'}{2} \right) - \frac{\alpha'}{2} \operatorname{vers}^{-1} \frac{2u}{\alpha'},$$

$$(8) \qquad y_1 = -\frac{2}{\alpha'} \sqrt{v^2 + \alpha' v}\,(u+v) \cos V_3 + \frac{4}{\alpha'} \int \sqrt{v^2 + \alpha' v} \cos V_3 \, dv,$$

$$z_1 = \frac{2}{\alpha'} \sqrt{v^2 + \alpha' v}\,(u+v) \sin V_3 - \frac{4}{\alpha'} \int \sqrt{v^2 + \alpha' v} \sin V_3 \, dv,$$

where

$$V_3 = \frac{\alpha'}{2} \int \frac{\sqrt{v^2 + \alpha' v - V}}{(v^2 + \alpha' v)\sqrt{V}} dv.$$

Likewise for those associated with (7), we have

$$ds_1^2 = (u+v)^2 \left(\frac{du^2}{\alpha' u} + \frac{dv^2}{V} \right),$$

$$x_1 = 2 \sqrt{\frac{u}{\alpha'}} \left(\frac{u}{3} - v \right),$$

$$(9) \qquad y_1 = -2 \sqrt{\frac{v}{\alpha'}}\,(u+v) \cos V_3 + \frac{4}{\sqrt{\alpha'}} \int \sqrt{v} \cos V_3 \, dv,$$

$$z_1 = 2 \sqrt{\frac{v}{\alpha'}}\,(u+v) \sin V_3 - \frac{4}{\sqrt{\alpha'}} \int \sqrt{v} \sin V_3 \, dv,$$

where

$$V_3 = \int \frac{\sqrt{\alpha' v - V}}{2v\sqrt{V}} dv.$$

It follows from a theorem due to DARBOUX,* that any surface, S_1, defined by (8) or (9), can be so placed with regard to its associate surface, S, defined by (6) or (7), that at corresponding points the normals to the surfaces and the tangents to the lines of curvature will be respectively parallel.

* Loc. cit., vol. 2, p. 243.

The lines $v =$ const. on the BONNET surfaces are circles, therefore the corresponding lines on (8) and (9), namely, one set of the lines of curvature, must be plane.

The problem of determining isothermic surfaces having one set of the lines of curvature plane has been solved in its most general form by DARBOUX.* Of three possible exceptional or limiting cases,† which DARBOUX did not consider, one has been treated by ADAM,‡ another leads to the BONNET surfaces, and the third to their associates. This problem, then, is completely solved by the addition of the surfaces defined by (8) and (9).

§ 8. *Discussion of the associates of the Bonnet surfaces.* The principle of inversion applied to the surfaces (8) and (9) gives little information with regard to them, but we do know of course that the resulting surfaces will be isothermic and will have spherical lines for one set of the lines of curvature.

A discussion of equations (8) and (9) shows that the plane lines of curvature on the surfaces defined by them lie in a system of planes which envelope a cylindrical surface, the form of the latter depending entirely upon the form of the arbitrary function V. These lines are in fact the intersections of the planes and a system of cylindrical surfaces depending upon v as a parameter, whose elements are perpendicular to the elements of the cylinder enveloped by the planes.

For the surfaces defined by (8), the cross-section of these cylinders are either trochoids with their base lines parallel to the elements of the enveloped cylinder or curves whose equations are of the form

$$x = a\sqrt{2by + y^2} - \frac{\alpha_1}{2}\log\left(y + b + \sqrt{2by + y^2}\right),$$

according as the constant α' appearing in (8) is real or imaginary. The plane lines on the surfaces (9) are nodal cubic curves.

§ 9. *Surfaces having both sets of the lines of curvature plane.* The surfaces corresponding to the linear element

$$ds^2 = (u + v)^2\left(\frac{du^2}{\alpha u^2 + \alpha' u + \alpha''} + \frac{dv^2}{\beta v^2 + \beta' v + \beta''}\right),$$

when

* Excluding surfaces of revolution, DARBOUX reduced the general problem to the simultaneous solution of the equations

$$\frac{\partial h}{\partial u} = Ue^h + U_1 e^{-h}, \qquad \frac{\partial^2 h}{\partial u^2} + \frac{\partial^2 h}{\partial v^2} = 0,$$

where $e^h = \sqrt{\lambda}$ and U and U_1 are independent functions of u. He solved the problem when $U \neq U_1 \neq 0$, and ADAM considered the case $U = U_1$. The case $U_1 = 0$ leads to the BONNET surfaces, and $U = 0$ to their associates (DARBOUX, loc. cit., vol. 4, p. 222).

† See preceding note.

‡ Annales scientifiques de l'Ecole Normal supérieure, series 3, vol. 10 (1893), p. 319.

$$4(\alpha+\beta)(\alpha''+\beta'')-(\alpha'-\beta')^2=0,$$

have plane lines for both sets of the lines of curvature, since on the corresponding BONNET surfaces both sets are circles.* It is clear that surfaces obtained from these by inversion will be isothermic and that the lines of curvature will be spherical. The condition imposed upon the constants is satisfied when

1°. $\alpha=-1,\ \beta\neq 1,\ \alpha'\neq\beta',\ \beta''\neq 0,$
2°. $\alpha=-1,\ \beta=1,\ \alpha'=\beta',\ \beta''\neq 0,$
3°. $\alpha=-1,\ \beta\neq 1,\ \alpha'=\beta',\ \beta''=0,$
4°. $\alpha=0,\ \beta\neq 0,\ \alpha'\neq\beta',\ \beta''\neq 0,$
5°. $\alpha=0,\ \beta=0,\ \alpha'=\beta'.$

All other cases lead to imaginary surfaces.

In each of these cases, the integration indicated in (8) or (9) can be performed, the resulting functions being trigonometric or logarithmic. Case 1° is the general case. The lines $v=$ const. are the same as in the general case for (8), and the lines $u=$ const. are transcendental plane curves.

Case 2° leads to a minimal surface, which was first derived by DARBOUX † in connection with the general subject of minimal surfaces.

In case 3° the expression for the linear element becomes

$$ds^2=(u+v)^2\left(\frac{du^2}{\alpha' u-u^2}+\frac{dv^2}{\beta v^2+\alpha' v}\right),$$

and the cartesian coördinates are

$$\begin{aligned} x&=\frac{2}{\alpha'}\sqrt{\alpha' u-u^2}\cdot v-\int\frac{udu}{\sqrt{\alpha' u-u^2}},\\ y&=\frac{2}{\alpha'}\sqrt{\beta v^2+\beta' v}\cdot u-\int\frac{vdv}{\sqrt{\beta v^2+\beta' v}},\\ z&=\frac{2}{\alpha'}\sqrt{1-\beta}\cdot uv. \end{aligned}\tag{10}$$

The two systems of planes in which the lines of curvature lie envelope cylindrical surfaces whose elements are perpendicular to one another.

(α) If, however, the constant β is greater than unity, the only real points on the surface are along the lines $u=0$ and $v=0$.

(β) If $\beta=1$ the surface is a plane and the lines of reference are like those on the DARBOUX minimal surface of the preceding case.

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* BONNET, loc. cit., pp. 151–2.

Loc. cit., vol. 1, p. 315.

(γ) If $\beta = 0$, the lines $u =$ const. are nodal cubic curves and the lines $v =$ const. are trochoids.

(δ) If $0 < \beta < 1$, the lines of curvature are similar to those on the DARBOUX minimal surface.

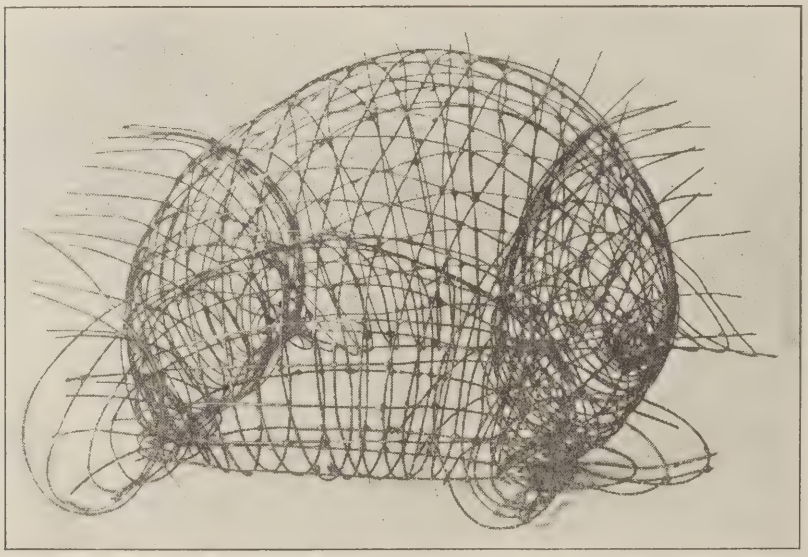
(ϵ) Finally, if β is negative, both sets of the lines of curvature are trochoids which have loops.

A model of one of these surfaces, the one corresponding to

$$ds^2 = (u + v)^2 \left(\frac{du^2}{2u - u^2} + \frac{dv^2}{2v - v^2} \right),$$

has been made by the writer. This surface is divided into segments corresponding to the repeating parts of the lines of curvature. On the four sides of each segment to which others are joined are "rolls," formed by the loops of the lines of curvature, which taper uniformly from the middle of the side of the section, and reduce to points at the ends.

The whole surface is of such a form that it might be placed between two parallel planes, in such a way that one plane would touch the surface in a system



of isolated points (circular points on the surface, each being the crest of a segment) and so that the other would touch it along a double system of straight lines, which divide the plane into a net-work of squares, the sides of each being the locus of the points common to the plane and one segment of the surface.

In case 4° one set of the lines of curvature on the surface are nodal cubic curves, and the other consists of trochoids, or logarithmic curves. Finally, case 5° leads to the well known minimal surface of ENNEPER, an algebraic surface

of the ninth order, on which both sets of the lines of curvature are nodal cubic curves.

III. *Surfaces associated with the quadrics and with the sphere referred to confocal ellipses.*

§ 10. *Solution* $3°(a)$ *of equation* (3). BONNET * undertook to prove that the only surfaces having as their linear element the expression

$$ds^2 = (u+v)\left(\frac{du^2}{U} + \frac{dv^2}{V}\right),$$

when referred to lines of curvature, were the quadrics; but in solving (3), for the case $K=1$, he overlooked † solution $3°$ (a) for this value of K, and considered merely the solution $3°(b)$. Since however the solution overlooked leads to the sphere (as may be shown by computing the total curvature), his final results are not affected.

The lines of reference for the sphere are confocal spherical ellipses.‡

The expressions for the cartesian coördinates of the minimal § surface associated with the sphere may be obtained from its coördinates by a simple integration. They are as follows:

$$(11)\qquad \begin{aligned} x_1 &= \frac{1}{\sqrt{(a-b)(a-c)}} \log\left(\frac{\sqrt{a-v}+\sqrt{a-u}}{\sqrt{u-v}}\right), \\ y_1 &= \frac{1}{\sqrt{(b-a)(b-c)}} \log\left(\frac{\sqrt{b-v}+\sqrt{b-u}}{\sqrt{u-v}}\right), \\ z_1 &= \frac{1}{\sqrt{(c-a)(c-b)}} \log\left(\frac{\sqrt{v-c}+\sqrt{u-c}}{\sqrt{u-v}}\right), \end{aligned}$$

where $c < v < b < u < a$, and where the corresponding linear element is

$$ds_1^2 = \frac{1}{(u-v)}\left(\frac{du^2}{(u-a)(u-b)(u-c)} + \frac{dv^2}{-(v-a)(v-b)(v-c)}\right).$$

§ 11. *Solutions* $3°$ (b) *of equation* (3). BONNET has shown that the surfaces corresponding to this solution of (3) when $K=1$ are the quadrics.

We obtain the expressions for the cartesian coördinates of the associated surfaces by integration, from those obtained by him for the quadrics,‖ in the usual

* Loc. cit., p. 118–132.

† Loc. cit., p. 126.

‡ DARBOUX, loc. cit., vol. 1, p. 242.

§ DARBOUX, loc. cit., vol. 2, p. 244.

‖ Loc. cit., p. 115.

way. Corresponding to the linear element written in the form

$$ds_1^2 = \frac{1}{\mu^2 - \nu^2}\left(\frac{(\rho^2 - \mu^2)\,d\mu^2}{(\mu^2 - b^2)(c^2 - \mu^2)} + \frac{(\rho^2 - \nu^2)\,d\nu^2}{(b^2 - \nu^2)(c^2 - \nu^2)}\right),$$

we find the following systems of equations:

$$x_1 = \frac{\rho}{2bc}\log\left(\frac{\mu - \nu}{\mu + \nu}\right),$$

$$y_1 = \frac{1}{b}\sqrt{\frac{\rho^2 - b^2}{c^2 - b^2}}\sin^{-1}\sqrt{\frac{\mu^2 - b^2}{\mu^2 - \nu^2}},$$

$$z_1 = \frac{1}{2c}\sqrt{\frac{\rho^2 - c^2}{c^2 - b^2}}\log\left(\frac{\sqrt{c^2 - \nu^2} + \sqrt{c^2 - \mu^2}}{\sqrt{c^2 - \nu^2} - \sqrt{c^2 - \mu^2}}\right),$$

$$\nu^2 < b^2 < \mu^2 < c^2 < \rho^2;$$

$$x_1 = \frac{\rho}{2bc}\log\left(\frac{\mu - \nu}{\mu + \nu}\right),$$

$$y_1 = \frac{1}{b}\sqrt{\frac{\rho^2 - b^2}{c^2 - b^2}}\sin^{-1}\sqrt{\frac{\mu^2 - b^2}{b^2 - \nu^2}},$$

$$z_1 = \frac{1}{c}\sqrt{\frac{c^2 - \rho^2}{c^2 - b^2}}\sin^{-1}\sqrt{\frac{\mu^2 - c^2}{\mu^2 - \nu^2}},$$

$$\nu^2 < b^2 < \rho^2 < c^2 < \mu^2;$$

$$x_1 = \frac{\rho}{2bc}\log\left(\frac{\nu - \mu}{\nu + \mu}\right),$$

$$y_1 = \frac{1}{2b}\sqrt{\frac{\rho^2 - b^2}{c^2 - b^2}}\log\left(\frac{\sqrt{\nu^2 - b^2} + \sqrt{\mu^2 - b^2}}{\sqrt{\nu^2 - b^2} - \sqrt{\mu^2 - b^2}}\right),$$

$$z_1 = \frac{1}{c}\sqrt{\frac{c^2 - \rho^2}{c^2 - b^2}}\sin^{-1}\sqrt{\frac{c^2 - \mu^2}{\nu^2 - \mu^2}},$$

$$\rho^2 < b^2 < \mu^2 < c^2 < \nu^2.$$

The surfaces defined by the last three sets of equations correspond respectively to the ellipsoid, the hyperboloid of one sheet, and the hyperboloid of two sheets. The parameters can be eliminated in each of the three cases, giving an equation of the form $f(x_1, y_1, z_1) = 0$, where the non-algebraic functions involved are trigonometric or logarithmic.

PURDUE UNIVERSITY.

THE EXPANSION OF A FUNCTION IN TERMS OF NORMAL FUNCTIONS.*

BY

MAX MASON.

In an article published in these TRANSACTIONS † the author has proved the existence of normal functions for the differential equation

$$y'' + p(x)y' + [\lambda A(x) - B(x)]y = 0 \tag{1}$$

and the boundary conditions

$$\begin{cases} a_1 y(a) + a_2 y(b) + a_3 y'(a) + a_4 y'(b) = 0, \\ b_1 y(a) + b_2 y(b) + b_3 y'(a) + b_4 y'(b) = 0, \end{cases} \tag{2}$$

by a method which sets in evidence the properties of the normal functions as solutions of minimum problems. It was shown that this point of view is useful in deriving oscillation theorems, i. e., in investigating the zeros of the functions.

It is the object of this paper to show that the minimal properties of the normal functions may be used to prove in a simple manner the possibility of expanding a function in terms of normal functions. ‡

The coefficients p, A, B are supposed to be continuous in the interval (a, b). In addition the following assumptions are made, the determinants $a_i b_k - a_k b_i$ of the coefficients of the equations (2) being denoted by d_{ik}:

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* Presented to the Society September 3 and December 28, 1906, in less general form. Received for publication June 27, 1907.

† Vol. 7 (1906), p. 337.

‡ Recent articles of special importance on this subject are those of HILBERT, *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, Nachrichten der K. Gesellschaft der Wissenschaften zu Göttingen, 1904, 1905, 1906, and KNESER, Mathematische Annalen, vol. 63 (1907), p. 477. These articles are based on the theory of integral equations, as developed by HILBERT and SCHMIDT. KNESER removes the restrictions imposed by HILBERT, that the function be continuous with its first and second derivatives, and satisfy the same boundary conditions as the normal functions. In all work on this subject the boundary conditions for the normal functions are special cases of equations (2), above, except in the important case of singularities of the differential equation. The function A is supposed positive in all previous work, except in the fifth memoir of HILBERT's series, in which, by considering a new class of integral equations, the restriction is removed. This restriction will not be made in the present article.

$$(\alpha) \qquad B(x) \geqq 0 \qquad (a \leqq x \leqq b),$$

$$(\beta) \qquad d_{24} = d_{13} e^{\int_a^b p dx},$$

(γ) *No two of the quantities* d_{12}, d_{14}, d_{23}, d_{43} *have opposite signs.*

Equation (β) insures that any two functions u, v which satisfy the boundary conditions (2) also satisfy the equation *

$$(3) \qquad [P(uv' - vu')]_a^b = 0 \qquad \left(P = e^{\int_a^x p dx}\right)$$

On account of (γ) the quantity

$$-[Pyy']_a^b$$

is positive or zero for any function y which satisfies (2). Under these assumptions the following theorem holds: †

There exists an infinite series of normal parameter values λ_n *and corresponding solutions (normal functions)* y_n *of the differential equation* (1) *and the boundary conditions* (2). *If* A *changes sign in* (a, b), *the values* λ_n *include an infinite series of positive terms,* $\lambda_1 \leqq \lambda_2 \leqq \lambda_3 \leqq \lambda_4 \leqq \cdots$, *increasing without limit, and an infinite series of negative terms,* $\lambda_{-1} \geqq \lambda_{-2} \geqq \lambda_{-3} \geqq \lambda_{-4} \geqq \cdots$, *decreasing without limit.*‡ *The function* y_n *satisfies the conditions* §

$$\int_a^b PAy^2 dx = \pm 1,$$

$$\int_a^b PAy_i y dx = 0 \qquad [i = \pm 1, \pm 2, \cdots, \pm (n-1)],$$

and gives to the expression

$$J(y) = \int_a^b (y'^2 + By^2) P dx - [yy'P]_a^b$$

its least possible value consistent with these conditions and with equations (2). *This minimum value of* J *is* $\pm \lambda_n$.

On the basis of these properties the following theorem will be proved:

A function $f(x)$ *which satisfies the boundary conditions* (2), *is continuous within* (a, b), *and has a derivative which is continuous except at a finite number of points, may be expanded in a uniformly and absolutely converging series of the form*

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* See page 349 of the article cited.

† Ibid., page 356.

‡ If A does not change sign, there are no normal parameter values of sign opposite to that of A. In case $B \equiv 0$, λ_1 or λ_{-1} may be zero.

§ The upper or lower signs are to be taken according as λ_n is positive or negative.

$$c_1 y_1 + c_2 y_2 + c_3 y_3 + \cdots$$
$$+ c_{-1} y_{-1} + c_{-2} y_{-2} + c_{-3} y_{-3} + \cdots,$$

where

$$c_{\pm n} = \pm \int_a^b f A y_{\pm n} dx.$$

§ 1. *A property of the constants* c_i.

By consideration of the expression J it will be proved that the series

$$\sum_{i=1}^{\infty} c_i^2 \lambda_i, \qquad \sum_{i=-1}^{-\infty} c_i^2 \lambda_i$$

are convergent.

If u and v are two functions satisfying the boundary conditions, then

$$J(u+v) = J(u) + J(v) + 2J_1(u, v),$$

where

$$J_1(u, v) = \int_a^b (u'v' + Buv) P dx - [uv'P]_a^b.$$

Then, since $J(y_i) = \lambda_i$ for positive subscripts,

$$(4)\quad J\left(f - \sum_{i=1}^{n} c_i y_i\right) = J(f) + \sum_{i=1}^{n} c_i^2 \lambda_i - 2\sum_{i=1}^{n} c_i J_1(f, y_i) + 2\sum_{i,k=1}^{n} c_i c_k J_1(y_i, y_k).$$

Now the function y_i satisfies the differential equation (1) for $\lambda = \lambda_i$, or on multiplying (1) by $P = e^{\int_a^x p dx}$,

$$(Py_i')' + (\lambda_i A - B) P y_i = 0.$$

Therefore,

$$J_1(f, y_i) = \int_a^b (f'y_i' + Bfy_i) P dx - [fy_i' P]_a^b$$
$$= \int_a^b f[BPy_i - (Py_i')'] dx = \lambda_i \int_a^b PAfy_i dx = c_i \lambda_i.$$

Similarly,

$$J_1(y_i, y_k) = \lambda_i \int_a^b PAy_i y_k dx = 0.$$

Since the left member of equation (4) is positive or zero, it follows that whatever be n,

$$J(f) \geqq \sum_{i=1}^{n} c_i^2 \lambda_i.$$

But the terms of the series are positive. Therefore the series

$$\sum_{i=1}^{\infty} c_i^2 \lambda_i$$

is convergent. The convergence of the series with negative subscripts is proved in the same way by considering $J(f - \sum_{i=-1}^{-n} c_i y_i)$.

§ 2. *Convergence of the series.*

Consider the terms with positive subscripts, and write

$$\phi_{n,m} = \sum_{i=n}^{n+m} |c_i y_i|.$$

If n be taken sufficiently large, each function y_i will have at least one zero α_i in the interval (a, b),* and

$$\phi_{n,m} = \sum_{i=n}^{n+m} \left| c_i \int_{\alpha_i}^{x} y_i' \, dx \right|.$$

Now $r^2 + s^2 \geqq 2rs$ for any two numbers r, s, and therefore

$$\phi_{n,m}^2 \leqq 2 \sum_{i=n}^{n+m} c_i^2 \left(\int_{\alpha_i}^{x} y_i' \, dx \right)^2.$$

Furthermore, for all values of the parameter μ, the expression

$$\int_{\alpha_i}^{x} (\mu + y_i')^2 \, dx = \mu^2 (x - \alpha_i) + 2\mu \int_{\alpha_i}^{x} y_i' \, dx + \int_{\alpha_i}^{x} y_i'^2 \, dx$$

has the sign of $x - \alpha_i$. The discriminant of this form is then not negative, that is,

$$\left(\int_{\alpha_i}^{x} y_i' \, dx \right)^2 \leqq (x - \alpha_i) \int_{\alpha_i}^{x} y_i'^2 \, dx \leqq (b - a) \int_{a}^{b} y_i'^2 \, dx.$$

But

$$\int_{a}^{b} y_i'^2 \, dx \leqq J(y_i) = \lambda_i,$$

and therefore

$$\phi_{n,m}^2 \leqq 2 \sum_{i=n}^{n+m} c_i^2 \lambda_i.$$

Since the series

$$\sum_{i=1}^{\infty} c_i^2 \lambda_i$$

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* This follows at once from one of the simplest of STURM's well-known results. It may be proved for the case here considered as follows: Let u be a function having α and β for consecutive zeros in an interval where $A > 0$, and satisfying (1) for $\lambda = \bar{\lambda}$ inside (α, β). The existence of such a function is assured by the theorem stated in the introduction. From the differential equations for u and y_i it follows that

$$[y_i u' P]_{\alpha}^{\beta} = (\lambda_i - \bar{\lambda}) \int_{\alpha}^{\beta} P A u y_i \, dx.$$

Determine the sign of u to be positive in (α, β). Then $u'(\beta) < 0$, $u'(\alpha) > 0$, since u cannot vanish together with its derivative. Now $A > 0$ in (α, β), and therefore it follows from the above equation that y_i must change sign in (α, β) if $\lambda_i > \bar{\lambda}$.

is convergent, it follows that the series

$$c_1 y_1 + c_2 y_2 + c_3 y_3 + \cdots$$

converges uniformly and absolutely. The convergence of the series of terms with negative subscripts is proved in the same way.

§ 3. *The series represents the function.*

Since the series converges uniformly, the difference

$$f - \sum_{i=-\infty}^{\infty} c_i y_i = g$$

is a continuous function. On multiplying this equation by PAy_i and integrating from a to b, the equation

$$\int_a^b PAy_i g dx = 0 \tag{5}$$

results, whatever be i. This equation is sufficient to insure that g is identically zero. It will be shown that on assumption of the contrary a function y may be formed, which has a continuous derivative and satisfies the boundary conditions (2) and the equations

$$\int_a^b PAy_i y dx = 0 \qquad (i = \pm 1, \pm 2, \pm 3, \cdots), \tag{6}$$

$$\int_a^b PAy^2 dx = \pm 1. \tag{7}$$

Suppose such a function exists. The value of the expression

$$J = \int_a^b P(y'^2 + By^2) dx - [yy'P]_a^b$$

is less than λ_n and $-\lambda_{-n}$ if n be taken sufficiently large, since these values increase without limit with n. Now y satisfies all the conditions of the minimum problem of which either y_n or else y_{-n} is the solution, according as (7) holds with upper or lower sign. But it gives to J a value less than the smallest value λ_n or $-\lambda_{-n}$ consistent with the conditions. Such a function y can therefore not exist.

It remains to show that if the difference g is not identically zero, a function y satisfying the boundary conditions (2) and the equations (6) and (7) can be formed. Let $\bar{\lambda}$ be a value which is not a normal parameter value. Then there exists a solution of the equation

$$(Pu')' + (\bar{\lambda}A - B)Pu = PAg$$

under the boundary conditions (2). From this equation and

$$(Py_i')' + (\lambda_i A - B)Py_i = 0$$

it follows, on account of (5), that

$$\int_a^b \{y_i(Pu')' - u(Py_i')'\}\,dx = (\lambda_i - \bar{\lambda})\int_a^b PAy_i u dx.$$

The left member vanishes after an integration by parts, on account of equation (3). Therefore, since $\bar{\lambda} \neq \lambda_i$, the equations (6) are satisfied by the function u. On multiplying u by a constant the equation (7) may be satisfied also, unless

$$\int_a^b PAu^2\,dx = 0.$$

Suppose, however, that this equation holds. Solve the equation

$$(Pv')' + (\bar{\lambda}A - B)Pv = PAu$$

under the boundary conditions (2). This function satisfies (6), the proof being the same as before, except that g is replaced by u. Hence the desired function y may be formed by multiplying v by a constant so that (7) will hold, unless

$$\int_a^b PAv^2\,dx = 0.$$

Suppose again that this is the case. Consider then the function

$$w = u - v,$$

which satisfies (2) and (6). Now

$$\int_a^b PAw^2\,dx = -2\int_a^b PAuv dx = -2\int_a^b \{v(Pv')' - BPv^2\}\,dx = 2J(v) > 0.$$

The required function y may, therefore, be formed in any case, unless g is identically zero. It follows that the series represents the function.

At a later time I hope to extend this method to more general cases.

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SUR LES OPÉRATIONS LINÉAIRES*

(TROISIÈME NOTE†)

PAR

MAURICE FRÉCHET

Définition du champ. Considérons un champ de fonctions de la variable x définies dans l'intervalle $(0, 2\pi)$. Je supposerai que si deux fonctions appartiennent au champ, il en est de même de leur somme. A toute fonction du champ, $f(x)$, nous pourrons faire correspondre un nombre bien déterminé U_f. Nous définirons ainsi une opération dans ce champ de fonctions. Nous dirons que cette opération est *distributive*, si quelles que soient les fonctions $f_1(x)$, $f_2(x)$ du champ, on a:

$$U_{f_1} + U_{f_2} = U_{f_1+f_2}.$$

On en conclut en particulier que l'on a pour toute opération distributive:

$$cU_f = U_{cf} \tag{1}$$

quelle que soit la constante rationnelle c. Pour que cette relation ait lieu même pour c irrationnel, il suffit que U_f satisfasse à une certaine condition complémentaire. Nous allons énoncer plus loin cette condition complémentaire, mais nous l'énoncerons d'une manière particulière pour chacun des champs de fonctions que nous allons examiner.

§ 1. *Opérations linéaires dans le champ (M) des fonctions mesurables et bornées.*

Définition. Supposons qu'une opération U_f soit définie pour toute fonction $f(x)$ mesurable‡ et bornée dans l'intervalle $(0, 2\pi)$. Nous dirons que cette opération est *continue* dans (M), si l'on a:

$$U_f = \lim_{n=\infty} U_{f_n},$$

lorsque $f(x), f_1(x), f_2(x), \cdots$ sont des fonctions du champ (M) qui sont bornées dans leur ensemble et que $f_n(x)$ tend vers $f(x)$ sauf peut être en un

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† Voir Transactions of the American Mathematical Society, vol. 5 (1904), pp. 493–499 et vol. 6 (1905), pp. 134–140.

‡ Voir les *Leçons sur les séries trigonométriques* par H. Lebesgue, p. 8.

ensemble de valeurs de x de mesure nulle. Nous appellerons opération *linéaire* toute opération distributive et continue. Pour une telle opération l'égalité (1) aura évidemment lieu quelle que soit la constante c.

Il est facile de donner des exemples d'opérations linéaires dans le champ (M). Telle est par exemple l'opération:

$$U_f = \int_0^{2\pi} f(x)h(x)\,dx,$$

où $h(x)$ est une fonction mesurable et bornée choisie une fois pour toutes.* Au contraire l'opération distributive $U_f = f(\pi)$ par exemple n'est pas continue à notre sens actuel.

Inversement, étant donnée de façon quelconque une opération linéaire dans le champ (M), nous allons chercher à en donner une expression analytique. Pour cela, nous nous appuierons sur quelques résultats de la théorie des séries trigonométriques.

1°. Toute fonction $f(x)$ mesurable et bornée a une série de Fourier bien déterminée, ce que l'on exprime par la notation symbolique:

$$(2)\quad f(x) \sim \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + \cdots + (a_p \cos px + b_p \sin px) + \cdots.$$

Cette notation qui n'implique nullement l'égalité des deux membres est équivalente aux relations:

$$(2)\qquad a_p = \frac{1}{\pi}\int_0^{2\pi} f(y)\cos py\,dy, \quad b_p = \frac{1}{\pi}\int_0^{2\pi} f(y)\sin py\,dy \quad (p=0, 1, 2, \cdots).$$

2°. En utilisant les sommes de Fejer:

$$(3)\quad \sigma_n(x) \equiv \frac{\frac{na_0}{2} + \cdots + (n-p)(a_p \cos px + b_p \sin px) + \cdots + [a_{n-1}\cos(n-1)x + b_{n-1}\sin(n-1)x]}{n},$$

on sait qu'on a pour toute fonction $f(x)$ mesurable et bornée de 0 à 2π:

$$(4)\qquad f(x) = \lim_{n=\infty} \sigma_n(x),$$

sauf peut être pour un ensemble de valeurs de x de mesure nulle.†
De plus, la quantité $\sigma_n(x)$ reste comprise quels que soient n et x entre les limites supérieure et inférieure de $f(x)$.‡

* Lebesgue, loc. cit., p. 10, 14.
† Loc. cit., p. 92, 96.
‡ Loc. cit., p. 98.

3°. En appelant $s_n(x)$ la somme des $n+1$ premiers termes de la série de Fourier, on sait même que si $f(x)$ est une fonction à variation bornée* dans l'intervalle $(0, 2\pi)$, on a:

$$f(x) = \lim_{n=\infty} s_n(x), \tag{5}$$

sauf peut être en un ensemble de valeurs de x de mesure nulle.† De plus la quantité $s_n(x)$ est comprise entre deux nombres fixes quels que soient n et x.‡

Développement de U_f. Appliquons l'opération U_f aux deux membres de l'égalité (3). On aura:

$$U_f = \lim_{n=\infty} \left[\frac{n\frac{a_0 u_0}{2\pi} + \cdots + (n-p)p(a_p v_p - b_p u_p) + \cdots + (n-1)(a_{n-1}v_{n-1} - b_{n-1}u_{n-1})}{n} \right] \tag{6}$$

en posant:

$$\frac{u_0}{\pi} = U_1, \qquad u_p = -\frac{1}{p} U_{\sin px}, \qquad v_p = \frac{1}{p} U_{\cos px} \quad (p = 1, 2, \cdots). \tag{7}$$

On pourra écrire la formule (6) plus simplement en mettant d'une façon générale une relation telle que:

$$\lambda = \lim_{n=\infty} \frac{n\alpha_0 + \cdots + (n-p)\alpha_p + \cdots + \alpha_{n-1}}{n}$$

sous la forme:

$$\lambda \doteq \alpha_0 + \alpha_1 + \cdots + \alpha_n + \cdots.$$

On voit alors que *pour toute fonction $f(x)$ mesurable et bornée déterminée par* (1), *on a:*

$$U_f \doteq \frac{a_0 u_0}{2\pi} + (a_1 v_1 - b_1 u_1) + \cdots + p(a_p v_p - b_p u_p) + \cdots, \tag{8}$$

où $u_0, u_1, v_1, \cdots$ sont des constantes indépendantes de la fonction $f(x)$ et déterminées par les formules (7). De plus lorsque $f(x)$ est une fonction à variation bornée, on peut remplacer dans le développement (8) le signe $\doteq$ par le signe $=$.

Une propriété importante de ce développement est *d'être unique.* D'une façon plus précise, si l'on obtient par un moyen quelconque la formule:

$$U_f \doteq \frac{a_0 u_0'}{2\pi} + \cdots + p(a_p v_p' - b_p u_p') + \cdots, \tag{9}$$

* Loc. cit., p. 3.

† Loc. cit., p. 73.

‡ M. FATOU a bien voulu me faire savoir qu'on peut obtenir cette propriété en appliquant l'analyse classique de DIRICHLET à une différence de fonctions positives non croissantes.

où les u', v' sont des constantes indépendantes de $f(x)$, ces constantes doivent respectivement coïncider avec les u, v. Il suffit pour s'en assurer d'appliquer l'expression (9) au cas où $f(x)$ est l'une des fonctions 1, $\cos px$, $\sin px$, en tenant compte de (7).

Signification des coefficients u_p, v_p. Nous allons montrer que les coefficients $u_0, u_1, v_1, \cdots$ ne peuvent être pris arbitrairement quand l'opération U_f n'est pas déterminée. Pour cela, appelons $\phi_{x_0}(x)$ la fonction qui est égale à $(x-x_0)/2+\pi$ pour $0 \leqq x < x_0$, à $\pi/2$ pour $x = x_0$ et à $(x-x_0)/2$ pour $x_0 < x \leqq 2\pi$. C'est une fonction à variation bornée qui a pour série de Fourier:

$$\frac{\pi}{2} + (\sin x_0 \cos x - \cos x_0 \sin x) + \cdots + \left(\frac{\sin px_0 \cos px - \cos px_0 \sin px}{p}\right) + \cdots.$$

On a donc d'après ce qui précède:

$$U_{\phi_{x_0}(x)} = \frac{u_0}{2} + (u_1 \cos x_0 + v_1 \sin x_0) + \cdots + (u_p \cos px_0 + v_p \sin px_0) + \cdots,$$

ce qui exige d'abord que les quantités u, v soient telles que le second membre soit une série convergente. La somme de cette série est une certaine fonction de x_0, soit $u(x_0)$. Je dis que cette fonction est continue. En effet, on a:

$$u(x_0') - u(x_0) = U_{[\phi_{x_0'}(x) - \phi_{x_0}(x)]}.$$

Il suffit donc de montrer que lorsque x_0' tend vers x_0, la fonction $\phi_{x_0'}(x) - \phi_{x_0}(x)$ (qui reste comprise entre -2π et $+2\pi$ quels que soient x, x_0, x_0') tend vers zéro sauf peut être en un ensemble de valeurs de x de mesure nulle. Or cela est évident, d'après la définition de $\phi_{x_0}(x)$. Ainsi la série:

$$(10) \qquad u(x) = \frac{u_0}{2} + (u_1 \cos x + v_1 \sin x) + \cdots + (u_p \cos px + v_p \sin px) + \cdots$$

représente une fonction continue. Par suite, les quantités u, v sont les coefficients de FOURIER de cette fonction* qui est aussi de période 2π. En résumé, *les coefficients $u_0, u_1, v_1, \cdots$ du développement* (8) *d'une opération linéaire sont les coefficients de Fourier d'une certaine fonction continue et de période 2π, $u(x)$.* Cette fonction $u(x)$ est déterminée connaissant l'opération U_f par la formule:

$$u(x_0) = U_{\phi_{x_0}(x)}.$$

Représentation de U_f comme limite d'une intégrale double. Réciproquement, pour exprimer U_f connaissant la fonction $u(x)$ il suffit de remplacer dans la formule (8) les u, v, a, b par leurs expressions comme coefficients de Fourier de $u(x)$ et $f(x)$. Par suite *à toute opération linéaire dans le champ* (M)

* Loc. cit., p. 124.

correspond une fonction continue et de période 2π, $u(x)$, *déterminée par la formule:*

$$u(x_0) = U_{\phi_{x_0}(x)}, \tag{11}$$

telle que l'on puisse mettre U_f *sous la forme:*

$$U_f = \lim_{n=\infty} \int_0^{2\pi} \int_0^{2\pi} f(x) u(y) \theta_n(y-x)\, dx dy, \tag{12}$$

où les fonctions $\theta_n(y-x)$ *sont parfaitement déterminées indépendamment de la fonction* $f(x)$ *et de l' opération* U_f *par la formule:*

$$\theta_n(y) = \frac{\dfrac{n}{2\pi} + (n-1)\sin y + \cdots + p(n-p)\sin py + \cdots + (n-1)\sin(n-1)y}{n\pi^2}. \tag{13}$$

Dans le cas où $f(x)$ est à variation bornée, on peut remplacer les $\theta_n(y)$ par les fonctions plus simples:

$$\begin{aligned} \psi_n(y) &= \frac{1}{\pi^2}\left[\frac{1}{2\pi} + \sin y + \cdots + p \sin py + \cdots + (n-1)\sin(n-1)y\right] \\ &= \frac{1}{2\pi^3} + \frac{n\sin(n-1)y - (n-1)\sin ny}{4\pi^2 \sin^2 \dfrac{y}{2}}. \end{aligned} \tag{14}$$

La représentation (12) *de l'opération* U_f *est unique.* Autrement dit, si l'on peut former par un moyen quelconque une fonction continue $v(y)$ telle que l'on ait quelle que soit la fonction $f(x)$:

$$U_f = \lim_{n=\infty} \int_0^{2\pi} \int_0^{2\pi} f(x) v(y) \theta_n(y-x)\, dx dy, \tag{15}$$

on a nécessairement $v(y) \equiv u(y)$. En effet, il suffit d' appliquer les formules (12) et (15) pour $f(x) \equiv 1, \cos px, \sin px$ pour voir que $v(y)$ et $u(y)$ ont les mêmes séries de Fourier.*

Etude de la fonction génératrice. Ainsi à toute opération U_f linéaire dans le champ (M) correspond une fonction continue bien déterminée $u(y)$ et réciproquement si l'on connait $u(y)$ l'opération U_f est exprimée par la formule (12). Nous pouvons donc dire que la fonction $u(y)$ est *la fonction génératrice de* U_f. Il est manifeste que les propriétés de l'opération U_f doivent se refléter dans celles de la fonction continue $u(y)$; de sorte que l'étude de l'opération U_f est ramenée à l'étude d'une fonction continue: sa fonction génératrice. Un premier résultat de cette méthode sera donné par le théorème suivant:

* Loc. cit., p. 37.

La condition nécessaire et suffisante pour qu'une opération U_f linéaire dans le champ (M) soit de la forme:

$$U_f = \int_0^{2\pi} f(x)h(x)\,dx, \tag{16}$$

où $h(x)$ est une fonction mesurable et bornée, est que la fonction génératrice de U_f soit une fonction continue de période 2π qui puisse être considérée comme l'intégrale indéfinie d'une fonction sommable et bornée.

1°. La condition est nécessaire. En effet, si U_f est de la forme (16), sa fonction génératrice est:

$$u(y) = U_{\phi_y(x)} = \int_0^{2\pi} h(x)\phi_y(x)\,dx = \int_0^{2\pi}\left(\frac{x-y}{2}\right)h(x)\,dx + \pi\int_0^y h(x)\,dx$$

d'après la définition de $\phi_y(x)$. On a donc:

$$u(y) = \pi H(y) - \frac{y}{2}[H(2\pi) - H(0)] + M,$$

en posant:

$$H(y) = \int_0^y h(x)\,dx, \qquad M = \int_0^{2\pi}\frac{x}{2}h(x)\,dx.$$

Par conséquent, on a bien $u(2\pi) = u(0)$ et $u(y)$ est l'intégrale indéfinie de la fonction mesurable et bornée:

$$\pi h(y) - \frac{H(2\pi) - H(0)}{2}.$$

2°. La condition est suffisante. En effet, supposons:

$$u(y) = \int k(y)\,dy, \qquad u(2\pi) = u(0),$$

$k(y)$ étant une fonction mesurable et bornée. Ecrivons la série de Fourier de $k(y)$:

$$k(y) \sim (A_1\cos y + B_1\sin y) + \cdots + (A_n\cos ny + B_n\sin ny) + \cdots,$$

où le terme constant s'annule puisque $u(y)$ est de période 2π. Si l'on calcule maintenant les coefficients de Fourier de $u(y)$ en fonction de ceux de $k(y)$, on trouve:

$$u(y) \sim \frac{A_0}{2} + (-B_1\cos y + A_1\sin y) + \cdots + \left(\frac{-B_n\cos ny + A_n\sin ny}{n}\right) + \cdots.$$

D'où:

$$\int_0^{2\pi} u(y)\theta_n(y-x) = \frac{1}{\pi n}\left\{\frac{n}{2\pi}A_0 + \cdots + (n-p)(A_p\cos px + B_p\sin px) + \cdots \right.$$
$$\left. + [A_{n-1}\cos(n-1)x + B_{n-1}\sin(n-1)x]\right\}.$$

La formule (12) s'écrit donc:

$$U_f = \lim_{n=\infty} \int_0^{2\pi} f(x) \Sigma_n(x)\, dx \tag{17}$$

avec:

$$\Sigma_n(x) = \frac{1}{\pi n} \left\{ \frac{n}{2\pi} A_0 + \cdots + (n-p)(A_p \cos px + B_p \sin px) + \cdots \right.$$
$$\left. + [A_{n-1} \cos(n-1)x + B_{n-1} \sin(n-1)x] \right\}.$$

Or on sait que cette quantité tend vers

$$\frac{1}{\pi}\left(\frac{A_0}{2\pi} + k(y)\right)$$

sauf peut être pour un ensemble de valeurs de x de mesure nulle et de plus qu'elle reste bornée comme sa limite. Par suite,* on peut écrire la formule (17):

$$U_f = \int_0^{2\pi} f(x) h(x)\, dx,$$

en appelant $h(x)$ la fonction mesurable et bornée:

$$h(x) \equiv \frac{1}{\pi}\left[\frac{A_0}{2\pi} + k(y)\right].$$

§ 2. *Opérations linéaires dans le champ* (Ω) *des fonctions sommables et de carrés sommables.*

On peut arriver à un résultat plus précis en employant la notion de *convergence en moyenne* introduite par M. Riesz † et M. Fischer ‡ dans le champ (Ω) des fonctions sommables et de carrés sommables. On dira que $f_n(x)$ converge en moyenne vers $f(x)$ et l'on écrira:

$$f(x) \sim \lim f_n(x)$$

si $f(x)$ et $f_n(x)$ sont des fonctions de (Ω) telles que l'intégrale:

$$\int_0^{2\pi} [f_n(x) - f(x)]^2\, dx$$

tende vers zéro avec $1/n$. Nous dirons alors qu'une opération U_f est continue dans le champ Ω si U_{f_n} tend vers U_f quand $f_n(x)$ converge en moyenne vers $f(x)$.

Théorème. A toute opération U_f linéaire dans le champ (Ω), on peut faire correspondre une fonction $k(x)$ du même champ, telle que l'on ait:

$$U_f = \int_0^{2\pi} f(x) k(x)\, dx. \tag{16}$$

* Loc. cit., pp. 14, 15.

† Comptes Rendus, du 12 Nov., 1906; 18 Mars, et 8 Avril, 1907.

‡ Comptes Rendus, du 13 et 27 Mai, 1907.

En effet, on voit facilement qu'avec les notations (2) on a:

$$f(x) \sim \lim_{n=\infty} \left(\frac{a_0}{2} + a_1 \cos x + \cdots + b_n \sin nx \right).$$

Donc:

$$(8') \qquad U_f = \pi \left(\frac{a_0 \alpha_0}{2} + (a_1 \alpha_1 + b_1 \beta_1) + \cdots + (a_n \alpha_n + b_n \beta_n) + \cdots \right)$$

si l'on pose:

$$\pi \alpha_0 = U_1, \cdots, \qquad \pi \alpha_n = U_{\cos nx}, \qquad \pi \beta_n = U_{\sin nx}, \cdots.$$

Or d'après un théorème de M. Riesz* la condition nécessaire et suffisante pour qu'il existe une fonction du champ Ω, $k(x)$, telle que:

$$k(x) \sim \frac{\alpha_0}{2} + \cdots + \alpha_n \cos nx + \beta_n \sin nx + \cdots$$

est que la série $\sum(\alpha_n^2 + \beta_n^2)$ converge. D'autre part si $u(x)$ et $k(x)$ sont des fonctions de Ω, on sait d'apres M. Fatou que l'on a:

$$\int_0^{2\pi} f(x) k(x)\, dx = \pi \left(\frac{a_0 \alpha_0}{2} + \cdots + (a_n \alpha_n + b_n \beta_n) + \cdots \right).$$

Notre théorème sera donc démontré si nous prouvons que la série $\sum(\alpha_n^2 + \beta_n^2)$ converge. Mais, d'après la définition de U_f, la série $\sum(a_n \alpha_n + b_n \beta_n)$ doit être convergente toutes les fois que l'on peut écrire les a_n, b_n sous la forme (2), $f(x)$ étant une fonction de (Ω), c'est à dire toutes les fois que $\sum(a_n^2 + b_n^2)$ converge. Or si $\sum(\alpha_n^2 + \beta_n^2)$ ne convergeait pas, il faudrait que l'une des séries $\sum \alpha_n^2$, $\sum \beta_n^2$ diverge. Supposons par exemple que $\sum \beta_n^2$ diverge; on sait alors † qu'en posant $s_n = \beta_1^2 + \cdots + \beta_n^2$, la série $\sum(\beta_n^2/s_n)$ converge et la série $\sum(\beta_n^2/\sqrt{s_n})$ diverge. Donc si l'on posait:

$$\phi(x) \sim \frac{\beta_2}{\sqrt{s_1}} \sin x + \cdots + \frac{\beta_n}{\sqrt{s_n}} \sin nx + \cdots,$$

on voit que la formule (8)′ ne s'appliquerait plus quoique $f(x)$ soit une fonction de (Ω). On arrive donc à une contradiction, ce qui démontre le théorème.

Inversement, on voit facilement que si $k(x)$ désigne une fonction *quelconque* de (Ω), la formule (16) définit une opération linéaire dans (Ω). En effet, elle

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* Loc. cit.

† Encyclopédie des sciences mathématiques, vol. 1, p. 224, No. 7.

M. Pringsheim a bien voulu me faire remarquer l'usage de cette propriété pour démontrer la propriété suivante utilisée ici:

Si la série $\sum \beta_n v_n$ converge quand les v_n restant fixes, les β_n sont des nombres quelconques tels que $\sum \beta_n^2$ converge, la série $\sum v_n^2$ est nécessairement convergente.

est évidemment distributive; d'autre part si f et f_n sont des fonctions quelconques de (Ω), on aura

$$(U_f - U_{f_n})^2 = \left(\int_0^{2\pi} (f - f_n) k(x)\, dx\right)^2 \leqq \int_0^{2\pi} k^2 dx \cdot \int_0^{2\pi} (f - f_n)^2 dx.$$

Donc si $f_n(x)$ converge en moyenne vers $f(x)$, U_{f_n} tend vers U_f.

§ 3. *Opérations linéaires dans le champ des fonctions continues.*

Définition. Nous distinguerons plusieurs cas selon que l'opération est définie dans tout le champ des fonctions continues de 0 à 2π ou bien dans le champ C_n des fonctions ayant n dérivées continues ou bien dans le champ C_ω des fonctions indéfiniment dérivables de 0 à 2π ou enfin dans le champ E des fonctions holomorphes de 0 à 2π. Enfin nous appellerons respectivement C', C'_n, C'_ω, E' le champ de celles des fonctions de C, C_n, C_ω ou E qui sont de période 2π ainsi que leurs dérivées.

Nous dirons alors avec M. Hadamard que deux fonctions ont un voisinage d'ordre n défini par le nombre ϵ quand, de 0 à 2π, ces fonctions et leurs n premières dérivées différent en valeur absolue de moins de ϵ (n pouvant être nul, entier ou infini). Cela étant, une opération U_f définie dans l'un des champs C, C_n, C_ω ou E s'appellera opération *continue* si $U_f - U_{f_p}$ est infiniment petit en même temps que le voisinage (d'ordre 0, n, ∞, ∞ respectivement) de $f(x)$ et de $f_p(x)$. Une opération distributive et continue sera dite *linéaire* et dans tous les cas l'égalité (1) sera vérifiée même pour c irrationnel.

Montrons d'abord que l'on peut ramener la considération de chacun de ces champs à celle des champs E' et E'_ω.

Supposons une opération U_f définie dans le champ C_n; on pourra représenter toute fonction de ce champ sous la forme:

$$f(x) = f(0) + xf'_0 + \cdots + \frac{x^{n-1}}{(n-1)!} f_0^{(n-1)} + \int_0^x \cdots \int_0^x f_x^{(n)} (dx)^n.$$

On aura donc:

$$U_f = A_0 f(0) + A_1 f'_0 + \cdots + A_{n-1} f_0^{(n-1)} + V_{f_x^{(n)}},$$

$A_0, \cdots, A_{n-1}$ étant des constantes indépendantes de la fonction $f(x)$ et $V_{\phi(x)}$ une opération linéaire dans le champ des fonctions continues $\phi(x)$.

Soit $V_{\phi(x)}$ une opération linéaire quelconque dans le champ C des fonctions continues. Si l'on pose:

$$\psi(x) \equiv f(x) + x\left[\frac{f(0) - f(2\pi)}{2\pi}\right],$$

la fonction $\psi(x)$ est continue et de période 2π. On a donc:

$$V_f = B\,[f(0) - f(2\pi)] + W_{\psi(x)},$$

B étant une constante et $W_{\psi(x)}$ une opération linéaire définie dans le champ C' des fonctions continues et de période 2π.

Enfin si $f(x)$ est une fonction indéfiniment dérivable, on peut trouver une fonction $g(x)$ représentable par un développement de Taylor uniformément convergent et telle que $f(x) - g(x) \equiv h(x)$ soit une fonction admettant une infinité de dérivées toutes de période 2π comme $h(x)$. Si donc U est une opération définie dans le champ C_ω, on peut écrire:

$$U_f = U_{g(x)} + U_{h(x)}.$$

Or, on a:

$$g(x) = g(\pi) + (x-\pi)g'_\pi + \cdots + \frac{(x-\pi)^n}{n!} g^{(n)}(\pi) + \cdots.$$

On a donc:

$$U_g = m_0 g(\pi) + \cdots + m_n g^{(n)}(\pi) + \cdots,$$

$m_0, \cdots, m_n$ étant des constantes indépendantes de la fonction $g(x)$. En résumé tous les cas examinés se ramènent à celui où le champ est C' ou C'_ω.

Fonction génératrice. Soit U_f une opération linéaire définie dans l'un des champs C, C_n, C_ω, E, C', C'_n, C'_ω, E'. Tous ces champs contiennent les fonctions:

$$\tfrac{1}{2} + r\cos x + \cdots + r^p \cos px,$$

$$r\sin x + \cdots + r^p \sin px,$$

quels que soient l'entier p et le nombre r compris entre -1 et $+1$. Ils contiennent aussi leurs limites quand n tend vers l'infini; et le voisinage d'ordre *infini* de ces fonctions avec leurs limites tend vers zéro.

En leur appliquant l'opération U_f, on voit en posant:

$$2\lambda_0 = U_1, \cdots, \qquad \lambda_n = U_{\cos nx}, \qquad \mu_n = U_{\sin nx}, \cdots,$$

que les deux séries:

$$\lambda_0 + \lambda_1 r + \cdots + \lambda_n r^n + \cdots,$$

$$\mu_1 r + \cdots + \mu_n r^n + \cdots,$$

soient convergentes pour $|r| < 1$. Par conséquent la série:

$$\kappa(z) = \lambda_0 + (\lambda_1 + i\mu_1)z + \cdots + (\lambda_n + i\mu_n)z^n + \cdots$$

converge et représente une fonction holomorphe dans le cercle $|z| < 1$ du plan complexe des z.

Ainsi, à l'opération linéaire U_f correspond une fonction holomorphe dans S et bien déterminée: $\kappa(z)$. C'est ce que j'appellerai *la fonction génératrice* de

U_f*. Pour légitimer cette définition, je montrerai que l'on peut exprimer une opération linéaire au moyen de sa fonction génératrice.

En effet, d'après ce que nous avons vu plus haut, on peut se borner au cas des fonctions qui sont de période 2π ainsi que leurs dérivées éventuelles. On peut considérer une telle fonction $f(x)$ comme la valeur pour $r=1$ d'une fonction $F(r, x)$ harmonique et régulière en tout point de coordonnées polaires x et $r<1$. Si $a_0, a_1, b_1, \cdots$ sont les coefficients de Fourier de $f(x)$, on aura:

$$F(r,x)=\frac{a_0}{2}+r(a_1\cos x+b_1\sin x)+\cdots+r^n(a_n\cos nx+b_n\sin nx)+\cdots.$$

D'où:

$$U_F=\frac{1}{\pi}\int_0^{2\pi}f(x)\cdot[\lambda_0+\cdots+r^n(\lambda_n\cos nx+\mu_n\sin nx)+\cdots]\,dx,$$

ou:

$$U_F=\frac{1}{\pi}\cdot\int_0^{2\pi}f(x)P[\kappa(re^{-ix})]\,dx,$$

en désignant d'une manière générale par $P(A+iB)$ la partie réelle A de tout nombre imaginaire $A+iB$.

Si maintenant on fait tendre r vers 1 par valeurs plus petites, on voit qu'en définitive, *U_f sera déterminée par sa fonction génératrice au moyen de la formule:*

$$U_f=\lim_{r=1}\frac{1}{\pi}\int_0^{2\pi}f(x)\cdot P[\kappa(re^{-ix})]\,dx, \tag{19}$$

ou:

$$U_f=\lim_{r=1}P\left\{\frac{1}{\pi}\int_0^{2\pi}f(x)\kappa(re^{-ix})\right\}dx.$$

D'ailleurs, inversement il est très facile d'exprimer la fonction génératrice connaissant U. En effet, on a:

$$P\kappa(re^{i\phi})=\lambda_0+\cdots+r^n(\lambda_n\cos n\phi-\mu_n\sin n\phi)+\cdots=U_{\frac{1}{2}+\ldots+r^n\cos n(\phi-x)+\ldots}.$$

D'où:

$$P\kappa(z)=U_{P1\frac{1}{2}\left[\frac{1+ze^{ix}}{1-ze^{ix}}\right]}.$$

En définitive, on a:

$$\kappa(z)=U_{P1\frac{1}{2}\left[\frac{1+e^{ix}z}{1-e^{ix}z}\right]}-iU_{P\frac{i}{2}\left[\frac{1+ze^{ix}}{1-ze^{ix}}\right]}.$$

Si $\kappa(z)$ est la fonction génératrice d'une opération linéaire dans l'un des champs C', C'_n, C'_ω, cette opération peut s'exprimer sous la forme (19). Mais

* Il ne peut y avoir de confusion dans ce qui suit entre les deux définitions non équivalentes que j'ai données dans des cas distincts pour la fonction génératrice.

il est important de remarquer que même *si l' on se donne a priori pour* $\kappa(z)$ *une fonction holomorphe quelconque dans le cercle* S, *la formule* (19) *définit une opération linéaire au moins dans un certain champ.* En effet, elle donne une valeur bien déterminée de U_f lorsque f est une fonction du champ D_ρ des fonctions $f(x)$ telles que la fonction harmonique correspondante $F(r, x)$ soit régulière même pour $1 < r < \rho$. Car, en prenant: $1 < \rho_1 < \rho$ et appliquant la formule (19) à:

$$F(r, x) = \frac{a_0}{2} + \cdots + \left(\frac{r}{\rho_1}\right)^n \rho_1^n (a_n \cos nx + b_n \sin nx) + \cdots;$$

on trouve l'expression:

$$U_f = \frac{1}{\pi} \int_0^{2\pi} F(\rho_1, x) P\kappa\left(\frac{e^{-ix}}{\rho_1}\right) dx.$$

De plus cette opération est bien distributive et continue.

Mais, selon les propriétés de la fonction $\kappa(z)$, il peut arriver que la formule (19) donne pour U_f une opération linéaire définie dans un champ plus ou moins grand.

On sait que la fonction $\kappa(z)$ est holomorphe dans le cercle S. Il est donc à prévoir que les circonstances les plus importantes dépendront des singularités de $\kappa(z)$ sur le contour de S.

Effet des singularités de $\kappa(z)$. 1°. On est donc amené à considérer d'abord le cas où $\kappa(z)$ est holomorphe dans un cercle de rayon supérieur à 1. Dans ce cas la partie réelle, $k(x)$, de $\kappa(z)$ sur le cercle S est une fonction continue de x et on a:

$$\kappa(z) = \tfrac{1}{2} \int_0^{2\pi} k(-x) \frac{1 + ze^{ix}}{1 - ze^{ix}} dx.$$

Remarquons que cette formule détermine une fonction $\kappa(z)$ holomorphe dans S dans le cas général où $k(x)$ est une fonction mesurable et bornée quelconque et arrêtons nous à ce cas général. En introduisant cette expression de $\kappa(z)$ dans la formule (19), il est facile de voir que l'opération U_f sera définie même dans tout le champ (M) sous la forme:

$$U_f = \int_0^{2\pi} f(x) k(x) dx.$$

2°. Supposons que la fonction $\kappa(z)$ n'ait sur S qu'un nombre fini de points singuliers et que ces points singuliers soient des pôles. Alors on pourra mettre $\kappa(z)$ sous la forme:

$$\kappa(z) = H(z) + \sum_{p=1}^{p=m} \left\{ A_0^{(p)} \left(\frac{1 + ze^{iq_p}}{1 - ze^{iq_p}}\right) + \cdots + A_{n_p}^{(p)} D_{q_p}^{(n_p)} \left(\frac{1 + ze^{iq_p}}{1 - ze^{iq_p}}\right) \right\},$$

où $H(z)$ est une fonction holomorphe au delà de S, où les quantités $A_s^{(p)}$ sont

des constantes et où on désigne par $D_q^{(n_p)}$ la dérivée d'ordre n_p prise par rapport à l'argument de l'un des pôles. Les quantités q_p sont réelles, bornons-nous d'abord au cas où les $A_s^{(p)}$ sont réels et appelons $h(x)$ la partie réelle de $H(e^{-ix})$. On aura alors:

$$U_f = \int_0^{2\pi} f(x)h(x)\,dx + \sum_{p=1}^{p=m} \{ A_0^{(p)} f(q_p) + \cdots + A_{n_p}^{(p)} f^{(n_p)} q_p \},$$

et U_f sera ainsi une opération linéaire bien définie dans le champ C'_m. Mais la formule (19) n'aura plus de sens si on l'applique à une fonction quelconque du champ C'. Prenons par exemple le cas particulier où:

$$\kappa(z) = \frac{Aize^{iq}}{2(1 - ze^{iq})^2},$$

A et q étant deux constantes réelles et:

$$f(x) = \sin x + \cdots + \frac{\sin nx}{n^2} + \cdots.$$

La formule (19) donnerait:

$$U_f = \lim_{r=1} L\sqrt{1 - 2r\cos q + r^2}.$$

Si donc q est quelconque, on aura en général:

$$U_f = L\sqrt{2(1 - \cos q)};$$

mais si l'on a pris par exemple $q = 0$, on a:

$$U_f = \lim_{r=1} L(1 - r),$$

ce qui ne donne aucune valeur finie de U_f.

3°. On pourrait croire que puisque l'existence d'un pôle d'ordre m de $K(z)$ sur S restreint l'application de U_f aux fonctions du champ C'_m, l'existence d'un point singulier essentiel doit en restreindre encore l'application aux fonctions du champ C'_ω. Il n'en est pas ainsi comme le prouve l'exemple très général suivant.

Prenons:

$$\kappa(z) = \sum_{n=1}^{n=\infty} \frac{A_n}{2} \frac{1 + ze^{iq_n}}{1 - ze^{iq_n}},$$

où les A_n, q_n sont des nombres reéls quelconques, mais la série $\sum A_n$ étant absolument convergente. Si l'on prend $q_n = 1/n$ par exemple, on a bien une fonction holomorphe dans S et qui a sur S un point singulier $z = 1$. Or quels que soient les q_n, on a en appliquant la formule (19):

$$U_f = \sum_{n=1}^{n=\infty} A_n f(q_n),$$

et l'opération est définie pour toute fonction du champ C'. On voit donc qu'à ce point de vue, dans l'exemple $q_n = 1/n$, la singularité essentielle $z = 1$, n'est pour ainsi dire qu'une singularité artificielle. Elle ne présente aucune particularité propre en dehors de celles qu'elle possède comme limite de pôles simples. C'est là un point qui peut avoir son intérêt dans la théorie des fonctions analytiques.

BESANÇON, *Février*, 1907.

THE ELLIPTIC INTEGRAL IN ELECTROMAGNETIC THEORY*

BY

ALFRED GEORGE GREENHILL

The analytical complexity in the reduction of the elliptic integral in electromagnetism, as well as in most dynamical problems, arises in consequence of the appropriate integral of the third kind being of the circular form in LEGENDRE's terminology, and the elliptic parameter of JACOBI is then a fraction of the imaginary period.

The expression of the integral by the theta function would imply then a complex argument, and a table of the theta function would not be of complete utility unless made a triple entry table.

When required in a problem of electromagnetism it is the complete third elliptic integral which usually is sufficient for a solution, and this, as shown by LEGENDRE, can be expressed by elliptic integrals of the first and second kind, complete and incomplete; and for these the Table IX of LEGENDRE provides the material for a numerical evaluation.

As an important application we may cite the calculation of the mutual induction of two coaxial helices, employed in the ampère balance designed for weighing their electromagnetic attraction by the late Viriamu JONES and Professor AYRTON, and so arriving at an independent determination of the electrical units.

The present investigation was undertaken in the lifetime of Professor Viriamu JONES, with the object of exhibiting the result of his complete third elliptic integral in its simplest form, suitable for immediate numerical computation, and also to reconcile the conflicting notation of different writers on the subject by adopting MAXWELL's *Electricity and Magnetism* as the standard.

Incidentally the quadric transformation of LANDEN is required here so frequently that a digression has been made on the theory, and an elucidation submitted of its essential geometrical interpretation.

The references in the course of the work will be chiefly to —

MAXWELL, *Electricity and Magnetism.*

WEBSTER, *Electricity and Magnetism.*

GRAY, *Absolute Measurements in Electricity and Magnetism.*

CAYLEY, Proceedings of the London Mathematical Society, vol. 6.

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Viriamu JONES, Philosophical Magazine (1889), Philosophical Transactions (1891), Proceedings of the Royal Society (1897).

MINCHIN, Philosophical Magazine (1893), (1894).

BURNSIDE, Messenger of Mathematics, vol. 20 (1891).

Coleridge FARR, Proceedings of the Royal Society (1898).

NAGAOKA, Journal of the Tokyo College of Science, 16 (1903).

COFFIN, ROSA, and COHEN, *Electromagnetic Integrals*, Bulletin of the Bureau of Standards (1906).

Alexander RUSSELL, *Magnetic Field of the Helix*, Philosophical Magazine, April, 1907.

Notation and preliminary theory.

1. The elliptic integrals of the I, II, and III kind (abbreviated in the sequel to I. E. I.; II. E. I.; III. E. I.) are composed of differential elements of the form

$$(1) \qquad \frac{ds}{\sqrt{S}}, \qquad (s-\sigma)\frac{ds}{\sqrt{S}}, \qquad \frac{1}{s-\sigma}\frac{ds}{\sqrt{S}},$$

$$(2) \qquad S = 4\cdot s - s_1\cdot s - s_2\cdot s - s_3, \qquad s_1 > s_2 > s_3;$$

but for analytical simplification it is convenient to normalise them to zero degree by an appropriate factor, so that the elements may be written

$$(3) \qquad \frac{\sqrt{(s_1-s_3)}\,ds}{\sqrt{S}}, \qquad \frac{s-\sigma}{\sqrt{(s_1-s_3)}}\frac{ds}{\sqrt{S}}, \qquad \frac{\frac{1}{2}\sqrt{\Sigma}}{s-\sigma}\frac{ds}{\sqrt{S}},$$

$$(4) \qquad \Sigma = 4\cdot\sigma - s_1\cdot\sigma - s_2\cdot\sigma - s_3;$$

and in the circular form of the III. E. I. the expression Σ is negative and the normalising factor is changed to $\frac{1}{2}\sqrt{(-\Sigma)}$.

The same normalising is required with the elementary circular or hyperbolic integral which arises when S is of the second or first degree, in consequence of s_1, or s_1 and s_2 being made infinite, so that we take

$$(5) \qquad S = 4\cdot s - s_2\cdot s - s_3, \qquad \text{or} \qquad 4\cdot s - s_3;$$

and the integral corresponding to the III. E. I. becomes

$$(6) \qquad \int \frac{\sqrt{(s_2-\sigma\cdot\sigma-s_3)}}{s-\sigma}\frac{ds}{\sqrt{(4\cdot s - s_2\cdot s - s_3)}} = \tan^{-1}\sqrt{\left(\frac{s_2-\sigma}{\sigma-s_3}\cdot\frac{s-s_3}{s-s_2}\right)},$$

$$(7) \qquad \int \frac{\sqrt{(s_3-\sigma)}}{s-\sigma}\frac{ds}{\sqrt{(4\cdot s - s_3)}} = \sin^{-1}\sqrt{\frac{s-s_3}{s-\sigma}} = \tan^{-1}\sqrt{\frac{s-s_3}{s_3-\sigma}},$$

with corresponding hyperbolic-logarithmic forms.

The standard III. E. I. adopted in the sequel is written

$$\int \frac{\frac{1}{2}\sqrt{(-\Sigma)}}{s-\sigma}\,\frac{ds}{\sqrt{S}}, \tag{8}$$

in which the sign of Σ is to be changed in the infrequent case which arises when the hyperbolic-logarithmic III. E. I. is to be employed; and it is preferable to keep to this form of the integral of an algebraic function, and to settle the sequence of magnitude of

$$\sigma,\ s,\ s_1,\ s_2,\ s_3,$$

before proceeding to a reduction in the notation of WEIERSTRASS, JACOBI, and LEGENDRE to their standard form, if requisite for comparison.

In an electromagnetic application it is the complete III. E. I. which gives the solution, and then the limits of integration are ∞, s_1, s_2, s_3, $-\infty$; and now the great theorem discovered by LEGENDRE (*Fonctions elliptiques*, I, chap. 23) enables us to express the result by incomplete integrals of the I and II kind, the I. E. I. and II. E. I., on which a digression is made at the outset.

2. The jacobian quarter periods, K and K', are defined by the complete integrals

$$K=\int_{s_1,\,s_3}^{\infty,\,s_2}\frac{\sqrt{(s_1-s_3)}\,ds}{\sqrt{S}},\qquad K'=\int_{s_2,\,-\infty}^{s_1,\,s_3}\frac{\sqrt{(s_1-s_3)}\,d\sigma}{\sqrt{(-\Sigma)}}; \tag{1}$$

and the incomplete integrals may be written

$$hK=\int_{s,\,s_3}^{\infty,\,s}\frac{\sqrt{(s_1-s_3)}\,ds}{\sqrt{S}},\qquad fK'=\int_{\sigma,\,-\infty}^{s_1,\,\sigma}\frac{\sqrt{(s_1-s_3)}\,d\sigma}{\sqrt{(-\Sigma)}}, \tag{2}$$

where h and f are real proper fractions defined by the ratios

$$h=\int_{s,\,s_3}^{\infty,\,s}\div\int_{s_1,\,s_3}^{\infty,\,s_2}\frac{ds}{\sqrt{S}},\qquad f=\int_{\sigma,\,-\infty}^{s_1,\,\sigma}\div\int_{s_2,\,-\infty}^{s_1,\,s_3}\frac{d\sigma}{\sqrt{(-\Sigma)}}. \tag{3}$$

In the jacobian notation

$$k^2=\frac{s_2-s_3}{s_1-s_3},\qquad k'^2=\frac{s_1-s_2}{s_1-s_3}, \tag{4}$$

and when

$$\infty>s>s_1,\qquad \text{or}\qquad s_2>s>s_3,$$

$$\operatorname{sn}^2 hK=\frac{s_1-s_3}{s-s_3},\qquad \text{or}\qquad \frac{s-s_3}{s_2-s_3}, \tag{5}$$

$$\operatorname{cn}^2 hK=\frac{s-s_1}{s-s_3},\qquad \text{or}\qquad \frac{s_2-s}{s_2-s_3}, \tag{6}$$

(7) $$\mathrm{dn}^2\, hK = \frac{s - s_2}{s - s_3}, \quad \text{or} \quad \frac{s_1 - s}{s_1 - s_3};$$

while with

$$s_1 > \sigma > s_2, \quad \text{or} \quad s_3 > \sigma > -\infty,$$

(8) $$\mathrm{sn}^2\, fK' = \frac{s_1 - \sigma}{s_1 - s_2}, \quad \text{or} \quad \frac{s_1 - s_3}{s_1 - \sigma},$$

(9) $$\mathrm{cn}^2\, fK' = \frac{\sigma - s_2}{s_1 - s_2}, \quad \text{or} \quad \frac{s_3 - \sigma}{s_1 - \sigma},$$

(10) $$\mathrm{dn}^2\, fK' = \frac{\sigma - s_3}{s_1 - s_3}, \quad \text{or} \quad \frac{s_2 - \sigma}{s_1 - \sigma}.$$

It is the advantage of the use of the fraction h or f of the period K or K' that the corresponding modulus k or k' is indicated thereby and need not be written down.

In the notation given already in equations (2) the incomplete I. E. I. can be written

(11) $$hK = \int_{s_1,\, s_3}^{\infty,\, s} \frac{\sqrt{(s_1 - s_3)}\, ds}{\sqrt{S}} = \mathrm{sn}^{-1} \sqrt{\frac{s_1 - s_3}{s - s_3}}, \qquad \mathrm{sn}^{-1} \sqrt{\frac{s - s_3}{s_2 - s_3}};$$

(12) $$(1 - h)K = \int_{s_1,\, s}^{s,\, s_2} \frac{\sqrt{(s_1 - s_3)}\, ds}{\sqrt{S}} = \mathrm{sn}^{-1} \sqrt{\frac{s - s_1}{s - s_2}}, \qquad \mathrm{dn}^{-1} \sqrt{\frac{s_1 - s_2}{s_1 - s}};$$

and for the complementary modulus k',

(13) $$fK' = \int_{\sigma,\, -\infty}^{s_1,\, \sigma} \frac{\sqrt{(s_1 - s_3)}\, d\sigma}{\sqrt{(-\Sigma)}} = \mathrm{sn}^{-1} \sqrt{\frac{s_1 - \sigma}{s_1 - s_2}}, \qquad \mathrm{sn}^{-1} \sqrt{\frac{s_1 - s_3}{s_1 - \sigma}};$$

(14) $$(1 - f)K' = \int_{s_2,\, \sigma}^{\sigma,\, s_3} \frac{\sqrt{(s_1 - s_3)}\, d\sigma}{\sqrt{(-\Sigma)}} = \mathrm{sn}^{-1} \sqrt{\frac{\sigma - s_2}{\sigma - s_3}}, \qquad \mathrm{sn}^{-1} \sqrt{\frac{s_3 - \sigma}{s_2 - \sigma}}.$$

3. Besides the first elliptic integral (I. E. I.), given already in § 2 (1), (2), (11), (12), (13), (14), we shall require the second elliptic integral (II. E. I.) complete and incomplete.

The jacobian zeta function which expresses the incomplete II. E. I. may be efined by the standard integral

(1) $$\int_{s_3}^{s} \frac{s_1 - s}{\sqrt{(s_1 - s_3)}} \frac{ds}{\sqrt{S}} = \int_0^h \mathrm{dn}^2\, hK \cdot K dh = E\, \mathrm{am}\, hK = hH + \mathrm{zn}\, hK,$$

or

(2) $$\int_{s_2}^{\sigma} \frac{\sigma - s_3}{\sqrt{(s_1 - s_3)}} \frac{d\sigma}{\sqrt{(-\Sigma)}} = \int_0^f \mathrm{dn}^2 fK' \cdot K' df = E\, \mathrm{am}\, fK' = fH' + \mathrm{zn}\, fK';$$

where zn u in GLAISHER's notation denotes the jacobian function Zu; also H and H' denote the complete II. E. I. to moduli k and k', defined by

$$(3) \qquad H = \int_0^1 \mathrm{dn}^2\, hK \cdot K dh, \qquad H' = \int_0^1 \mathrm{dn}^2\, fK' \cdot K' df,$$

corresponding to K and K', the complete I. E. I.

The letter H is used instead of the usual E, as E will be required for the quadric transformation.

With GLAISHER's notation

$$(4) \qquad \mathrm{zs}\, u = \mathrm{zn}\, u + \frac{d}{du} \log \mathrm{sn}\, u,$$

the incomplete II. E. I. in the regions

$$\infty > s > s_1 > s_2 > s > s_3,$$

is given by

$$(5) \quad \int_{s_1}^{s} \frac{s - s_1}{\sqrt{(s_1 - s_3)}} \frac{ds}{\sqrt{S}} = \int_{s}^{s_2} \frac{s_2 - s}{s - s_3} \frac{\sqrt{(s_1 - s_3)}\, ds}{\sqrt{S}} = -(1-h)H + \mathrm{zs}\, hK;$$

$$(6) \quad \int \frac{s - s_2}{\sqrt{(s_1 - s_3)}} \frac{ds}{\sqrt{S}} = k^2 \int \frac{s_1 - s}{s - s_3} \frac{\sqrt{(s_1 - s_3)} ds}{\sqrt{S}} = -(1-h)(H - k'^2 K) + \mathrm{zs}\, hK;$$

$$(7) \quad \int \frac{s - s_3}{\sqrt{(s_1 - s_3)}} \frac{ds}{\sqrt{S}} = \int \frac{s_2 - s_3}{s - s_3} \frac{\sqrt{(s_1 - s_3)}\, ds}{\sqrt{S}} = (1-h)(K - H) + \mathrm{zs}\, hK;$$

the integrals being infinite at the upper limit $s = \infty$, or the lower limit $s = s_3$, where $h = 0$ and zs hK is infinite; and so also

$$(8) \quad \int_{s, s_1}^{\infty, s} \frac{s - s_2}{s - s_3} \frac{\sqrt{(s_1 - s_3)}\, ds}{\sqrt{S}} = \int_{s_3, s}^{s, s_2} \frac{s_1 - s}{\sqrt{(s_1 - s_3)}} \frac{ds}{\sqrt{S}}$$
$$= hH + \mathrm{zn}\, hK, \ (1-h)H - \mathrm{zn}\, hK;$$

$$(9) \quad \int \frac{s - s_1}{s - s_3} \frac{\sqrt{(s_1 - s_3)}\, ds}{\sqrt{S}} = \int \frac{s_2 - s}{\sqrt{(s_1 - s_3)}} \frac{ds}{\sqrt{S}}$$
$$= h(H - k'^2 K) + \mathrm{zn}\, hK, \ (1-h)(H - k'^2 K) - \mathrm{zn}\, hK;$$

$$(10) \quad \int \frac{s_2 - s_3}{s - s_3} \frac{\sqrt{(s_1 - s_3)}\, ds}{\sqrt{S}} = \int \frac{s - s_3}{\sqrt{(s_1 - s_3)}} \frac{ds}{\sqrt{S}}$$
$$= h(K - H) - \mathrm{zn}\, hK, \ (1-h)(K - H) + \mathrm{zn}\, hK.$$

Similarly, in the regions

$$s_1 > \sigma > s_2 > s_3 > \sigma > -\infty,$$

$$(11)\quad \int_{\sigma,\, s_2}^{s_1,\, \sigma} \frac{s_1-\sigma}{\sqrt{(s_1-s_3)}}\frac{d\sigma}{\sqrt{(-\Sigma)}} = \int_{-\infty,\, \sigma}^{\sigma,\, s_3} \frac{s_1-s_2}{s_1-\sigma}\frac{\sqrt{(s_1-s_3)}\,d\sigma}{\sqrt{(-\Sigma)}}$$
$$= f(K'-H') - \mathrm{zn}\, fK',\ (1-f)(K'-H') + \mathrm{zn}\, fK';$$

$$(12)\quad \int \frac{\sigma-s_2}{\sqrt{(s_1-s_3)}}\frac{d\sigma}{\sqrt{(-\Sigma)}} = k'^2 \int \frac{s_3-\sigma}{s_1-\sigma}\frac{\sqrt{(s_1-s_3)}d\sigma}{\sqrt{(-\Sigma)}}$$
$$= f(H'-k^2K') + \mathrm{zn}\, fK',\ (1-f)(H'-k^2K') - \mathrm{zn}\, fK';$$

$$(13)\quad \int \frac{\sigma-s_3}{\sqrt{(s_1-s_3)}}\frac{d\sigma}{\sqrt{(-\Sigma)}} = \int \frac{s_2-\sigma}{s_1-\sigma}\frac{\sqrt{(s_1-s_3)}\,d\sigma}{\sqrt{(-\Sigma)}}$$
$$= fH' + \mathrm{zn}\, fK',\ (1-f)H' - \mathrm{zn}\, fK';$$

$$(14)\quad \int_{s_2}^{\sigma} \frac{s_1-s_2}{s_1-\sigma}\frac{\sqrt{(s_1-s_3)}d\sigma}{\sqrt{(-\Sigma)}} = \int_{\sigma}^{s_3} \frac{s_1-\sigma}{\sqrt{(s_1-s_3)}}\frac{d\sigma}{\sqrt{(-\Sigma)}}$$
$$= (1-f)(K'-H') + \mathrm{zs}\, fK';$$

$$(15)\quad \begin{aligned} k'^2 \int \frac{s_3-\sigma}{s_1-\sigma}\frac{\sqrt{(s_1-s_3)}d\sigma}{\sqrt{(-\Sigma)}} &= \int \frac{s_2-\sigma}{\sqrt{(s_1-s_3)}}\frac{d\sigma}{\sqrt{(-\Sigma)}} \\ &= -(1-f)(H'-k^2K') + \mathrm{zs}\, fK', \end{aligned}$$

$$(16)\quad \int \frac{s_2-\sigma}{s_1-\sigma}\frac{\sqrt{(s_1-s_3)}d\sigma}{\sqrt{(-\Sigma)}} = \int \frac{s_3-\sigma}{\sqrt{(s_1-s_3)}}\frac{d\sigma}{\sqrt{(-\Sigma)}}$$
$$= -(1-f)H' + \mathrm{zs}\, fK';$$

these integrals (14), (15), (16) being infinite when the upper limit $\sigma = s_1$, or the lower limit $\sigma = -\infty$, where $f = 0$ and $\mathrm{zs}\, fK'$ is infinite.

Putting $h = 1$ or $f = 1$ in any of these forms will give the corresponding complete II. E. I., noticing that $\mathrm{zn}\, K'$ and $\mathrm{zs}\, K'$ are zero.

4. The incomplete III. E. I. requires the theta and eta functions of JACOBI, and BURKHARDT has given in his *Elliptische Functionen*, § 126, a method of approximating rapidly to the numerical value for given elliptic argument.

But in the circular III. E. I. this argument will be complex of the form $hK + fK'i$, where h and f are real fractions, so that great care will be required with BURKHARDT's method in separating the real and imaginary parts of the theta function.

When f, however, is a rational fraction,

$$(1)\qquad f = \tfrac{1}{2},\ \tfrac{1}{3},\ \tfrac{1}{4},\ \tfrac{1}{5},\ \cdots,$$

a quotient of theta functions, such as

$$(2) \qquad \frac{\Theta(hK+fK'i)}{\Theta(hK-fK'i)}, \qquad \frac{\Theta 0\,\Theta(hK+fK'i)}{\Theta hK\,\Theta fK'i},$$

can be expressed algebraically by elliptic functions of hK; a collection of the simplest results will be found in the Philosophical Transactions (1904), under the title, *The third elliptic integral and the ellipsotomic problem.*

5. Make a start now with the complete circular III. E. I.

$$(1) \qquad A,\ B = \int_{s_1,\, s_3}^{\infty,\, s_2} \frac{\frac{1}{2}\sqrt{(-\Sigma)}}{s \sim \sigma} \frac{ds}{\sqrt{S}},$$

in which the quantities range in the order of magnitude

$$(2) \qquad \infty > s > s_1 > \sigma > s_2 > s > s_3 > -\infty$$

and this integral (1) is by Legendre's theorem to be expressed by incomplete integrals, I. E. I. and II. E. I.

Provided with the notation above we shall find in the sequel

$$(3) \qquad A = \int_{s_1}^{\infty} \frac{\frac{1}{2}\sqrt{(-\Sigma)}}{s-\sigma} \frac{ds}{\sqrt{S}} = \tfrac{1}{2}\pi(1-f) - K\,\mathrm{zn}\,fK',$$

$$(4) \qquad B = \int_{s_3}^{s_2} \frac{\frac{1}{2}\sqrt{(-\Sigma)}}{\sigma-s} \frac{ds}{\sqrt{S}} = \tfrac{1}{2}\pi f + K\,\mathrm{zn}\,fK',$$

$$(5) \qquad A + B = \tfrac{1}{2}\pi.$$

To calculate the numerical value we have to determine the co-modular angle θ and the amplitude angle of ϕ from the relations (2) and (8), § 2,

$$(6) \qquad \sin^2\theta = k^2 = \frac{s_1 - s_2}{s_1 - s_3}, \qquad \sin^2\phi = \frac{s_1 - \sigma}{s_1 - s_2},$$

and then from Table IX of Legendre

$$(7) \qquad f = F\phi \div F(\tfrac{1}{2}\pi),$$

$$(8) \qquad \mathrm{zn}\,fK' = E\phi - fE(\tfrac{1}{2}\pi),$$

$$(9) \qquad K = F(\tfrac{1}{2}\pi, k).$$

The result in the Weierstrassian notation is given in Schwarz's *Formeln*, §§ 59, 60.

The relation in (5) is the equivalent of Minchin's equation (21), p. 212, Philosophical Magazine, February, 1894; it is proved immediately by the substitution

$$(10)\qquad \left(s - s_3,\ \frac{s_1 - s_3 \cdot s_2 - s_3}{s - s_3}\right)$$

in A, (1), which leaves $ds/\sqrt{S}$ unaltered except for sign, but changes the limits from ∞ and s_1 to s_3 and s_2, and thus makes

$$(11)\qquad A = \int_{s_3}^{s_2} \frac{\frac{1}{2}\sqrt{(-\Sigma)}}{\dfrac{s_1 - s_3 \cdot s_2 - s_3}{s - s_3} - \sigma + s_3} \frac{ds}{\sqrt{S}},$$

$$(12)\qquad A + B = \int \frac{\frac{1}{2}\sqrt{(-\Sigma)}\,[(s_1 - s_3)(s_2 - s_3) - (s - s_3)^2]}{[(s_1 - s_3 \cdot s_2 - s_3) - (s - s_3 \cdot \sigma - s_3)](\sigma - s)} \frac{ds}{\sqrt{S}}$$

$$= \cos^{-1}\sqrt{\frac{\sigma - s_3 \cdot s_1 - s \cdot s_2 - s}{[(s_1 - s_3)(s_2 - s_3) - (s - s_3)(\sigma - s_3)](\sigma - s)}}$$

$$= \sin^{-1}\sqrt{\frac{s_1 - \sigma \cdot \sigma - s_2 \cdot s - s_3}{[(s_1 - s_3 \cdot s_2 - s_3) - (s - s_3)(\sigma - s_3)](\sigma - s)}}$$

$$= \tfrac{1}{2}\sin^{-1}\left.\frac{\frac{1}{2}\sqrt{(-\Sigma)}\sqrt{S}}{[(s_1 - s_3 \cdot s_2 - s_3) - (s - s_3)(\sigma - s_3)](\sigma - s)}\right]_{s_3}^{s_2} = \tfrac{1}{2}\pi.$$

6. In the arrangement

$$(1)\qquad \infty > s > s_1 > s_2 > s > s_3 > \sigma > -\infty,$$

there are two remaining forms of the complete circular III. E. I., namely

$$(2)\qquad C,\ D = \int_{s_1,\ s_3}^{\infty,\ s_1} \frac{\frac{1}{2}\sqrt{(-\Sigma)}}{s - \sigma} \frac{ds}{\sqrt{S}};$$

and we shall find

$$(3)\qquad C = \int_{s_1}^{\infty} \frac{\frac{1}{2}\sqrt{(-\Sigma)}}{s - \sigma} \frac{ds}{\sqrt{S}} = K\,\mathrm{zs}\,fK' - \tfrac{1}{2}\pi(1 - f),$$

$$(4)\qquad D = \int_{s_3}^{s_2} \frac{\frac{1}{2}\sqrt{(-\Sigma)}}{s - \sigma} \frac{ds}{\sqrt{S}} = K\,\mathrm{zs}\,fK' + \tfrac{1}{2}\pi f,$$

where in Glaisher's notation

$$(5)\qquad \mathrm{zs}\,u = \mathrm{zn}\,u + \frac{\mathrm{cn}\,u\,\mathrm{dn}\,u}{\mathrm{sn}\,u} = \mathrm{zn}\,u + \frac{d}{du}\log \mathrm{sn}\,u = \frac{d}{du}\log Hu,$$

and also

$$(6)\qquad \mathrm{zc}\,u = \mathrm{zn}\,u + \frac{d}{du}\log \mathrm{cn}\,u = -\,\mathrm{zs}\,(K - u),$$

$$(7)\qquad \mathrm{zd}\,u = \mathrm{zn}\,u + \frac{d}{du}\log \mathrm{dn}\,u = -\,\mathrm{zn}\,(K - u).$$

Thus

(8) $$-C+D=\tfrac{1}{2}\pi,$$

which can be proved independently in the same manner as (12) § 5.

Hence we can write down the expression of the complete III. E. I. of the circular form, for any assigned region of s and σ, the result involving a zeta function to the complementary modulus; and this zeta function is an algebraical function when f is a rational fraction; the formula for the simplest fractions is given in the Philosophical Transactions (1904).

7. We have seen how A and B change place by the substitution

(1) $$\left(s-s_3,\ \frac{s_1-s_3\cdot s_2-s_3}{s-s_3}\right),$$

and the same substitution interchanges C and D.

So also A and C or B and D change place by the substitution

(2) $$\left(s-s_1,\ \frac{s_1-s_2\cdot s_1-s_3}{s-s_1}\right);$$

while the substitution

(3) $$\left(s-s_2,\ \frac{s_2-s_1\cdot s_2-s_3}{s-s_2}\right)$$

interchanges A and D, or B and C.

8. Two more forms are required to finish the series of the complete III. E. I., when it is finite and the parameter is a fraction of the real period, namely,

(1) $$E=\int_{s_1}^{\infty}\frac{\frac{1}{2}\sqrt{\Sigma}}{s-\sigma}\frac{ds}{\sqrt{S}}=K\,\mathrm{zn}\,fK,$$

(2) $$F=\int_{s_3}^{s_2}\frac{\frac{1}{2}\sqrt{S}}{s-\sigma}\frac{d\sigma}{\sqrt{\Sigma}}=K\,\mathrm{zs}\,hK,$$

in the arrangement

(3) $$\infty>s>s_1>s_2>\sigma>s_3>-\infty,$$

with

(4) $$h=\int_{s}^{\infty}\div\int_{s_1}^{\infty}\frac{ds}{\sqrt{S}},$$

(5) $$f=\int_{s_3}^{s}\div\int_{s_3}^{s_2}\frac{d\sigma}{\sqrt{\Sigma}}.$$

(Legendre, *Fonctions elliptiques*, chap. 23; Cayley, *Elliptic Functions*, §§ 178–186.)

9. The proof of his theorem that the complete III. E. I., circular or logarithmic, can be expressed by the I. and II. E. I. is given by Legendre in *Fonctions elliptiques*, chapters 23 and 24, and is reproduced in Cayley's *Elliptic*

Functions, chap. 5. We can abbreviate the demonstration considerably, and avoid also all introduction of an imaginary in the discussion of the circular form expressed by A, B C, and D by means of the simple lemma

$$(1)\qquad \sqrt{S}\,\frac{d}{ds}\,\frac{\frac{1}{2}\sqrt{S}}{s\sim\sigma}-\sqrt{(-\Sigma)}\,\frac{d}{d\sigma}\,\frac{\frac{1}{2}\sqrt{(-\Sigma)}}{s\sim\sigma}=s\sim\sigma,$$

proved immediately by performing the differentiation.

Integrating with respect to the elliptic differential elements $ds/\sqrt{S}$ and $d\sigma/\sqrt{(-\Sigma)}$ beween the limits s_3, s_2 of s, and σ, s_1 of σ,

$$\begin{aligned}
&\int\!\!\int\frac{d\sigma}{\sqrt{(-\Sigma)}}\,\frac{ds}{\sqrt{S}}\sqrt{S}\,\frac{d}{ds}\,\frac{\frac{1}{2}\sqrt{S}}{\sigma-s}-\int\!\!\int\frac{ds}{\sqrt{S}}\,\frac{d\sigma}{\sqrt{(-\Sigma)}}\sqrt{(-\Sigma)}\,\frac{d}{d\sigma}\,\frac{\frac{1}{2}\sqrt{(-\Sigma)}}{-s}\\
(2)\qquad &=\int_{\sigma}^{s_1}\left(\frac{\frac{1}{2}\sqrt{S}}{\sigma-s}\right)_{s_3}^{s_2}\frac{d\sigma}{\sqrt{(-\Sigma)}}\,(\text{or }0)+\int_{s_3}^{s_2}\frac{\frac{1}{2}\sqrt{(-\Sigma)}}{\sigma-s}\,\frac{ds}{\sqrt{S}}\,(\text{or }B)\\
&=\int\!\!\int(\sigma-s)\,\frac{ds}{\sqrt{S}}\,\frac{d\sigma}{\sqrt{(-\Sigma)}},
\end{aligned}$$

in which the variables are separated, so that

$$\begin{aligned}
B&=\int_{\sigma}^{s_1}\int_{s_3}^{s_2}(\sigma-s_3)\,\frac{ds}{\sqrt{S}}\,\frac{d\sigma}{\sqrt{(-\Sigma)}}-\int\!\!\int(s-s_3)\,\frac{ds}{\sqrt{S}}\,\frac{d\sigma}{\sqrt{(-\Sigma)}}\\
(3)\qquad &=K\int_{\sigma}^{s_1}\frac{\sigma-s_3}{\sqrt{(s_1-s_3)}}\,\frac{d\sigma}{\sqrt{(-\Sigma)}}-fK'\int_{s_3}^{s_2}\frac{s-s_3}{\sqrt{(s_1-s_3)}}\,\frac{ds}{\sqrt{S}}\\
&=K(fH'+\operatorname{zn}fK')-fK'(K-H)\\
&=\tfrac{1}{2}\pi f+K\operatorname{zn}fK',
\end{aligned}$$

by reason of LEGENDRE's relation

$$(4)\qquad \int_{s_2}^{s_1}\int_{s_3}^{s_2}(\sigma-s)\,\frac{ds}{\sqrt{S}}\,\frac{d\sigma}{\sqrt{(-\Sigma)}}=KH'+K'H-KK'=\tfrac{1}{2}\pi,$$

derivable from the relation in (5) § 5, and utilising theorems (13) and (10), § 3.

Taking the limits of s as s_3 and s gives a more general theorem

$$\begin{aligned}
&\int_{\sigma}^{s_1}\frac{\frac{1}{2}\sqrt{S}}{\sigma-s}\,\frac{d\sigma}{\sqrt{(-\Sigma)}}+\int_{s_3}^{s}\frac{\frac{1}{2}\sqrt{(-\Sigma)}}{\sigma-s}\,\frac{ds}{\sqrt{S}}\\
(5)\qquad &=hK\int_{\sigma}^{s_1}\frac{\sigma-s_3}{\sqrt{(s_1-s_3)}}\,\frac{d\sigma}{\sqrt{(-\Sigma)}}-fK'\int_{s_3}^{s}\frac{s-s_3}{\sqrt{(s_1-s_3)}}\,\frac{ds}{\sqrt{S}}\\
&=hK(fH'+\operatorname{zn}fK')-fK'\,[h(K-H)-\operatorname{zn}hK]\\
&=\tfrac{1}{2}\pi hf+hK\operatorname{zn}fK'-fK'\operatorname{zn}hK,
\end{aligned}$$

connecting incomplete III. E.I's., as in LEGENDRE's equation (i'), p. 133.

Putting $h = 1$ gives B as before; and putting $f = 1$ leads to a new theorem

$$(6) \qquad \int_{s_2}^{s_1} \frac{\frac{1}{2}\sqrt{S}}{\sigma - s} \frac{d\sigma}{\sqrt{(-\Sigma)}} = \tfrac{1}{2}\pi h - K' \operatorname{zn} hK.$$

10. When A is deduced in this manner we must keep clear of $s = \infty$ by writing the lemma (1), § 9, in the form

$$(1) \quad \sqrt{(-\Sigma)} \frac{d}{d\sigma} \frac{\frac{1}{2}\sqrt{(-\Sigma)}}{s - \sigma} - \sqrt{S} \frac{d}{ds}\left(\frac{\frac{1}{2}\sqrt{S}}{s - \sigma} - \frac{\frac{1}{2}\sqrt{S}}{s - s_3}\right) = \sigma - s_3 - \frac{s_1 - s_3 \cdot s_2 - s_3}{s - s_3},$$

in which at

$$(2) \qquad s = \infty, \qquad \operatorname{limit}\left(\frac{\frac{1}{2}\sqrt{S}}{s - \sigma} - \frac{\frac{1}{2}\sqrt{S}}{s - s_3}\right) = 0.$$

Integrating between the limits s, ∞ of s, and s_2, σ of σ, we have

$$(3) \quad \int_s^\infty \frac{\frac{1}{2}\sqrt{(-\Sigma)}}{s - \sigma} \frac{ds}{\sqrt{S}} + \int_{s_2}^{\sigma} \left(\frac{\frac{1}{2}\sqrt{S}}{s - \sigma} - \frac{\frac{1}{2}\sqrt{S}}{s - s_3}\right) \frac{d\sigma}{\sqrt{(-\Sigma)}}$$

$$= \int\int \left(\sigma - s_3 - \frac{s_1 - s_3 \cdot s_2 - s_3}{s - s_3}\right) \frac{ds}{\sqrt{S}} \frac{d\sigma}{\sqrt{(-\Sigma)}}$$

(in which the variables are separated)

$$= hK \int_{s_2}^{\sigma} \frac{\sigma - s_3}{\sqrt{(s_1 - s_3)}} \frac{d\sigma}{\sqrt{(-\Sigma)}} - (1 - f) K' \int_s^\infty \frac{s_2 - s_3}{s - s_3} \frac{\sqrt{(s_1 - s_3)}\, ds}{\sqrt{S}}$$

$$= hK[(1 - f)H' - \operatorname{zn} fK'] - (1 - f)K'[h(K - H) - \operatorname{zn} hK]$$

$$= \tfrac{1}{2}\pi h(1 - f) - hK \operatorname{zn} fK' + (1 - f)K' \operatorname{zn} hK,$$

utilising the theorems in (13) and (5), § 3.

Putting $s = s_1$, $h = 1$, gives the result for A,

$$(4) \quad A = \int_{s_1}^\infty \frac{\frac{1}{2}\sqrt{(-\Sigma)}}{s - \sigma} \frac{ds}{\sqrt{S}} = \int_{s_1}^\infty \int_{s_2}^{\sigma} \left(\sigma - s_3 - \frac{s_1 - s_3 \cdot s_2 - s_3}{s - s_3}\right) \frac{d\sigma}{\sqrt{(-\Sigma)}} \frac{ds}{\sqrt{S}},$$

and this by the substitution (10), § 5, is equivalent to

$$(5) \qquad A = \int_{s_2}^{\sigma} \int_{s_3}^{s_2} (\sigma - s) \frac{ds}{\sqrt{S}} \frac{d\sigma}{\sqrt{(-\Sigma)}},$$

while from (2), § 9,

$$(6) \qquad B = \int_{\sigma}^{s_1} \int_{s_3}^{s_2} (\sigma - s) \frac{ds}{\sqrt{S}} \frac{d\sigma}{\sqrt{(-\Sigma)}},$$

$$(7)\qquad A+B=\int_{s_2}^{s_1}\int_{s_3}^{s_2}(\sigma-s)\frac{ds}{\sqrt{S}}\frac{d\sigma}{\sqrt{(-\Sigma)}},$$

or from (12), § 5, and employing (12), (9), § 3,

$$(8)\qquad \tfrac{1}{2}\pi=\int\int[\sigma-s_2+s_2-s]\frac{ds}{\sqrt{S}}\frac{d\sigma}{\sqrt{(-\Sigma)}}$$
$$=K(H'-k^2K')+K'(H-k'^2K);$$

and thus LEGENDRE's relation is proved.

Putting $f=0$, $\sigma=s_1$ in (3) gives a new theorem,

$$(9)\ \int_{s_2}^{s_1}\frac{\frac{1}{2}\sqrt{S}}{s-\sigma}\frac{d\sigma}{\sqrt{(-\Sigma)}}=\tfrac{1}{2}\pi h+K'\operatorname{zn}hK+K'\sqrt{\frac{s-s_1\cdot s-s_2}{s-s_3\cdot s_1-s_3}}=\tfrac{1}{2}\pi h+K'\operatorname{zs}hK,$$

employing (5), (6), (7), § 2, and (4), § 3, or (5), § 6.

11. For C, integrate lemma (1), § 9, from s to ∞ and σ to s_3 in the form

$$(1)\ -\sqrt{(-\Sigma)}\frac{d}{d\sigma}\frac{\frac{1}{2}\sqrt{(-\Sigma)}}{s-\sigma}+\sqrt{S}\frac{d}{ds}\left(\frac{\frac{1}{2}\sqrt{S}}{s-\sigma}-\frac{\frac{1}{2}\sqrt{S}}{s-s_3}\right)=s_3-\sigma+\frac{s_1-s_3\cdot s_2-s_3}{s-s_3},$$

$$\int_s^\infty\frac{\frac{1}{2}\sqrt{(-\Sigma)}}{s-\sigma}\frac{ds}{\sqrt{S}}-\int_\sigma^{s_3}\left(\frac{\frac{1}{2}\sqrt{S}}{s-\sigma}-\frac{\frac{1}{2}\sqrt{S}}{s-s_3}\right)\frac{d\sigma}{\sqrt{(-\Sigma)}}$$
$$=\int\int\left(s_3-\sigma+\frac{s_1-s_3\cdot s_2-s_3}{s-s_3}\right)\frac{ds}{\sqrt{S}}\frac{d\sigma}{\sqrt{(-\Sigma)}}$$
$$(2)\qquad =hK\int_\sigma^{s_3}\frac{s_3-\sigma}{\sqrt{(s_1-s_3)}}\frac{d\sigma}{\sqrt{(-\Sigma)}}+(1-f)K'\int_s^\infty\frac{s_2-s_3}{s-s_3}\frac{\sqrt{(s_1-s_3)}\,ds}{\sqrt{S}}$$
$$=hK[\operatorname{zs}fK'-(1-f)H']+(1-f)K'[h(K-H)-\operatorname{zn}hK]$$
$$=-\tfrac{1}{2}\pi h(1-f)-(1-f)K'\operatorname{zn}hK+hK\operatorname{zs}fK',$$

employing (16), (10), § 3; from which C is deduced by putting $h=1$, $s=s_1$,

$$(3)\qquad C=-\tfrac{1}{2}\pi(1-f)+K\operatorname{zs}fK'.$$

In D we must keep clear of $\sigma=s_1$ by writing the lemma

$$(4)\quad \sqrt{(-\Sigma)}\frac{d}{d\sigma}\left(\frac{\frac{1}{2}\sqrt{(-\Sigma)}}{s-\sigma}-\frac{\frac{1}{2}\sqrt{(-\Sigma)}}{s_1-\sigma}\right)-\sqrt{S}\frac{d}{ds}\cdot\frac{\frac{1}{2}\sqrt{S}}{s-\sigma}=s_1-s-\frac{s_1-s_2\cdot s_1-s_3}{s_1-\sigma}$$

and then integrate it between s_3 and s, $-\infty$ and σ, so that

$$\int_{s_3}^{s}\left(\frac{\frac{1}{2}\sqrt{(-\Sigma)}}{s-\sigma}-\frac{\frac{1}{2}\sqrt{(-\Sigma)}}{s_1-\sigma}\right)\frac{ds}{\sqrt{S}}-\int_{-\infty}^{\sigma}\frac{\frac{1}{2}\sqrt{S}}{s-\sigma}\frac{d\sigma}{\sqrt{(-\Sigma)}}$$

$$(5)\qquad = fK'\int\frac{s_1-s}{\sqrt{(s_1-s_3)}}\frac{ds}{\sqrt{S}} - hK\int\frac{s_1-s_2}{s_1-\sigma}\sqrt{(s_1-s_3)}\frac{d\sigma}{\sqrt{(-\Sigma)}}$$

$$= fK'(hH + \operatorname{zn} hK) - hK[f(K'-H') - \operatorname{zn} fK']$$

$$= \tfrac{1}{2}\pi hf + fK' \operatorname{zn} hK + hK \operatorname{zn} fK',$$

employing (8), (11), § 3.

Put $h = 1$, $s = s_2$, and we obtain by means of (8), (9), (10), § 2,

$$(6)\quad D = \tfrac{1}{2}\pi f + K \operatorname{zn} fK' + K\sqrt{\frac{\sigma - s_2 \cdot \sigma - s_3}{s_1 - \sigma \cdot s_1 - s_3}} = \tfrac{1}{2}\pi f + K \operatorname{zs} fK'.$$

12. To obtain E and F, write the lemma

$$(1)\quad \sqrt{\Sigma}\frac{d}{d\sigma}\frac{\frac{1}{2}\sqrt{\Sigma}}{s-\sigma} + \sqrt{S}\frac{d}{ds}\left(\frac{\frac{1}{2}\sqrt{S}}{s-\sigma} - \frac{\frac{1}{2}\sqrt{S}}{s-s_3}\right) = \frac{s_1 - s_3 \cdot s_2 - s_3}{s - s_3} - \sigma + s_3,$$

and integrate between s, ∞ and s_3, σ; then

$$\int_s^\infty \frac{\frac{1}{2}\sqrt{\Sigma}}{s-\sigma}\frac{ds}{\sqrt{S}} - \int_{s_3}^\sigma \left(\frac{\frac{1}{2}\sqrt{S}}{s-\sigma} - \frac{\frac{1}{2}\sqrt{S}}{s-s_3}\right)\frac{d\sigma}{\sqrt{\Sigma}}$$

$$(2)\qquad = fK\int\frac{s_2-s_3}{s-s_3}\frac{\sqrt{(s_1-s_3)}\,ds}{\sqrt{S}} - hK\int\frac{\sigma-s_3}{\sqrt{(s_1-s_3)}}\frac{d\sigma}{\sqrt{\Sigma}}$$

$$= fK[h(K-H) - \operatorname{zn} hK] - hK[f(K-H) - \operatorname{zn} fK]$$

$$= hK \operatorname{zn} fK - fK \operatorname{zn} hK,$$

employing (10), § 3, and thus connecting E and F; for $h = 1$, $s = s_1$, makes

$$(3)\qquad E = K \operatorname{zn} fK,$$

and $f = 1$, $\sigma = s_2$, makes

$$(4)\qquad F = K \operatorname{zn} hK + K\sqrt{\frac{s - s_1 \cdot s - s_2}{s - s_3 \cdot s_1 - s_3}} = K \operatorname{zs} hK.$$

13. With s_1 and s_3 conjugate imaginary, and putting

$$(1)\qquad s_1 - s_2 \cdot s_3 - s_2 = M^2,$$

$$(2)\qquad K = \int_{s_2}^\infty \frac{\sqrt{M}ds}{\sqrt{S}}, \qquad hK = \int_{s_2}^s \frac{\sqrt{M}ds}{\sqrt{S}},$$

$$(3)\qquad K' = \int_{-\infty}^{s_2} \frac{\sqrt{M}d\sigma}{\sqrt{(-\Sigma)}}, \qquad fK' = \int_\sigma^{s_2} \frac{\sqrt{M}d\sigma}{\sqrt{(-\Sigma)}},$$

and then

$$(4)\qquad s - s_2 = M\frac{1 - \operatorname{cn} 2hK}{1 + \operatorname{cn} 2hK}, \qquad \sqrt{S} = M^{\frac{3}{2}}\frac{4 \operatorname{sn} 2hK \operatorname{dn} 2hK}{(1 + \operatorname{cn} 2hK)^2},$$

$$(5)\qquad s_2-\sigma=M\frac{1-\operatorname{cn}2fK}{1+\operatorname{cn}2fK},\qquad \sqrt{(-\Sigma)}=M^{\frac{3}{2}}\frac{4\operatorname{sn}2fK'\operatorname{dn}2fK'}{(1+\operatorname{cn}2fK')^2},$$

in which hK and fK' may be replaced occasionally by h and f in the sequel without confusion.

Employing a formula (Legendre I, p. 257),

$$(6)\qquad \int\frac{1-\cos\phi}{1+\cos\phi}\frac{d\phi}{\Delta\phi}=\frac{2\sin\phi\Delta\phi}{1+\cos\phi}+F\phi-2E\phi,$$

verified immediately by differentiation, and integrating the lemma (1) § 9,

$$\begin{aligned}\int_\sigma^{s_2}\frac{\frac{1}{2}\sqrt{S}}{s-\sigma}\frac{d\sigma}{\sqrt{(-\Sigma)}}+\int_{s_2}^{s}\frac{\frac{1}{2}(\sqrt{-\Sigma})}{s-\sigma}\frac{ds}{\sqrt{S}}&=\int_\sigma^{s_2}\int_{s_2}^{s}(s-\sigma)\frac{ds}{\sqrt{S}}\frac{d\sigma}{\sqrt{(-\Sigma)}}\\
&=fK'\int_{s_2}^{s}\frac{s-s_2}{M}\cdot\frac{\sqrt{M}ds}{\sqrt{S}}+hK\int_\sigma^{s_2}\frac{s_2-\sigma}{M}\frac{\sqrt{M}d\sigma}{\sqrt{(-\Sigma)}}\\
(7)\qquad &=fK'\int_0^h\frac{1-\operatorname{cn}2h}{1+\operatorname{cn}2h}Kdh+hK\int_0^f\frac{1-\operatorname{cn}2f}{1+\operatorname{cn}2f}K'df\\
&=fK'\left(\frac{\operatorname{sn}2h\operatorname{dn}2h}{1+\operatorname{cn}2h}+\tfrac{1}{2}hK-hH-\operatorname{zn}2h\right)\\
&\quad+hK\left(\frac{\operatorname{sn}2f\operatorname{dn}2f}{1+\operatorname{cn}2f}+\tfrac{1}{2}fK'-fH'-\operatorname{zn}2f\right).\end{aligned}$$

But when we wish to obtain the complete III. E. I., by putting $s=\infty$, $h=1$, we make both sides infinite.

The infinite part must then be cut out by deducting from both sides

$$(8)\qquad \int_\sigma^{s_2}\frac{\frac{1}{2}\sqrt{S}}{s-s_2}\frac{d\sigma}{\sqrt{(-\Sigma)}}=2fK'\frac{\operatorname{dn}2h}{\operatorname{sn}2h},$$

and now

$$\begin{aligned}(9)\qquad &\int_\sigma^{s}\left(\frac{\frac{1}{2}\sqrt{S}}{s-\sigma}-\frac{\frac{1}{2}\sqrt{S}}{s-s_2}\right)\frac{d\sigma}{\sqrt{(-\Sigma)}}+\int_{s_2}^{s}\frac{\frac{1}{2}\sqrt{(-\Sigma)}}{s-\sigma}\frac{ds}{\sqrt{S}}\\
&=fK'\left(\frac{-\operatorname{sn}2h\operatorname{dn}2h}{1-\operatorname{cn}2h}+\tfrac{1}{2}hK-hH-\operatorname{zn}2h\right)\\
&\quad+hK\left(\frac{\operatorname{sn}2f\operatorname{dn}2f}{1+\operatorname{cn}2f}+\tfrac{1}{2}fK'-fH'-\operatorname{zn}2f\right).\end{aligned}$$

Here, putting $s=\infty$, $h=1$, the first integral vanishes, and

$$\begin{aligned}(10)\qquad \int_{s_2}^{\infty}\frac{\frac{1}{2}\sqrt{(-\Sigma)}}{s-\sigma}\frac{ds}{\sqrt{S}}&=fK'(\tfrac{1}{2}K-H)\\
&\quad+K\left(\frac{\operatorname{sn}2f\operatorname{dn}2f}{1+\operatorname{cn}2f}+\tfrac{1}{2}fK'-fH'-\operatorname{zn}2f\right)\end{aligned}$$

$$= K\left(\frac{\operatorname{sn} 2f \operatorname{dn} 2f}{1 + \operatorname{cn} 2f} - \operatorname{zn} 2f\right) - \tfrac{1}{2}\pi f$$

$$= K\left(\frac{\operatorname{dn} 2f}{\operatorname{sn} 2f} - \operatorname{zs} 2f\right) - \tfrac{1}{2}\pi f.$$

Similarly with $f = 1$,

$$\int_{-\infty}^{s_2} \frac{\frac{1}{2}\sqrt{S}}{s - \sigma} \frac{d\sigma}{\sqrt{(-\Sigma)}} = K'\left(\frac{\operatorname{dn} 2h}{\operatorname{sn} 2h} - \operatorname{zs} 2h\right) - \tfrac{1}{2}\pi h. \tag{11}$$

14. To compare B with LEGENDRE's standard form of the III. E. I.,

$$\int_0^x \frac{1}{1 + n \sin^2 \chi} \frac{d\chi}{\Delta\chi}, \qquad \Delta\chi = \sqrt{(1 - k^2 \sin^2 \chi)}, \tag{1}$$

denoted by $\Pi\,(n, k, \chi)$ or Π, when complete, put

$$s - s_3 = (s_2 - s_3) \sin^2 \chi, \qquad s_2 - s = (s_2 - s_3) \cos^2 \chi, \qquad s_1 - s = (s_1 - s_3)\,\Delta^2 \chi, \tag{2}$$

$$k^2 = \frac{s_2 - s_3}{s_1 - s_3}, \qquad k'^2 = \frac{s_1 - s_2}{s_1 - s_3};$$

and then

$$B = \sqrt{\frac{s_1 - \sigma \cdot \sigma - s_2}{s_1 - s_3 \cdot \sigma - s_3}} \int \frac{1}{1 + n \sin^2 \chi} \frac{d\chi}{\Delta\chi} = \sqrt{\alpha}\,\Pi\,(n, k, \chi), \tag{3}$$

in which, with LEGENDRE's notation,

$$n = -\frac{s_2 - s_3}{\sigma - s_3}, \qquad 1 + n = \frac{\sigma - s_2}{\sigma - s_3}, \qquad 1 + \frac{k^2}{n} = \frac{s_1 - \sigma}{s_1 - s_3}, \tag{4}$$

$$\alpha = (1 + n)\left(1 + \frac{k^2}{n}\right) = \frac{s_1 - \sigma \cdot \sigma - s_2}{s_1 - s_3 \cdot \sigma - s_3}, \tag{5}$$

and LEGENDRE's Π requires the normalising factor $\sqrt{\alpha}$.

In the comparison of A, put

$$s - s_3 = \frac{s_1 - s_3}{\sin^2 \chi}, \qquad s - s_2 = (s_1 - s_3)\frac{\Delta^2 \chi}{\sin^2 \chi}, \qquad s - s_1 = (s_1 - s_3)\frac{\cos^2 \chi}{\sin^2 \chi}, \tag{6}$$

and then

$$A = \sqrt{\frac{s_1 - \sigma \cdot \sigma - s_2}{s_1 - s_3 \cdot \sigma - s_3}} \int \frac{-n' \sin^2 \chi}{1 + n' \sin^2 \chi} \frac{d\chi}{\Delta\chi} = \sqrt{\alpha}[\,\Pi\,(n', k, \chi) - F\chi\,] \tag{7}$$

in which

$$n' = -\frac{\sigma - s_3}{s_1 - s_3}, \qquad 1 + n' = \frac{s_1 - \sigma}{s_1 - s_3}, \qquad 1 + \frac{k^2}{n'} = \frac{\sigma - s_2}{\sigma - s_3}, \tag{8}$$

so that

$$(1 + n')\left(1 + \frac{k^2}{n'}\right) = \frac{s_1 - \sigma \cdot \sigma - s_2}{s_1 - s_3 \cdot \sigma - s_3} = \alpha, \tag{9}$$

the same as before; also

$$(10) \qquad nn' = k^2, \qquad \Pi(n) + \Pi(n') = K + \frac{\frac{1}{2}\pi}{\sqrt{\alpha}}.$$

But in all the subsequent calculations it is best to keep clear of the standard forms of LEGENDRE, JACOBI, or WEIERSTRASS, and to work with the III. E. I. as the integral of an algebraical function of the standard form in (8) § 1.

For example we find, putting the differential element,

$$(11) \qquad \frac{ds}{\sqrt{S}} = du,$$

$$(12) \qquad \frac{d^2y}{du^2} = \sqrt{S}\frac{d}{ds}\left(\sqrt{S}\frac{dy}{ds}\right) = S\frac{d^2y}{ds^2} + \tfrac{1}{2}\frac{dS}{ds}\frac{dy}{ds};$$

and if we take

$$(13) \qquad \log y = \tfrac{1}{2}\int \frac{\sqrt{S} - \sqrt{\Sigma}}{s - \sigma}\frac{ds}{\sqrt{S}},$$

we obtain the Lamé differential equation of the second order

$$(14) \qquad \frac{1}{y}\frac{d^2y}{du^2} = 2s + \sigma - s_1 - s_2 - s_3,$$

$$(15) \qquad S\frac{d^2y}{ds^2} + \tfrac{1}{2}\frac{dS}{ds}\frac{dy}{ds} - (2s + \sigma - s_1 - s_2 - s_3)y = 0.$$

Then the function

$$(16) \qquad z = \frac{d}{du}(ye^{\lambda u})$$

leads to the Lamé differential equation of the second order

$$(17) \qquad \frac{1}{z}\frac{d^2z}{du^2} = 6s - 3\sigma - s_1 - s_2 - s_3 + 3\lambda^2,$$

provided that

$$(18) \qquad \lambda^3 - \lambda(3\sigma - s_1 - s_2 - s_3) - \sqrt{\Sigma} = 0,$$

the spherical pendulum relation.

Landen's Transformation.

15. It will be convenient at this stage to collect together all the formulas of Landen's transformation of the second order, required in the sequel, adopting the notation employed previously for the elliptic argument, hK denoting an argument to modulus k and fK' to modulus k', h and f denoting real fractions, as thereby the modulus is indicated in the notation and need not be written down.

Without this it will be difficult to reconcile the notation of different writers on this subject, each adopting a method of his own irrespective of others.

Following MAXWELL in his *Electricity and Magnetism* (E. and M.) § 702, we select the LANDEN transformation connecting the new modulus c and c' with k and k' by the relation

$$(1)\qquad c' = \frac{1-k}{1+k}, \qquad c = \frac{2\sqrt{k}}{1+k}, \qquad k = \frac{1-c'}{1+c'},$$

$$k' = \frac{2\sqrt{c'}}{1+c'}, k'c = 2\sqrt{(kc')}, \qquad c > k, \qquad c' < k',$$

all different forms of the modular equation of the second order, connecting the modular angles γ and β shown in figures A and B, such that

$$(2)\qquad \begin{aligned} \cos\gamma = c' &= \frac{1-\sin\beta}{1+\sin\beta} = \tan^2(\tfrac{1}{4}\pi - \tfrac{1}{2}\beta) = \frac{BC}{AC}, \\ k = \sin\beta &= \frac{1-\cos\gamma}{1-\cos\gamma} = \tan^2\tfrac{1}{2}\gamma = \frac{AC}{AC'} = \frac{OC}{OA}. \end{aligned}$$

Then the complete elliptic integral or quarter-period $F(\frac{1}{2}\pi, c)$ or $F(c)$ is denoted by F, and $F(c')$ by F', K and K' denoting $F(k)$ and $F(k')$; and

$$(3)\qquad F = (1+k)K, \qquad K = \tfrac{1}{2}(1+c')F,$$

$$(4)\qquad F' = \tfrac{1}{2}(1+k)K', \qquad K' = (1+c')F',$$

$$(5)\qquad \frac{F'}{F} = \tfrac{1}{2}\frac{K'}{K}, \qquad \frac{K'}{K} = 2\frac{F'}{F}.$$

16. The geometrical interpretation of LANDEN's transformation is seen in fig. B, where we may put

$$(1)\qquad \omega = \operatorname{am} hF, \qquad \omega' = \operatorname{am}(1-h)F, \qquad \chi = \operatorname{am} 2hK,$$

$$(2)\qquad \theta = 2\omega = \chi + \chi', \qquad 2\omega' = \pi - \chi + \chi',$$

so that

$$(3)\qquad \chi = \tfrac{1}{2}\pi + \omega - \omega', \qquad \operatorname{am} 2hK = \tfrac{1}{2}\pi + \operatorname{am} hF - \operatorname{am}(1-h)F,$$

$$(4)\qquad \begin{aligned} \operatorname{sn} 2hK &= \cos[\operatorname{am} hF - \operatorname{am}(1-h)F] \\ &= \operatorname{cn} hF \operatorname{cn}(1-h)F + \operatorname{sn} hF \operatorname{sn}(1-h)F = (1+c')\frac{\operatorname{sn} hF \operatorname{cn} hF}{\operatorname{dn} hK}, \end{aligned}$$

LANDEN's first transformation; and putting $h = 0$ leads to (3) § 15.

Thence also

$$(5)\qquad \begin{aligned} \operatorname{cn} 2hK &= \frac{1-(1+c')\operatorname{sn}^2 hF}{\operatorname{dn} hF} = \frac{\operatorname{dn}^2 hF - c'}{(1-c')\operatorname{dn} hF} \\ &= \frac{\operatorname{dn} hF - \operatorname{dn}(1-h)F}{1-c'} = \frac{CQ - CQ'}{CA - CB} = \cos\chi, \end{aligned}$$

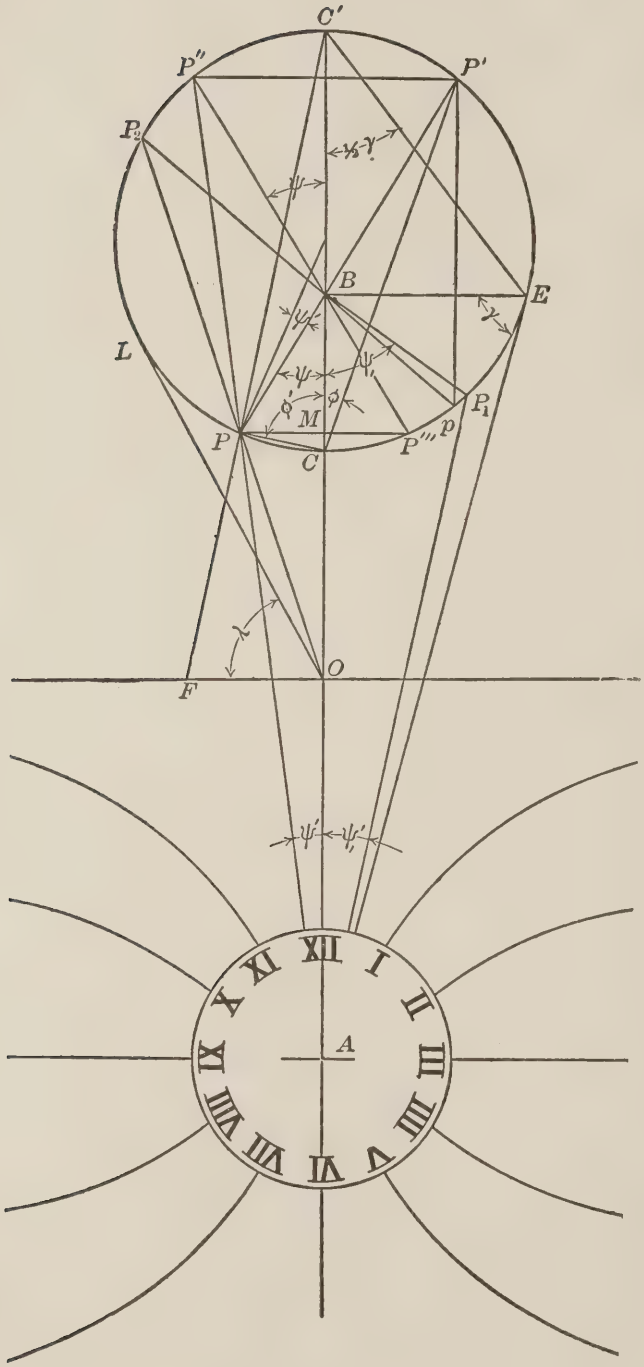

FIG. *A*.

the geometrical interpretation, since CQ, CQ', $CB = CA \operatorname{dn}(f, 1-f, 1)F$;

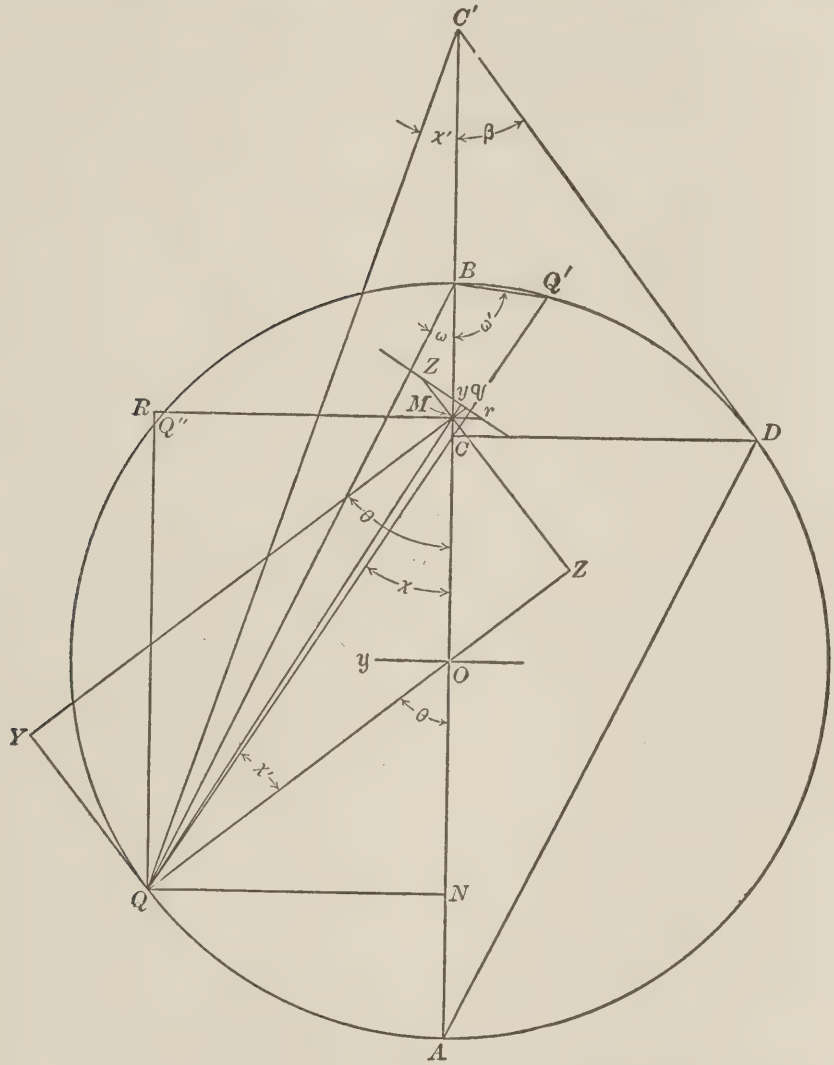

FIG. B.

$$\text{(6)} \qquad \operatorname{dn} 2hK = \frac{1-(1-c')\operatorname{sn}^2 hF}{\operatorname{dn} hF} = \frac{\operatorname{dn}^2 hF + c'}{(1+c')\operatorname{dn} hF} = \frac{QQ'}{AB}.$$

Equation (4) can be written

$$\text{(7)} \qquad \operatorname{sn}^2 2hK = \frac{1+c'}{1-c'}\,\frac{1-\operatorname{dn} 2hF}{1+\operatorname{dn} 2hF},$$

and thence

$$\text{(8)} \qquad \operatorname{dn} 2hF = \frac{1-k\operatorname{sn}^2 2hK}{1+k\operatorname{sn}^2 2hK},$$

which is LANDEN's II transformation, in conjunction with

$$\text{(9)} \qquad \operatorname{sn} 2hF = \frac{(1+k)\operatorname{sn} 2hK}{1+k\operatorname{sn}^2 2hK},$$

$$\text{(10)} \qquad \operatorname{cn} 2hF = \frac{\operatorname{cn} 2hK \operatorname{dn} 2hK}{1+k\operatorname{sn}^2 2hK}.$$

These may also be written, replacing $2h$ by h,

$$\text{(11)} \quad \operatorname{sn} hF = \tfrac{1}{2}\sqrt{(1+\operatorname{sn} 2hK\cdot 1+k\operatorname{sn} 2hK)} - \tfrac{1}{2}\sqrt{(1-\operatorname{sn} 2hK\cdot 1-k\operatorname{sn} 2hK)},$$

$$\text{(12)} \quad \operatorname{cn} hF = \tfrac{1}{2}\sqrt{(1+\operatorname{sn} 2hK\cdot 1-k\operatorname{sn} 2hK)} + \tfrac{1}{2}\sqrt{(1-\operatorname{sn} 2hK\cdot 1+k\operatorname{sn} 2hK)},$$

$$\text{(13)} \quad \begin{aligned} \operatorname{dn} hF &= \frac{\operatorname{dn} 2hK + k\operatorname{cn} 2hK}{1+k} = \frac{1-k}{\operatorname{dn} 2hK - k\operatorname{cn} 2hK} \\ &= \sqrt{\frac{1-k}{1+k}}\sqrt{\frac{\operatorname{dn} 2hK + k\operatorname{cn} 2hK}{\operatorname{dn} 2hK - k\operatorname{cn} 2hK}}. \end{aligned}$$

17. The associated transformation of the complementary period K' or F' can now be written down and interpreted geometrically in fig. A where

$$\text{(1)} \quad \psi = \operatorname{am} 2fF',\ \phi = \operatorname{am} fK',\ \phi' = \operatorname{am}(1-f)K',\ 2\phi = \psi+\psi',\ \pi - 2\phi' = \psi - \psi',$$

and as in LANDEN's II transformation in (11) § 16,

$$\sin\phi = \sin\tfrac{1}{2}(\psi+\psi') = \tfrac{1}{2}\sqrt{(1+\sin\psi\cdot 1+\sin\psi')} - \tfrac{1}{2}\sqrt{(1-\sin\psi\cdot 1-\sin\psi')},$$

$$\text{(2)} \quad \begin{aligned} \operatorname{sn} fK' &= \tfrac{1}{2}\sqrt{(1+\operatorname{sn} 2fF'\cdot 1+c'\operatorname{sn} 2fF')} - \tfrac{1}{2}\sqrt{(1-\operatorname{sn} 2fF'\cdot 1-c'\operatorname{sn} 2fF')} \\ &= \frac{(1+c')\operatorname{sn} fF'}{1+c'\operatorname{sn}^2 fF'}, \end{aligned}$$

as in (9) § 16 so that $f=0$ leads to (4) § 15;

$$\text{(3)} \quad \begin{aligned} \operatorname{cn} fK' &= \tfrac{1}{2}\sqrt{(1+\operatorname{sn} 2fF'\cdot 1-c'\operatorname{sn} 2fF')} + \tfrac{1}{2}\sqrt{(1-\operatorname{sn} 2fF'\cdot 1+c'\operatorname{sn} 2fF')} \\ &= \frac{\operatorname{cn} fF' \operatorname{dn} fF'}{1+c'\operatorname{sn}^2 fF'}, \end{aligned}$$

$$\text{(4)} \quad \begin{aligned} \operatorname{dn} fK' &= \frac{1-c'\operatorname{sn}^2 fF'}{1+c'\operatorname{sn}^2 fF'} = \frac{\operatorname{dn} 2fF' + c'\operatorname{cn} 2fF'}{1+c'}. \\ &= \frac{1-c'}{\operatorname{dn} 2fF' - c'\operatorname{cn} 2fF'} = \sqrt{\frac{1-c'}{1+c'}}\sqrt{\frac{\operatorname{dn} 2fF' + c'\operatorname{cn} 2fF'}{\operatorname{dn} 2fF' - c'\operatorname{cn} 2fF'}}, \end{aligned}$$

and, conversely,

$$\text{(5)} \qquad \operatorname{sn} 2fF' = (1+k)\frac{\operatorname{sn} fK' \operatorname{cn} fK'}{\operatorname{dn} fK'} = \sqrt{\frac{1+k}{1-k}}\sqrt{\frac{1-\operatorname{dn} 2fK'}{1+\operatorname{dn} 2fK'}},$$

derivable from (2, 3, 4), and the geometrical interpretation of

$$(6)\quad \psi = \tfrac{1}{2}\pi + \phi - \phi', \qquad \sin\psi = \cos(\phi - \phi'), \qquad \cos\psi = \sin(\phi' - \phi);$$

$$(7)\quad \begin{aligned} \operatorname{cn} 2fF' &= \frac{1-(1+k)\operatorname{sn}^2 fK'}{\operatorname{dn} fK'} = \frac{\operatorname{dn}^2 fK' - k}{(1-k)\operatorname{dn} fK'} \\ &= \frac{\operatorname{dn} fK' - \operatorname{dn}(1-f)K'}{1-k} = \frac{BP' - BP}{BC' - BC} = \cos\psi, \end{aligned}$$

$$(8)\quad \begin{aligned} \operatorname{dn} 2fF' &= \frac{1-(1-k)\operatorname{sn}^2 fK'}{\operatorname{dn} fK'} = \frac{\operatorname{dn}^2 fK' + k}{(1+k)\operatorname{dn} fK'} \\ &= \frac{\operatorname{dn} fK' + \operatorname{dn}(1-f)K'}{1+k} = \frac{BP' + BP}{BC' + BC} = \frac{PP'}{CC'} = \cos\psi', \end{aligned}$$

the geometrical interpretation.

The two circles in figures A and B, linked in perpendicular planes, may be taken to typify the magnetic and electric circuit, associated in a similar manner.

18. In LEGENDRE's notation, the II. E. I.

$$(1)\qquad E(\omega, c) = \int_0^\omega \sqrt{(1 - c^2 \sin^2\omega)}\, d\omega = \int \Delta\omega\, d\omega,$$

and the complete II. E. I., or $E(\frac{1}{2}\pi, c)$ or $E(c)$ is denoted by E, and $E(i)$ by E', while as before $E(\frac{1}{2}\pi, k)$ is denoted by H, and $E(k')$ by H'.

In JACOBI's notation, with $\omega = \operatorname{am} hF$, and as employed already in § 3,

$$(2)\qquad \int_0^h \operatorname{dn}^2 hF \cdot F dh = E \operatorname{am} hF = hE + \operatorname{zn} hF.$$

Squaring equation (13) § 16,

$$(3)\qquad \operatorname{dn}^2 hF = \frac{2\operatorname{dn}^2 2hK + 2k \operatorname{cn} 2hK \operatorname{dn} 2hK - k'^2}{(1+k)^2},$$

and then integrating with respect to h, there results

$$(4)\qquad \frac{hE + \operatorname{zn} hF}{F} = \frac{2hH + \operatorname{zn} 2hK + k \operatorname{sn} 2hK - k'^2 hK}{(1+k)^2 K},$$

$$(5)\qquad hE + \operatorname{zn} hF = \frac{2hH - k'^2 hK + \operatorname{zn} 2hK + k \operatorname{sn} 2hK}{1+k},$$

so that, putting $h = 1$,

$$(6)\qquad E = \frac{2H - k'^2 K}{1+k} = \frac{2H}{1+k} - (1-k)K,$$

(7) $$F-E=2\left(K-\frac{H}{1+k}\right),\qquad E-c'^2F=2\frac{H-(1-k)K}{1+k},$$

and conversely

(8) $$H=\frac{E+c'F}{1+c'},\qquad HK'=EF'+c'FF',$$

so that, as in Maxwell E. and M. § 701 when corrected,

(9) $$\frac{2}{c}(F-E)-cF=2\frac{K-H}{\sqrt{k}},$$

(10) $$2(F-E)-c^2F=4\frac{K-H}{1+k},$$

(11) $$(1+c'^2)E-2c'^2F=4\frac{(1+k^2)H-(1-k^2)K}{(1+k)^3}.$$

19. Then

(1) $$\operatorname{zn} hF=\frac{\operatorname{zn} 2hK+k\operatorname{sn} 2hK}{1+k},$$

which it is convenient to write

(2) $$F\operatorname{zn} hF=K\operatorname{zn} 2hK+Kk\operatorname{sn} 2hK,$$

and conversely

(3) $$\begin{aligned}K\operatorname{zn} 2hK&=F\operatorname{zn} hF-\tfrac{1}{2}Fc^2\frac{\operatorname{sn} hF\operatorname{cn} hF}{\operatorname{dn} hF}\\&=\tfrac{1}{2}F\operatorname{zn} hF-\tfrac{1}{2}F\operatorname{zn}(1-h)F\\&=\tfrac{1}{2}F\operatorname{zn} 2hF-\tfrac{1}{2}Fc^2\frac{\operatorname{sn} 2hF\operatorname{cn} hF}{1+\operatorname{dn} 2hF}.\end{aligned}$$

Squaring (4) § 17 and integrating with respect to f, we obtain the remaining relations

(4) $$H'=\frac{2E'}{1+c'}-(1-c')F',$$

(5) $$E'=\frac{H'+kK'}{1+k},\qquad E'F=H'K+kKK',$$

(6) $$\operatorname{zn} fK'=\frac{\operatorname{zn} 2fF'+c'\operatorname{sn} 2fF'}{1+c'},$$

which we write

(7) $$K\operatorname{zn} fK'=\tfrac{1}{2}F\operatorname{zn} 2fF'+\tfrac{1}{2}Fc'\operatorname{sn} 2fF',$$

(8) $$K\operatorname{zn}(1-f)K'=-\tfrac{1}{2}F\operatorname{zn} 2fF'+\tfrac{1}{2}Fc'\operatorname{sn} 2fF',$$

and conversely

$$F\operatorname{zn} 2fF'=2K\operatorname{zn} fK'-Kk'^2\frac{\operatorname{sn} fK'\operatorname{cn} fK'}{\operatorname{dn} fK'}$$

$$
\begin{aligned}
&= K\,\mathrm{zn}\,fK' - K\,\mathrm{zn}\,(1-f)K' \\
(9)\qquad &= K\,\mathrm{zn}\,2fK' - Kk'^2\frac{\mathrm{sn}\,2fK'\,\mathrm{cn}\,2fK'}{1+\mathrm{dn}\,2fK'} \\
&= K\,\mathrm{zs}\,2fK' - K\frac{\mathrm{cn}\,2fF'}{\mathrm{sn}\,2fK'},
\end{aligned}
$$

$$
(10)\qquad F\,2\mathrm{n}\,(1-2f)F' = K\,\mathrm{zs}\,(1-2f)K' - Kk\,\mathrm{tn}\,2fK',
$$

theorems required in the sequel for the quadric transformation of M, the mutual induction of a helix and a coaxial ring.

20. Integrating (3) § 19 with respect to h,

$$
(1)\qquad \log\frac{\Theta 2hK}{\Theta 0K} = 2\log\frac{\Theta hF}{\Theta 0F} + \log\mathrm{dn}\,hF,
$$

$$
(2)\qquad \frac{\Theta 2hK}{\Theta 0K} = \left(\frac{\Theta hF}{\Theta 0F}\right)^2\mathrm{dn}\,hF = \frac{\Theta hF}{\Theta 0F}\,\frac{\Theta(1-h)F}{\Theta F},
$$

the quadric transformation of the theta function.

Similarly by integration of (9), § 19, with respect to f,

$$
(3)\qquad \frac{\Theta 2fF'}{\Theta 0F'} = \left(\frac{\Theta fK'}{\Theta 0K'}\right)^2\mathrm{dn}\,fK' = \frac{\Theta fK'}{\Theta 0K'}\,\frac{\Theta(1-f)K'}{\Theta K'}.
$$

The same notation is useful for expressing the change in a theta function and eta function from imaginary to real argument and comodulus in the form

$$
(4)\qquad \frac{\Theta(2fK'i,k)}{\Theta(0,k)} = q^{-f^2}\frac{H(1-2f)K'}{HK'},\qquad \frac{H(2fK'i,k)}{H(K,k)} = iq^{-f^2}\frac{H2fK'}{HK'},
$$

where $q = \exp(-\pi K'/K)$. (Jacobi, *Fundamenta nova*, or *Werke*, I, p. 215; Cayley, *Elliptic Functions*, p. 151.)

These transformations may be used for the bisection of the elliptic function;

$$
\begin{aligned}
(5)\qquad \left(\frac{\Theta hF}{\Theta 0F}\right)^2 &= \frac{\Theta 2hK}{\Theta 0K}\,\frac{\mathrm{dn}\,2hK - k\,\mathrm{cn}\,2hK}{1-k} \\
&= \frac{1}{1-k}\,\frac{\Theta(1-2h)K}{\Theta 0K} - \frac{k}{1-k}\,\frac{H(1-2h)K}{HK},
\end{aligned}
$$

$$
(6)\qquad \left(\frac{\Theta hF}{\Theta F}\right)^2 = \frac{1}{1+k}\,\frac{\Theta(1-2h)K}{\Theta 0K} - \frac{k}{1+k}\,\frac{H(1-2h)K}{HK}.
$$

Algebraical form of Landen's transformations.

21. The algebraical equivalent of our first quadric transformation in § 16 is (TANNERY and MOLK, *Fonctions elliptiques*, formulas XXII, XXIV)

$$(1)\qquad m^2(s-s_1)=\frac{[t-t_1-\sqrt{(t_1-t_2\cdot t_1-t_3)}]^2}{4(t-t_1)},$$

$$(2)\qquad m^2(s-s_2)=\frac{[t-t_1+\sqrt{(t_1-t_3\cdot t_2-t_3)}]^2}{4(t-t_1}),$$

$$(3)\qquad m^2(s-s_3)=\frac{t-t_2\cdot t-t_3}{4(t-t_1)}=\frac{T}{16(t-t_1)^2},$$

so that the graph of a relation (s, t) is a hyperbola; and

$$(4)\qquad 2m\sqrt{(s_1-s_3)}=\sqrt{(t_1-t_3)}-\sqrt{(t_1-t_2)},$$

$$(5)\qquad 2m\sqrt{(s_2-s_3)}=\sqrt{(t_1-t_3)}-\sqrt{(t_1-t_2)},$$

the equivalent of the modular equation (1) § 15, with k, k' in terms of s_1, s_2, s_3, as in (4) § 2, and c, c' the equivalent for t_1, t_2, t_3. Also

$$(6)\qquad m^2\frac{ds}{dt}=\frac{(t-t_1)^2-(t_1-t_2\cdot t_1-t_3)}{4(t-t_1)^2},$$

$$(7)\qquad m^3\sqrt{S}=\frac{(t-t_1)^2-(t_1-t_2\cdot t_1-t_3)}{8(t-t_1)^2}\sqrt{T},$$

where

$$(8)\qquad S=4(s-s_1\cdot s-s_2\cdot s-s_3),\qquad T=4(t-t_1\cdot t-t_2\cdot t-t_3),$$

$$(9)\qquad \frac{ds}{m\sqrt{S}}=2\frac{dt}{\sqrt{T}}.$$

This algebraical result is obtained by substitution in (4), (5), (6), § 15, of

$$(10)\qquad \sin^2\chi=\operatorname{sn}^2 2hK=\frac{s_1-s_3}{s-s_3},\ \text{or}\ \frac{s-s_3}{s_2-s_3},\ \infty>s>s_1,\ \text{or}\ s_2>s>s_3,$$

$$(11)\qquad \sin^2\omega=\operatorname{sn}^2 hF'=\frac{t_1-t_3}{t-t_3},\ \text{or}\ \frac{t-t_3}{t_2-t_3},\ \infty>t>t_1,\ \text{or}\ t_2>t>t_3.$$

In the associated complementary transformation of § 18, writing σ and Σ for s and S, τ and T' for t and T, the region $s_1>\sigma>s_2$ is excluded, because $\sigma-s_1$ and $\sigma-s_2$ must have the same sign; but

$$(12)\qquad \sin^2\phi=\operatorname{sn}^2 fK'=\frac{s_1-s_3}{s_1-\sigma},\qquad s_3>\sigma>-\infty.$$

$$(13)\qquad \operatorname{sn}^2 fF' = \frac{t_1 - \tau}{t_1 - t_2}, \text{ or } \frac{t_1 - t_3}{t_1 - \tau}, \; t_1 > \tau > t_2, \text{ or } t_3 > \tau > -\infty,$$

and these substituted in (2), (3), § 17, will satisfy the algebraical relation above.

On applying this algebraical quadric transformation to our III. E. I.,

$$(14)\qquad m^3 \sqrt{(-\Sigma)} = \frac{(\tau - t_1)^2 - (t_1 - t_2 \cdot t_1 - t_3)}{8(\tau - t_1)^2} \sqrt{(-T')},$$

$$(15)\qquad m^2(s - \sigma) = \frac{(t - \tau)\left[(t - t_1)(\tau - t_1) - (t_1 - t_2 \cdot t_1 - t_3)\right]}{4(t - t_1)(\tau - t_1)},$$

$$(16)\qquad \frac{2m\sqrt{(-\Sigma)}}{s - \sigma} = \frac{\sqrt{(-T')}}{t - \tau} - \frac{(t_1 - t_2 \cdot t_1 - t_3)\sqrt{(-T')}}{(\tau - t_1)\left[(t - t_1 \cdot \tau - t_1) - (t_1 - t_2 \cdot t_1 - t_3)\right]}$$

$$= \frac{\sqrt{(-T')}}{t - \tau} + \frac{\sqrt{(-T')}}{u - \tau} + \frac{\sqrt{(-T')}}{\tau - t_1},$$

and on putting

$$(17)\qquad t - t_1 = \frac{t_1 - t_2 \cdot t_1 - t_3}{u - t_1}, \qquad \frac{dt}{\sqrt{T}} = -\frac{du}{\sqrt{U}},$$

$$(18)\qquad (t - t_1 \cdot \tau - t_1) - (t_1 - t_2 \cdot t_1 - t_3) = (t_1 - t_2 \cdot t_1 - t_3)\frac{\tau - u}{u - t_1},$$

so that

$$(19)\qquad \frac{\sqrt{(-\Sigma)}}{s - \sigma}\frac{ds}{\sqrt{S}} = \frac{\sqrt{(-T')}}{t - \tau}\frac{dt}{\sqrt{T}} - \frac{\sqrt{(-T')}}{u - \tau}\frac{du}{\sqrt{U}} - \frac{\sqrt{(-T')}}{t_1 - \tau}\frac{dt}{\sqrt{T}}.$$

The hyperbolic graph connecting s and t being drawn it shows that as t increases from t_1 to ∞, u diminishes from ∞ to t_1 and s diminishes from ∞ to s_1 and rises again to infinity so that

$$(20)\qquad \int_{s_1}^{\infty} \frac{\sqrt{(-\Sigma)}}{s - \sigma}\frac{ds}{\sqrt{S}} = -\int_{t_1}^{\infty} \frac{\sqrt{(-T')}}{t - \tau}\frac{dt}{\sqrt{T}} + \sqrt{\left(\frac{\tau - t_1 \cdot \tau - t_3}{t_1 - \tau \cdot t_1 - t_3}\right)}\int \frac{\sqrt{(t_1 - t_3)}\,dt}{\sqrt{T}},$$

$$(21)\qquad 2C(fK') = -2A(fF') + \frac{\operatorname{cn} fF' \operatorname{dn} fF'}{\operatorname{sn} fF'} F = -A(fF') + C(fF').$$

But as t increases from t_3 to t_2, u diminishes from t_2 to t_3, and s increases from s_3 to s_2 and diminishes again to s_3, so that

$$(22)\qquad \int_{s_3}^{s_2} \frac{\sqrt{(-\Sigma)}}{s - \sigma}\frac{ds}{\sqrt{S}} = \int_{t_3}^{t_2} \frac{\sqrt{(-T')}}{t - \tau}\frac{dt}{\sqrt{T}} + \sqrt{\left(\frac{\tau - t_2 \cdot \tau - t_3}{t_1 - \tau \cdot t_1 - t_3}\right)}\int \frac{\sqrt{(t_1 - t_3)}\,dt}{\sqrt{T}},$$

$$(23)\qquad 2D(fK') = 2B(fF') + \frac{\operatorname{cn} fF' \operatorname{dn} fF'}{\operatorname{sn} fF'} F = B(fF') + D(fF').$$

Making use of the expressions for A, B, C, D in § 5, § 6, equations (21), (23) are equivalent to the single relation

$$(24) \qquad 2K \operatorname{zs} fK' = F \operatorname{zs} fF' + F \operatorname{zn} fF',$$

as before in § 19.

22. For the transformation from t back to s in the III. E. I., we take

$$(1) \qquad \frac{t-t_1}{t_1-\tau} = \frac{\sqrt{(s-s_2)}+\sqrt{(s-s_1)}}{\sqrt{(s-s_2)}-\sqrt{(s-s_1)}} \cdot \frac{\sqrt{(s_1-\sigma)}-\sqrt{(s_2-\sigma)}}{\sqrt{(s_1-\sigma)}+\sqrt{(s_2-\sigma)}},$$

$$(2) \qquad \frac{t-\tau}{t_1-\tau} = 2\,\frac{\sqrt{(s-s_2)}\sqrt{(s_1-\sigma)}-\sqrt{(s-s_1)}\sqrt{(s_2-\sigma)}}{[\sqrt{(s-s_2)}-\sqrt{(s-s_1)}]\,[\sqrt{(s_1-\sigma)}+\sqrt{(s_2-\sigma)}]},$$

$$(3) \qquad \frac{t_1-\tau}{t-\tau} = \frac{[\sqrt{(s_1-\sigma)}+\sqrt{(s_2-\sigma)}]\,[\sqrt{(s_1-s_2)}\sqrt{(s-s_1)}] \times [s-s_2)\sqrt{(s-\sigma)}+\sqrt{(s-s_1)}\sqrt{(s_2-\sigma)}]}{2(s_1-s_2)(s-\sigma)}$$

$$= \tfrac{1}{2} + \frac{\sqrt{(s_1-\sigma\cdot s_2-\sigma)}-\sqrt{(s-s_1\cdot s-s_2)}}{2(s-\sigma)},$$

$$(4) \qquad \frac{\frac{1}{2}\sqrt{(-T')}}{t_1-\tau} = 2\sqrt{(s_3-\sigma)},$$

$$(5) \qquad \frac{\frac{1}{2}\sqrt{(-T')}}{t-\tau}\,\frac{dt}{\sqrt{T}} = \left[\tfrac{1}{2} + \frac{\sqrt{(s_1-\sigma\cdot s_2-\sigma)}-\sqrt{(s-s_1\cdot s-s_2)}}{2(s-\sigma)}\right]\frac{\sqrt{(s_3-\sigma)}\,ds}{\sqrt{S}},$$

$$(6) \qquad \int_{t_1,\,t_3}^{\infty,\,t_2} \frac{\frac{1}{2}\sqrt{(-T')}}{t-\tau}\,\frac{dt}{\sqrt{T}}$$

$$= \sqrt{\frac{s_3-\sigma}{s_1-s_3}} \int_{s_1,\,s_3}^{\infty,\,s_2} \frac{\sqrt{(s_1-s_3)}\,ds}{\sqrt{S}} + \int \frac{\frac{1}{2}\sqrt{(-\Sigma)}}{s-\sigma}\,\frac{ds}{\sqrt{S}} - \tfrac{1}{2}\int \frac{\sqrt{(s_3-\sigma)}\,ds}{(s-\sigma)\sqrt{(s-s_3)}},$$

the last integral being zero as s ranges forward and back again.

According to the region of τ,

$$(7) \qquad t_1 > \tau > t_2, \qquad \text{or} \qquad t_2 > \tau > -\infty,$$

and with

$$s_3 > \sigma > -\infty,$$

the relation (6) is equivalent to

$$(8) \qquad A(fF') = K\frac{\operatorname{cn} fK'}{\operatorname{sn} fK'} - C(fK'),$$

$$(9) \qquad -B(fF') = K\frac{\operatorname{cn} fK'}{\operatorname{sn} fK'} - D(fK'),$$

$$(10) \qquad C(fF') = K\frac{\operatorname{cn} fK'}{\operatorname{sn} fK'} + C(fK'),$$

$$D(fF') = K\frac{\operatorname{cn} fK'}{\operatorname{sn} fK'} + D(fK'), \tag{11}$$

and these reduce to the single relation

$$F \operatorname{zn} fF' = K \operatorname{zs} fK' - K\frac{\operatorname{cn} fK'}{\operatorname{sn} fK'}. \tag{12}$$

23. The algebraical equivalent of the second quadric transformation in (8), (9), (10), § 16, is in a similar way equivalent to a hyperbolic graph between x and s,

$$M^2(x - t_1) = \frac{s - s_1 \cdot s - s_2}{s - s_3} = \frac{\frac{1}{4}S}{(s - s_3)^2}, \tag{1}$$

$$M^2(x - t_2) = \frac{[s - s_3 - \sqrt{(s_1 - s_3 \cdot s_2 - s_3)}]}{s - s_3}, \tag{2}$$

$$M^2(x - t_3) = \frac{[s - s_3 + \sqrt{(s_1 - s_3 \cdot s_2 - s_3)}]^2}{s - s_3}, \tag{3}$$

and as in the modular equation (1) § 15,

$$M\sqrt{(t_1 - t_3)} = \sqrt{(s_1 - s_3)} + \sqrt{(s_2 - s_3)}, \tag{4}$$

$$M\sqrt{(t_1 - t_2)} = \sqrt{(s_1 - s_3)} - \sqrt{(s_2 - s_3)}. \tag{5}$$

Also $Mm = 1$, and

$$M^2\frac{dx}{ds} = \frac{(s - s_3)^2 - (s_1 - s_3 \cdot s_2 - s_3)}{(s - s_3)^2}, \tag{6}$$

$$M^3\sqrt{X} = \frac{(s - s_3)^2 - (s_1 - s_3 \cdot s_2 - s_3)}{(s - s_3)^2}\sqrt{S}, \tag{7}$$

$$\frac{dx}{M\sqrt{X}} = \frac{ds}{\sqrt{S}}, \qquad X = 4(x - t_1 \cdot x - t_2 \cdot x - t_3). \tag{8}$$

The region $t_2 > x > t_3$ is now excluded, because $x - t_2$ and $x - t_3$ must have the same sign; but

$$\sin^2\xi = \operatorname{sn}^2 2hF = \frac{t_1 - t_3}{x - t_3}, \qquad \infty > x > t_1, \tag{9}$$

$$\sin^2\chi = \operatorname{sn}^2 2hK = \frac{s_1 - s_3}{s - s_3} \text{ or } \frac{s - s_3}{s_2 - s_3}, \qquad \infty > s > s_1 \text{ or } s_2 > s > s_3; \tag{10}$$

and these substituted in (8), (9), (10) § 16 will give the algebraical result above.

In the associated complementary transformation with σ, Σ for s, S and x', X' for x, X,

$$(11)\qquad \sin^2\phi = \operatorname{sn}^2 fK' = \frac{s_1-\sigma}{s_1-s_2} \text{ or } \frac{s_1-s_3}{s_1-\sigma},\ s_1>\sigma>s_2,\ \text{or } s_3>\sigma>-\infty,$$

$$(12)\qquad \sin^2\psi = \operatorname{sn}^2 2fK' = \frac{t_1-x'}{t_1-t_2} \text{ or } \frac{t_1-t_3}{t_1-x'},\ t_1>x'>t_2,\ \text{or } t_3>x'>-\infty,$$

and the algebraical relation is satisfied when these are substituted in (5) § 17.

In the transformation of the III. E. I.,

$$(13)\qquad M^2(x-x') = \frac{(s-\sigma)[(s-s_3\cdot\sigma-s_3)-(s_1-s_3\cdot s_2-s_3)]}{(s-s_3)(\sigma-s_3)},$$

$$(14)\qquad M^3\sqrt{(-X')} = \frac{(\sigma-s_3)^2-(s_1-s_3\cdot s_2-s_3)}{(\sigma-s_3)^2}\sqrt{(-\Sigma)},$$

$$(15)\qquad \begin{aligned}\frac{M\sqrt{(-X')}}{x-x'} &= \frac{(s-s_3)[(\sigma-s_3)^2-(s_1-s_3\cdot s_2-s_3)]\sqrt{(-\Sigma')}}{(\sigma-s_3)[(s-s_3\cdot\sigma-s_3)-(s_1-s_3\cdot s_2-s_3)](s-\sigma)}\\ &= \frac{\sqrt{(-\Sigma)}}{s-\sigma}-\frac{\sqrt{(-\Sigma'')}}{s-\sigma''},\end{aligned}$$

where

$$(16)\qquad \sigma''-s_3=\frac{s_1-s_3\cdot s_2-s_3}{\sigma-s_3},\qquad \sigma''-s_2=\frac{s_2-s_3\cdot s_1-\sigma}{\sigma-s_3},\qquad \sigma''-s_1=\frac{s_1-s_3\cdot s_2-\sigma}{\sigma-s_3}.$$

In the arrangement

$$(17)\qquad s_1>\sigma>s_2,\qquad s_1>\sigma''>s_2,$$

$$(18)\qquad \operatorname{sn}^2 f''K' = \frac{s_1-\sigma''}{s_1-s_2} = \frac{s_1-s_3\,\sigma-s_2}{s_1-s_2\,\sigma-s_3} = \frac{\operatorname{cn}^2 fK'}{\operatorname{dn}^2 fK'},$$

and in the arrangement

$$(19)\qquad s_3>\sigma>-\infty,\qquad s_3>\sigma''>-\infty,$$

$$(20)\qquad \operatorname{sn}^2 f''K' = \frac{s_1-s_3}{s_1-\sigma''} = \frac{s_3-\sigma}{s_2-\sigma} = \frac{\operatorname{cn}^2 fK'}{\operatorname{dn}^2 fK'},$$

so that

$$(21)\qquad f''+f=1.$$

The integral relation

$$(22)\qquad \int\frac{\frac{1}{2}\sqrt{(-X')}}{x-x'}\frac{dx}{\sqrt{X}} = \int\frac{\frac{1}{2}\sqrt{(-\Sigma)}}{s-\sigma}\frac{ds}{\sqrt{S}} - \int\frac{\frac{1}{2}\sqrt{(-\Sigma'')}}{s-\sigma''}\frac{ds}{\sqrt{S}},$$

is now equivalent to

$$(23)\qquad A(2fF') = A(fK')-A(1-f)K',\ \text{or } = -B(fK')+B(1-f)K',$$

implying the relation

$$(24)\qquad F\,\mathrm{zn}\,2fF' = K\,\mathrm{zn}\,fK' - K\,\mathrm{zn}\,(1-f)K',$$

as in (9) § 19; or

$$(25)\qquad C(2fF') = C(fK') - C(1-f)K', \text{ or } = D(fK') - D(1-f)K',$$

implying the equivalent to (24), when (5), (7), (8), § 17 are employed:

$$(26)\qquad F\,\mathrm{zs}\,2fF' = K\,\mathrm{zs}\,fK' - K\,\mathrm{zs}\,(1-f)K'.$$

Thus in transformation I, hF and fF' become changed to $2hK$ and fK', and in transformation II, $2hK$ and fK' are changed into $2hF$ and $2fF'$, a successive application being equivalent to a duplication of hF and fF'.

24. To proceed from s to x by means of the II transformation, take

$$(1)\qquad \frac{s-s_3}{\sigma-s_3} = \frac{\sqrt{(x-t_3)}+\sqrt{(x-t_2)}}{\sqrt{(x-t_3)}-\sqrt{(x-t_2)}}\cdot\frac{\sqrt{(x'-t_3)}-\sqrt{(x'-t_2)}}{\sqrt{(x'-t_3)}+\sqrt{(x'-t_2)}},$$

$$(2)\qquad \frac{s-\sigma}{\sigma-s_3} = 2\,\frac{\sqrt{(x-t_2\cdot x'-t_3)}-\sqrt{(x-t_3\cdot x'-t_2)}}{[\sqrt{(x-t_3)}-\sqrt{(x-t_2)}]\,[\sqrt{(x'-t_3)}+\sqrt{(x'-t_2)}]},$$

$$(3)\qquad \frac{\sigma-s_3}{s-\sigma} = \frac{\sqrt{(x'-t_2\cdot x'-t_3)}+\sqrt{(x-t_2\cdot x-t_3)}}{2(x-x')} - \tfrac{1}{2},$$

$$(4)\qquad \frac{\frac{1}{2}\sqrt{(-\Sigma)}}{\sigma-s_3} = N\sqrt{(t_1-x')},$$

$$(5)\qquad \frac{\sqrt{(-\Sigma)}}{s-\sigma}\,\frac{ds}{\sqrt{S}} = \left[\frac{\sqrt{(x'-t_2\cdot x'-t_3)}+\sqrt{(x-t_2\cdot x-t_3)}}{x-x'} - 1\right]\frac{\sqrt{(t_1-x')}\,dx}{\sqrt{X}},$$

$$(6)\qquad \int_{s_1,\,s_2}^{\infty,\,s_3}\frac{\sqrt{(-\Sigma)}}{s-\sigma}\,\frac{ds}{\sqrt{S}} = \int_{t_1}^{\infty}\frac{\frac{1}{2}\sqrt{(-X')}}{x-x'}\,\frac{dx}{\sqrt{X}}$$

$$+\tfrac{1}{2}\int_{t_1}^{\infty}\frac{\sqrt{(t_1-x')}\,dx}{(x-x')\sqrt{(x-t_1)}} - \sqrt{\frac{t_1-x'}{t_1-t_3}}\int_{t_1}^{\infty}\frac{\sqrt{(t_1-t_3)}\,dx}{\sqrt{X}},$$

of which the second integral is $\frac{1}{2}\pi$, and the third is $Fc'\,\mathrm{sn}\,2fF'$ or $F\,\mathrm{sn}\,2fF'$, according as

$$t_1 > x > t_2 \text{ or } t_3 > x > -\infty;$$

so that in the region $s_1 > \sigma > s_2$, $t_1 > x > t_2$,

$$(7)\qquad 2A(fK') = A(2fF') + \tfrac{1}{2}\pi - Fc'\,\mathrm{sn}\,2fF',$$

$$(8)\qquad 2B(fK') = A(2fF') - \tfrac{1}{2}\pi - Fc'\,\mathrm{sn}\,2fF',$$

equivalent to

$$(9)\qquad 2K\,\mathrm{zn}\,fK' = F\,\mathrm{zn}\,2fF' + Fc'\,\mathrm{sn}\,2fF',$$

as in (7) § 19; and in the region $s_3 > \sigma > -\infty$, $t_3 > x' > -\infty$,

$$2C(fK') = C(2fF') - \tfrac{1}{2}\pi - \frac{F'}{\operatorname{sn} 2fF'}, \tag{10}$$

$$2D(fK') = C(2fF') + \tfrac{1}{2}\pi - \frac{F'}{\operatorname{sn} 2fF'}, \tag{11}$$

equivalent, as in (7) § 19, to

$$2K \operatorname{zs} fK' = F \operatorname{zs} 2fF' - F \operatorname{ns} 2fF'. \tag{12}$$

25. Another form of LANDEN's transformation may be added here; put

$$k = \operatorname{th} \alpha,\ k' = \operatorname{sech} \alpha,\ \text{then}\ c' = e^{-2\alpha}, \tag{1}$$

$$c' = \operatorname{th} \delta,\ c = \operatorname{sech} \delta,\ \text{then}\ k = e^{-2\delta}, \tag{2}$$

and then, as a form of the modular equation of the second order,

$$(e^{2\alpha} - 1)(e^{2\delta} - 1) = 2, \qquad \operatorname{sh} 2\alpha \operatorname{sh} 2\delta = 1. \tag{3}$$

Now with $\omega = \operatorname{am} hF$, put $\Delta\omega = \operatorname{th} \zeta$,

$$\begin{gathered} \sin \omega = \frac{\operatorname{ch} \delta}{\operatorname{ch} \zeta}, \qquad \cos \omega = \frac{\sqrt{(\operatorname{ch}^2 \zeta - \operatorname{ch}^2 \delta)}}{\operatorname{ch} \zeta}, \qquad \operatorname{cn}(1-h)F = \frac{\operatorname{sh} \delta}{\operatorname{sh} \zeta}, \\ Fdh = \frac{d\omega}{\Delta\omega} = -\frac{\operatorname{ch} \delta d\zeta}{\sqrt{(\operatorname{ch}^2 \zeta - \operatorname{ch}^2 \delta)}}, \end{gathered} \tag{4}$$

and from (7) § 16,

$$k \operatorname{sn}^2 hK = e^{-2\zeta}, \qquad \operatorname{sn} hK = e^{-\zeta+\delta}. \tag{5}$$

In the plane xOz of Fig. A, v and w of § 24 are elliptic coördinates, to employ on WEIR's azimuth diagram; while ϕ and k or α are dipolar coördinates, suitable for a stereographic projection or chart, with poles at A and B, in which 2ϕ is the longitude of P, $2\phi'$ of P' both on latitude λ, where

$$\sin \lambda = \operatorname{th} 2\alpha, \qquad \cos \lambda = \operatorname{sech} 2\alpha, \qquad \tan \lambda = \operatorname{sh} 2\alpha, \qquad \tan \tfrac{1}{2}\lambda = \operatorname{th} \alpha = k. \tag{6}$$

We can put, for the conformal representation of the stereographic projection,

$$z + xi = a \tan(\phi + \alpha i), \tag{7}$$

$$\frac{x}{a} = \frac{\operatorname{sh} 2\alpha}{\operatorname{ch} 2\alpha + \cos 2\phi}, \qquad \frac{z}{a} = \frac{\sin 2\phi}{\operatorname{ch} 2\alpha + \cos 2\phi}, \tag{8}$$

$$\frac{PA, PB}{a} = \frac{r_1, r_2}{a} = \frac{e^{\pm\alpha}}{\sqrt{[\tfrac{1}{2}(\operatorname{ch} 2\alpha + \cos 2\phi)]}}, \tag{9}$$

and if OPP_2 is a straight line

$$\phi + \phi_2 = \tfrac{1}{2}\pi. \tag{10}$$

But if APP'' is a straight line

$$\psi + \psi'' = \pi, \tag{11}$$

and ψ, ψ' are biangular coördinates of P, so that $\psi'' = \operatorname{am} 2(1-f)F'$, with the value of ψ, ϕ, ϕ' in (1) § 17; also

$$BP = \frac{a(1-k)}{\operatorname{dn} fK'}, \qquad AP = \frac{a(1+k)}{\operatorname{dn} fK'}, \tag{12}$$

$$BP' = BP'' = \frac{a(1-k)}{\operatorname{dn}(1-f)K'}, \qquad AP' = AP'' = \frac{a(1+k)}{\operatorname{dn}(1-f)K'}, \tag{13}$$

$$OC \cdot OC' = a^2 = OB^2, \qquad PB \cdot BP' = OB^2 \frac{(1-k)^2}{k}, \tag{14}$$

$$OC' = \frac{a}{k}, \qquad OC = ak, \qquad C'B = a\left(\frac{1}{k} - 1\right), \qquad BC = a(1-k). \tag{15}$$

Coefficient of mutual induction of two coaxial circles.

26. Apply the preceding method to MAXWELL's § 701 (E. and M.). The mutual induction of a short element of wire at P perpendicular to the plane APB of fig. A and of a circular wire on a diameter AB in the perpendicular plane of fig. B, is, for the arc AQ where $AOQ = \theta$,

$$\int \frac{\cos\epsilon}{PQ} ds = \int_0^\theta \frac{a\cos(\pi - \theta)\, d\theta}{\sqrt{(a^2 + 2aA\cos\theta + A^2 + b^2)}}, \tag{1}$$

an integral composed of incomplete I and II. E. I.'s: and this must be integrated round the circle AQB to obtain the vector potential G at P perpendicular to the plane APB; and then multiplied by $2\pi A$ to obtain M for the two circles in parallel planes.

According to our method of leaving the algebraical expression intact, we first substitute

$$PQ^2 = a^2 + 2aA\cos\theta + A^2 + b^2 = m^2(t_1 - t), \tag{2}$$

and supposing $t = t_3$ and t_2 at A and B, where $\theta = 0$ and π,

$$PA^2 = r_1^2 = (a+A)^2 + b^2 = m^2(t_1 - t_3), \qquad PB^2 = r_2^2 = (a-A)^2 + b^2 = m^2(t_1 - t_2), \tag{3}$$

$$\begin{aligned} PQ^2 - r_2^2 &= 2aA(1 + \cos\theta) = m^2(t_2 - t), \\ r_1^2 - PQ^2 &= 2aA(1 - \cos\theta) = m^2(t - t_3), \end{aligned} \tag{4}$$

so that

$$4aA\cos\theta = m^2(t_2 + t_3 - 2t), \qquad r_1^2 - r_2^2 = 4aA = m^2(t_2 - t_3), \tag{5}$$

$$\frac{r_2}{r_1} = \sqrt{\frac{t_1 - t_2}{t_1 - t_3}} = c', \qquad \frac{2\sqrt{(aA)}}{r_1} = \sqrt{\frac{t_2 - t_3}{t_1 - t_3}} = c; \tag{6}$$

$$(7)\quad \theta = 2\omega = 2\tan^{-1}\sqrt{\frac{t-t_3}{t_2-t}},\ d\theta = \frac{dt}{\sqrt{(t_2-t\cdot t-t_3)}},\ \frac{d\theta}{PQ} = \frac{2}{r_1}\frac{\sqrt{(t_1-t_3)}dt}{\sqrt{T}},$$

$$(8)\quad \begin{aligned} M &= \int_0^{2\pi}\frac{2\pi aA\cos(\pi-\theta)\,d\theta}{PQ} = 2\pi r_1\int_{t_3}^{t_2}\frac{2t-t_2-t_3}{\sqrt{(t_1-t_3)}}\frac{dt}{\sqrt{T}} \\ &= 4\pi r_1\int\frac{t-t_3}{\sqrt{(t_1-t_3)}}\frac{dt}{\sqrt{T}} - 2\pi r_1\frac{t_2-t_3}{t_1-t_3}\int\frac{\sqrt{(t_1-t_3)}\,dt}{\sqrt{T}}. \end{aligned}$$

and then by (10) § 3 and (1), (4) § 2,

$$(9)\qquad M = 2\pi r_1[2(F-E)-c^2F],$$

as in MAXWELL, with the original sign changed; and by the quadric transformation employing (10) § 18

$$(10)\quad M = 8\pi r_1\frac{K-H}{1+k} = 4\pi(r_1+r_2)(K-H) = 8\pi\sqrt{(aA)}\frac{K-H}{\sqrt{k}},$$

doubling MAXWELL's original result, and to be divided by 10^9 to bring it to henries.

MAXWELL's modular angle γ is now seen to be AEB in fig. A where AE is the tangent from A to the circle on the diameter CC', where PC, PC' are the bisectors of the angle APB; and then BE is at right angles to AB, so that A, B being limiting points of this circle

$$(11)\qquad \cos\gamma = \frac{EB}{EA} = \frac{PB}{PA} = \frac{r_2}{r_1} = c'.$$

Putting $k = \sin\beta$, so that β is the modular angle of the period K and K',

$$(12)\qquad \sin\beta = \frac{r_1-r_2}{r_1+r_2} = \frac{AC-CB}{AC+CB} = \frac{OC}{OA} = \frac{AC}{AC'},$$

so that β is the angle $OC'D$ in fig. B where CD is perpendicular to AB, and $C'D$ the tangent at D.

27. A geometrical interpretation can be given in LANDEN's manner of these transformations in the plane AQB of the circular disc or wire in fig. B, as well as in the perpendicular plane APB of fig. A, in which PC, PC', the bisectors of the angle APB, are axes of the elliptic cone whose vertex is P and circular base AQB.

The point F in fig. A where PC meets the axis of the circle AQB is the center of a sphere of which C and P are limiting points, so that PQ/QC is a constant ratio.

Now, ω denoting the angle ABQ in fig. B,

$$(1)\qquad PQ^2 = r_1^2\cos^2\omega + r_2^2\sin^2\omega,$$

and at D, where PDC' is the tangent plane of the cone,

$$(2)\qquad \cos 2\omega = -\frac{OC}{OB} = -\frac{r_1 - r_2}{r_1 + r_2}, \qquad \cos^2\omega = \frac{r_2}{r_1 + r_2}, \qquad \sin^2\omega = \frac{r_1}{r_1 + r_2},$$

$$(3)\qquad ABD = \operatorname{am} \tfrac{1}{2}F, \qquad PD^2 = r_1 r_2, \qquad CD^2 = AC \cdot CB = \frac{4r_1 r_2 a^2}{(r_1 + r_2)^2},$$

so that, round the circle AQB,

$$(4)\qquad \frac{PQ}{QC} = \frac{PD}{DC} = \frac{r_1 + r_2}{2a},$$

and so also

$$(5)\qquad \frac{PQ}{QC'} = \frac{r_1 - r_2}{2a}, \qquad \frac{QC}{QC'} = \frac{r_1 - r_2}{r_1 + r_2} = k.$$

On fig. B the angles QCA, $QC'A$ or CQO are denoted by χ, χ', and QBA, $Q'BA$ by ω, ω'; and we put, as in § 16,

$$(6)\qquad \chi = \operatorname{am} 2hK, \qquad \omega = \operatorname{am} hF, \qquad \omega' = \operatorname{am}(1-h)F,$$

$$(7)\qquad PQ = r_1 \operatorname{dn} hF, \qquad CQ = \frac{2ar_1}{r_1 + r_2} \operatorname{dn} hF = CA \operatorname{dn} hF,$$

and then, since $OQC = \chi'$, $OQC' = \chi$, $\sin\chi' = k \sin\chi$, $\cos\chi' = \operatorname{dn} 2hK$,

$$(8)\qquad CQ,\ CQ' = a\cos\chi' \pm ak\cos\chi = a(\operatorname{dn} 2hK \pm k \operatorname{cn} 2hK),$$

$$(9)\qquad PQ,\ PQ' = \tfrac{1}{2}(r + r_2)(\cos\chi' \pm k\cos\chi), \qquad PQ \cdot PQ' = r_1 r_2,$$

$$(10)\qquad QQ' = 2a\cos\chi' = 2a \operatorname{dn} 2hK,$$

$$(11)\qquad \sin\chi = \frac{a\sin\theta}{CQ} = (r_1 + r_2)\frac{\sin\omega\cos\omega}{PQ},$$

$$(12)\qquad (r_1 + r_2)^2 \operatorname{cec}^2\chi = r_1^2 \operatorname{cec}^2\omega + r_2^2 \sec^2\omega,$$

$$(13)\qquad (r_1 + r_2)\frac{\cos\chi}{\sin\chi} = r_1 \cot\omega - r_2 \tan\omega,$$

$$(14)\qquad (r_1 + r_2)\frac{\cos\chi'}{\sin\chi} = r_1 \cot\omega + r_2 \tan\omega,$$

$$(15)\qquad \sin^2\omega = \tfrac{1}{2}(1 - \cos\theta) = \tfrac{1}{2}(1 + \sin\chi\sin\chi' - \cos\chi\cos\chi'),$$

$$(16)\qquad \sin\omega = \tfrac{1}{2}\sqrt{(1 + \sin\chi \cdot 1 + \sin\chi')} - \tfrac{1}{2}\sqrt{(1 - \sin\chi \cdot 1 - \sin\chi')},$$

etc., the geometrical equivalent of the LANDEN transformation in § 16.

28. In the transformation of M to MAXWELL's second form in § 702, we notice that, from elementary geometrical considerations,

$$(1)\quad \frac{ad\theta}{CQ}=\frac{d\chi}{\cos\chi'},\quad \frac{ad\theta}{PQ}=\frac{2a}{r_1+r_2}\frac{d\chi}{\cos\chi'},\qquad \sin\chi'=k\sin\chi,\qquad \cos\chi'=\Delta\chi,$$

$$M=\int_0^{2\pi}\frac{-2\pi aA\cos\theta d\theta}{PQ}=\frac{-8\pi aA}{r_1+r_2}\int_0^{\pi}\cos(\chi+\chi')\frac{d\chi}{\cos\chi'}$$

$$(2)\qquad =\frac{8\pi aA}{r_1+r_2}\int\left(\frac{k\sin^2\chi}{\cos\chi'}-\cos\chi\right)d\chi$$

$$=\frac{8\pi aA}{r_1-r_2}\int_0^{\pi}\left(\frac{1-\Delta^2\chi}{\Delta\chi}-k\cos\chi\right)d\chi=4\pi(r_1+r_2)(K-H).$$

Hence another method in place of MAXWELL's § 702 for drawing the lines of magnetic force $M=$ constant, employing WEIR's Azimuth Diagram covered by confocal conics for which $r_1\pm r_2$ is constant.

Denoting WEIR's hour angle by α and latitude angle by λ, then

$$(3)\quad \sin\alpha=\frac{r_1-r_2}{2a},\quad \cos\lambda=\frac{2a}{r_1+r_2},\quad k=\sin\alpha\cos\lambda,\quad \frac{M}{8\pi a}=\frac{K-H}{\cos\lambda}.$$

Supposing the curved line CPC' in fig. A to represent for a moment a line of magnetic force of constant M, starting from C where $\lambda=0$, and orthogonal to the lines of constant Ω, such as those shown radiating from A,

$$(4)\qquad k=\sin\alpha,\qquad \frac{M}{8\pi a}=K-H.$$

At C' where $\alpha=90°$,

$$(5)\qquad k=\cos\lambda,\qquad \frac{M}{8\pi a}=\frac{K-H}{k}.$$

To find an intermediate point P on CPC' assume an arbitrary value of k and $K-H$, less than the value at C, and calculate λ and α from the relation

$$(6)\qquad \cos\lambda=(K-H)\div\frac{M}{8\pi a},$$

$$(7)\qquad \sin\alpha=\frac{k}{\cos\lambda}=\frac{M}{8\pi a}\div\frac{K-H}{k}.$$

We thus require a table of $K-H$ and $(K-H)/k$, say for every five degrees of the modular angle $\theta=\sin^{-1}k$; the work of redrawing MAXWELL's figure XVIII of the lines of magnetic force for a circular current is being carried out on this method.

The meridians and parallels of latitude of a stereographic projection on fig. A, with poles at A and B, will map out the electromagnetic field of a straight current through A and the return through B perpendicular to the plane, by means of the longitude for magnetic potential, and then $M = 2\alpha = \log(r_1/r_2)$.

29. Putting $\cos\theta = c$, the algebraical form of the quadric substitution is

$$(1) \qquad y \text{ or } y - y_3 = m^2(s - s_3) = \frac{a^2 \sin^2\theta}{PQ^2} = \sin^2 QPN,$$

of which the graph is a hyperbola in the coördinates (c, y);

$$(2) \qquad y_1 - y = m^2(s_1 - s) = \left[\frac{a(c - c_1)}{PQ}\right]^2,$$

$$(3) \qquad y_2 - y = m^2(s_2 - s) = \left[\frac{a(c - c_2)}{PQ}\right]^2,$$

because

$$(4) \qquad a^2c^2 + 2aAyc + (a^2 + A^2 + b^2)y - a^2 = 0,$$

$$(5) \qquad (ac + Ay)^2 = A^2(y_1 - y)(y_2 - y),$$

$$(6) \qquad y_1 = \left(\frac{r_1 + r_2}{2A}\right)^2 = \left(\frac{2a}{r_1 - r_2}\right)^2, \qquad c_1 = -\frac{Ay_1}{a} = -\frac{r_1 + r_2}{r_1 - r_2} = -\sqrt{\frac{y_1}{y_2}},$$

$$(7) \qquad y_2 = \left(\frac{r_1 - r_2}{2A}\right)^2 = \left(\frac{2a}{r_1 + r_2}\right)^2, \qquad c_2 = -\frac{Ay_2}{a} = -\frac{r_1 - r_2}{r_1 + r_2} = -\sqrt{\frac{y_2}{y_1}},$$

$$(8) \qquad y_1y_2 = \left(\frac{a}{A}\right)^2, \qquad c_1c_2 = 1,$$

$$(9) \qquad c = \frac{-y + \sqrt{(y_1 - y \cdot y_2 - y)}}{\sqrt{(y_1y_2)}},$$

$$c - c_1 = \frac{\sqrt{(y_1 - y)}[\sqrt{(y_1 - y)} + \sqrt{(y_2 - y)}]}{\sqrt{(y_1y_2)}},$$

$$c - c_2 = \frac{\sqrt{(y_2 - y)}[\sqrt{(y_1 - y)} + \sqrt{(y_2 - y)}]}{\sqrt{(y_1y_2)}},$$

$$\frac{PQ}{a} = \frac{\sqrt{(y_1 - y)} + \sqrt{(y_2 - y)}}{\sqrt{(y_1y_2)}}, \qquad \frac{CQ}{a} = \frac{\sqrt{(y_1 - y)} + \sqrt{(y_2 - y)}}{\sqrt{y_1}},$$

$$\frac{PQ}{A} = \sqrt{(y_1 - y)} + \sqrt{(y_2 - y)},$$

$$\sin\theta = \frac{\sqrt{y}[\sqrt{(y_1 - y)} - \sqrt{(y_2 - y)}]}{\sqrt{(y_1y_2)}},$$

$$\tan\tfrac{1}{2}\theta = \frac{(\sqrt{y_1} + \sqrt{y_2})\sqrt{y}}{\sqrt{y_2}\sqrt{(y_1 - y)} + \sqrt{y_1}\sqrt{(y_2 - y)}}.$$

Taking logarithmic differentials of $\tan \frac{1}{2}\theta$ and $\sin\theta$,

$$(10)\qquad \frac{d\theta}{\sin\theta}\quad \frac{\sqrt{(y_1y_2)}\,dy}{2y\sqrt{(y_1-y\cdot y_2-y)}},\qquad \cos\theta\frac{d\theta}{\sin\theta}=\frac{dy}{2y}+\frac{dy}{2\sqrt{(y_1-y\cdot y_2-y)}},$$

and multiplying by $\sqrt{y}=a\sin\theta/PQ$,

$$(11)\qquad \frac{ad\theta}{PQ}=\frac{\sqrt{(y_1y_2)}\,dy}{\sqrt{Y}},\qquad \frac{a\cos\theta d\theta}{PQ}=\frac{dy}{2\sqrt{y}}+\frac{ydy}{\sqrt{Y}},$$

$$(12)\qquad Y=4y\cdot y_1-y\cdot y_2-y;$$

and thence the integrals in the algebraical form for determining MAXWELL's second expression for M in terms of K and H, noticing that y increases from zero to y_2 and back again to zero, as θ increases from 0 to π.

With the transformation of (4) § 25

$$\frac{M}{4\pi\sqrt{(aA)}}=\int^{\infty}\frac{2\,\mathrm{ch}^2\delta-\mathrm{ch}^2\zeta}{\mathrm{ch}^2\zeta}\frac{d\zeta}{\sqrt{(\mathrm{ch}^2\zeta-\mathrm{ch}^2\delta)}}$$

$$(13)\qquad =\sqrt{2}\int\frac{2\,\mathrm{ch}\,2\delta-\mathrm{ch}\,2\zeta+1}{\mathrm{ch}\,2\zeta+1}\frac{d\zeta}{\sqrt{(\mathrm{ch}\,2\zeta-\mathrm{ch}\,2\delta)}}$$

$$=\sqrt{2}\int^{\infty}\left[\frac{e^{-2\zeta}}{\sqrt{(\mathrm{ch}\,2\zeta-\mathrm{ch}\,2\delta)}}+\tfrac{1}{2}\frac{d}{d\zeta}\frac{\sqrt{(\mathrm{ch}\,2\zeta-\mathrm{ch}\,2\delta)}}{e^{2\zeta}+1}\right]d\zeta,$$

of which the second term vanishes.

Components of electromagnetic force of a circular current.

30. We can now express the components of M, and also of the magnetic potential Ω, in terms of E and F, or H and K, so as to be able to select the simplest form. Thus

$$\frac{dM}{db}=\int_0^{2\pi}\frac{2\pi aAb\cos\theta d\theta}{PQ^3}=-\frac{2\pi b}{m}\int_{t_3}^{t_2}\frac{2t-t_2-t_3}{t_1-t}\frac{dt}{\sqrt{T}}$$

$$(1)\qquad =-\frac{2\pi b}{r_1}\int\left(\frac{t_1-t_3+t_1-t_2}{t_1-t}-2\right)\frac{\sqrt{(t_1-t_3)}\,dt}{\sqrt{T}}$$

$$=-\frac{2\pi b}{r_1}\left(\frac{1+c'^2}{c'^2}E-2F\right)=-\frac{2\pi br_1}{r_2^2}\left[(1+c'^2)E-2c'^2F\right],$$

by a theorem analogous to those in § 3,

$$(2)\qquad \int_{t_3}^{t}\frac{t_1-t_2}{t_1-t}\frac{\sqrt{(t_1-t_3)}\,dt}{\sqrt{T}}=\int_0^h \mathrm{dn}^2(1-h)F\cdot Fdh=hE-\mathrm{zn}\,(1-h)F.$$

Or, with

$$\cos\theta = \cos(\chi+\chi') = \frac{(\cos\chi' + k\cos\chi)(\cos\chi - k\cos\chi')}{k'^2}, \tag{3}$$

$$PQ = \tfrac{1}{2}(r_1+r_2)(\cos\chi' + k\cos\chi), \qquad d\theta = (\cos\chi' + k\cos\chi)\frac{d\chi}{\cos\chi'},$$

$$\begin{aligned}
\frac{dM}{db} &= \frac{32\pi aAb}{(r_1+r_2)^3 k'^2}\int_0^\pi \frac{\cos\chi - k\cos\chi'}{\cos\chi' + k\cos\chi}\,\frac{d\chi}{\cos\chi'} \\
&= \frac{32\pi aAb}{(r_1+r_2)^3 k'^4}\int [(1+k^2)\cos\chi\cos\chi' - k(\cos^2\chi + \cos^2\chi')]\,\frac{d\chi}{\cos\chi'} \\
&= -\frac{32\pi aAb}{(r_1+r_2)^3 kk'^4}\int [(1+k^2)\cos^2\chi' - (1-k^2)]\,\frac{d\chi}{\cos\chi'} \\
&= -\frac{16\pi b}{r_1+r_2}\,\frac{(1+k^2)H - (1-k^2)K}{k'^4}.
\end{aligned} \tag{4}$$

Next

$$\begin{aligned}
A\frac{dM}{dA} &= M + \int_0^{2\pi}\frac{2\pi aA\cos\theta(A^2 + aA\cos\theta)\,d\theta}{PQ^3} \\
&= M + \frac{A^2}{b}\frac{dM}{db} + \int\frac{2\pi(aA\cos\theta)^2\,d\theta}{PQ^3},
\end{aligned} \tag{5}$$

and similarly

$$a\frac{dM}{da} = M + \frac{a^2}{b}\frac{dM}{db} + \int\frac{2\pi(aA\cos\theta)^2\,d\theta}{PQ^3}, \tag{6}$$

of which the last integral C is given by

$$\begin{aligned}
C &= \tfrac{1}{2}\pi m\int_{t_2}^{t_2}\frac{(t_2+t_3-2t)^2}{t_1-t}\,\frac{dt}{\sqrt{T}} \\
&= \tfrac{1}{2}\pi r_1\left[\frac{(2t_1-t_2-t_3)^2}{t_1-t_2\;\;t_1-t_3}\int\frac{t_1-t_2}{t_1-t}\,\frac{\sqrt{(t_1-t_3)}\,dt}{\sqrt{T}}\right. \\
&\quad \left. - 4\,\frac{2t_1-t_2-t_3}{t_1-t_3}\int\frac{\sqrt{(t_1-t_3)}\,dt}{\sqrt{T}} + 4\int\frac{t_1-t}{\sqrt{(t_1-t_3)}}\,\frac{dt}{\sqrt{T}}\right] \\
&= \tfrac{1}{2}\pi r_1\left[\frac{(1+c'^2)^2}{c'^2}E - 4(1+c'^2)F + 4E\right] = \tfrac{1}{2}\pi r_1\frac{c^4}{c'^2}E - M,
\end{aligned} \tag{7}$$

by preceding theorems on the I. and II. E. I., so that

$$A\frac{dM}{dA} = \tfrac{1}{2}\pi r_1\frac{c^4}{c'^2}E + \frac{A^2}{b}\frac{dM}{db}, \tag{8}$$

$$a\frac{dM}{da} = \tfrac{1}{2}\pi r_1 \frac{c^4}{c'^2} E + \frac{a^2}{b}\frac{dM}{db}, \tag{9}$$

and the relation

$$a\frac{dM}{da} + A\frac{dM}{dA} + b\frac{dM}{db} = M \tag{10}$$

is satisfied, required in consequence of M being a homogeneous function of a, A, b, of the first degree. Otherwise

$$\begin{aligned} C &= \frac{32\pi a^2 A^2}{(r_1+r_2)^3}\int_0^\pi \frac{(\cos\chi - k\cos\chi')^2}{k'^4}\frac{d\chi}{\cos\chi'} \\ &= \frac{32\pi a^2 A^2}{(r_1+r_2)^3}\int \frac{(1+k^4)\cos^2\chi' - (1-k^2) - 2k^3\cos\chi\cos\chi'}{k^2k'^4}\frac{d\chi}{\cos\chi'} \\ &= \frac{64\pi a^2 A^2}{(r_1+r_2)^3}\frac{(1+k^4)H - (1-k^2)K}{k^2k'^4} \\ &= 4\pi(r_1+r_2)\frac{(1+k^4)H-(1-k^2)K}{k'^4}, \end{aligned} \tag{11}$$

$$M + C = 4\pi(r_1+r_2)\frac{k^2}{k'^4}(2H - k'^2K). \tag{12}$$

31. If Ω denotes the magnetic potential or conical angle at P of the circular ring AQB, the magnetic components are (E. and M., § 703),

$$\frac{d\Omega}{dA} = \frac{1}{2\pi A}\frac{dM}{db} = -\frac{4ab}{r_1^3}\frac{(1+c'^2)E - 2c'^2F}{c^2c'^2} \tag{1}$$

or

$$-\frac{32ab}{(r_1+r_2)^3}\frac{(1+k^2)H-(1-k^2)K}{kk'^4},$$

$$\frac{d\Omega}{db} = -\frac{1}{2\pi A}\frac{dM}{dA} = -\frac{M+C}{2\pi A^2} - \frac{1}{2\pi b}\frac{dM}{db} = -\frac{4a^2}{r_1^3}\frac{E}{c'^2} + \frac{1}{r_1}\frac{(1+c'^2)E - 2c'^2F}{c'^2} \tag{2}$$

or

$$-\frac{32a^2}{(r_1+r_2)^3}\frac{2H-k'^2K}{k'^4} + \frac{3}{r_1+r_2}\frac{(1+k^2)H-(1-k^2)K}{k'^4}.$$

Also, from the homogeneity of Ω, of zero dimension,

$$a\frac{d\Omega}{da} = -A\frac{d\Omega}{dA} - b\frac{d\Omega}{db} = \frac{b}{2\pi A^2}(M+C) = \frac{4a^2b}{r_1^3}\frac{E}{c'^2} \tag{3}$$

or

$$\frac{32a^2b}{(r_1+r_2)^3}\frac{2H-k'^2K}{k'^4}.$$

In the neighborhood of the axis, where A; $r_1 - r_2$, c, and k are small,

$$(4) \qquad F = \tfrac{1}{2}\pi\left(1 + \frac{c^2}{4} + \frac{9c^4}{64}\cdots\right), \qquad E = \tfrac{1}{2}\pi\left(1 - \frac{c^2}{4} - \frac{3c^4}{64}\cdots\right),$$

$$(5) \qquad F - E = \tfrac{1}{2}\pi(\tfrac{1}{2}c^2 + \tfrac{3}{16}c^4\cdots), \qquad E - c'^2 F = \tfrac{1}{2}\pi(\tfrac{1}{2}c^2 + \tfrac{13}{16}c^4\cdots),$$

$$(6) \qquad (1 + c'^2)E - 2c'^2 F = \frac{3\pi c^4}{16},$$

$$(7) \qquad \frac{2H - k'^2 K}{k'^4} = \tfrac{1}{2}\pi\left(1 + \frac{9k^2}{4}\cdots\right),$$

$$(8) \qquad \frac{dM}{db} = -\frac{6\pi^2 a^2 A^2 b}{r_1^5}, \qquad \text{or} \qquad -\frac{192\pi^2 a^2 A^2 b}{(r_1 + r_2)^5},$$

$$(9) \qquad M + C = \tfrac{1}{2}\pi r_1 \frac{16a^2 A^2}{r_1^4}\tfrac{1}{2}\pi\left(1 + \frac{3c^2}{4}\cdots\right) = \frac{4\pi^2 a^2 A^2}{r_1^3}\left(1 + \frac{3aA}{r_1^2}\right)$$

or

$$= 4\pi(r_1 + r_2)\frac{16a^2 A^2}{(r_1 + r_2)^4}\tfrac{1}{2}\pi(1 + \tfrac{9}{4}k^2) = \frac{32\pi^2 a^2 A^2}{(r_1 + r_2)^3}\left[1 + \frac{36a^2 A^2}{(r_1 + r_2)^4}\right],$$

$$(10) \qquad \frac{d\Omega}{dA} = \frac{3\pi a^2 Ab}{r_1^5}, \qquad \text{or} \qquad \frac{96\pi a^2 Ab}{(r_1 + r_2)^5},$$

$$(11) \qquad \frac{d\Omega}{db} = \frac{2\pi a^2}{r_1^3} + \frac{6\pi a^3 A}{r_1^5}, \qquad \text{or} \qquad \frac{16\pi a^2}{(r_1 + r_2)^3}\left[1 + \frac{36a^2 A^2}{(r_1 + r_2)^4}\right] - \frac{96\pi a^2 A^2}{(r_1 + r_2)^5}$$

On the axis itself, $A = 0$,

$$(12) \qquad \frac{d\Omega}{dA} = 0, \qquad \frac{d\Omega}{db} = \frac{2\pi a^2}{r_1^3}.$$

A simpler resolution of the magnetic force is into components G_1 and G_2, perpendicular to PA and PB, and we find (A. Russell, Philosophical Magazine, April, 1907),

$$(13) \qquad G_1 = \frac{F - E}{A} = \frac{2}{A}\left(K - \frac{H}{1 + k}\right), \qquad G_2 = \frac{E - c'^2 F}{c'A} = \frac{2}{A}\left(\frac{H}{1 - k} - K\right),$$

and on the axis $A = 0$, $G_1 = G_2 = \pi a / r_1^2$, but G_1 and G_2 are infinite along the wire.

A similar resolution of the gravitational force of the potential $\int a d\theta / PQ$ of the circular ring AB will show that it can be resolved into the components $2G_1$ and $2G_2$, but now in the direction of PA and PB.

Galvanometer constant of a circular coil.

32. To calculate the galvanometer constant G at a point on the axis of a coil of wire of n turns in the form of a ring of given cross section α, we have to evaluate the integral

$$(1) \qquad G = \frac{n}{\alpha}\int\int \frac{2\pi x^2 dxdy}{(x^2+y^2)^{\frac{3}{2}}} = \frac{2\pi n}{\alpha}\int \frac{ydx}{(x^2+y)^{\frac{1}{2}}},$$

taken over the area or round the perimeter.

For a circular section AQB, with the axis of the ring through C' in fig. B,

$$(2) \qquad G = \frac{2\pi n}{\pi a^2}\int_0^{2\pi} \frac{a^2 \sin^2\theta d\theta}{C'Q} = \int_0^{2\pi} \frac{2n \sin^2\theta d\theta}{C'Q},$$

and putting as before in § 26, with $OC' = a/k$,

$$(3) \quad C'Q^2 = m^2(t_1 - t), \qquad C'Q^2 - C'B^2 = \frac{2a^2}{k}(1+\cos\theta) = m^2(t_2 - t),$$

$$C'A^2 - C'Q^2 = \frac{2a^2}{k}(1-\cos\theta) = m^2(t - t_3),$$

we have

$$(4) \quad m^2(t_1 - t_3) = C'A^2 = a^2\left(\frac{1}{k}+1\right)^2, \qquad m^2(t_1 - t_2) = C'B^2 = a^2\left(\frac{1}{k}-1\right),$$

$$(5) \qquad c' = \sqrt{\frac{t_1 - t_2}{t_1 - t_3}} = \frac{1-k}{1+k},$$

$$(6) \qquad \begin{aligned} G &= \frac{2nk^2m^3}{a^4}\int_{t_3}^{t_2}(t_2 - t\cdot t - t_3)\frac{dt}{\sqrt{T}} \\ &= \frac{2nk^2m^3}{3a^4}\int\left[-2\sqrt{T}\frac{d}{dt}\sqrt{T} + (t_1 - t_3 + t_1 - t_2)(t_1 - t) - 2(t_1 - t_2\cdot t_1 - t_3)\right]\frac{dt}{\sqrt{T}} \\ &= \frac{2nk^2m^3}{3a^4}\left[(t_1 - t_3 + t_1 - t_2)\sqrt{(t_1 - t_3)}E - 2(t_1 - t_2)\sqrt{(t_1 - t_3)}F\right] \\ &= \frac{32n}{3AC'}\frac{(1+c'^2)E - 2c'^2F}{c^4}. \end{aligned}$$

The reduction can also be carried out with the functions K and H, by means of the angles χ and χ', and

$$(7)\qquad C'Q = \frac{a}{k}\cos\chi' + a\cos\chi, \qquad \frac{ad\theta}{C'Q} = k\frac{d\chi}{\cos\chi'},$$

$$(8)\quad G = \frac{4nk}{a}\int_0^\pi \sin^2(\chi+\chi')\frac{d\chi}{\cos\chi'} = \frac{8n}{3OC'}\frac{(1+k^2)H-(1-k^2)K}{k^2},$$

and G is given by the same expression when C' is moved to any point on the axis of the tore, the modular angle β of K and H being half the angle between the tangents from C' to the circular section of the tore.

33. The form of the result shows that the integral for G and dM/db is the same essentially; for the comparison of these and other integrals it is convenient at this stage to have the following integrals to quote in the sequel:

$$(1)\qquad \int_0^{2\pi}\frac{ad\theta}{PQ} = \frac{4aF}{r_1} = \frac{8aK}{r_1+r_2},$$

the potential of the ring AB; and as in § 31,

$$(2)\quad \int\frac{2aA(1+\cos\theta)d\theta}{PQ^3} = \frac{4}{r_1}\int_{t_3}^{t_2}\frac{t_2-t}{t_1-t}\frac{\sqrt{(t_1-t_3)}dt}{\sqrt{T}} = \frac{4}{r_1}(F-E) = 4A\frac{G_1}{r_1}$$

$$(3)\quad \int\frac{2aA(1-\cos\theta)d\theta}{PQ^3} = \frac{4}{r_1}\int\frac{t-t_3}{t_1-t}\frac{\sqrt{(t_1-t_3)}dt}{\sqrt{T}} = \frac{4}{r_1}\frac{E-c'^2F}{c'^2} = 4A\frac{G_2}{r_2}.$$

Since

$$\frac{d}{d\theta}\frac{2aA\sin\theta}{PQ} = \frac{2aA\cos\theta}{PQ} + \frac{2a^2A^2\sin^2\theta}{PQ^3}$$

$$(4)\qquad = \frac{PQ^2-\frac{1}{2}(r_1^2+r_2^2)}{PQ} + \frac{(r_1^2-PQ^2)(PQ^2-r_2^2)}{2PQ^3}$$

$$= \tfrac{1}{2}PQ - \tfrac{1}{2}\frac{r_1^2r_2^2}{PQ^3},$$

$$(5)\qquad \int_0^{2\pi}PQd\theta = \int\frac{r_1^2r_2^2}{PQ^3}d\theta = 4r_1E = 2(r_1+r_2)(2H-k'^2K),$$

$$(6)\qquad \int\frac{\cos\theta d\theta}{PQ} = \frac{4}{r_1}\frac{2E-(1+c'^2)F}{c^2} = -\frac{8}{r_1+r_2}\frac{K-H}{k}.$$

Also

$$(7)\qquad \begin{aligned}\frac{d}{d\theta}(2aA\sin\theta PQ) &= \left(PQ^2-\frac{r_1^2+r_2^2}{2}\right)PQ - \frac{(r_1^2-PQ^2)(PQ^2-r_2^2)}{2PQ}\\ &= \tfrac{3}{2}PQ^3-(r_1^2+r_2^2)PQ+\frac{r_1^2r_2^2}{2PQ},\end{aligned}$$

$$3\int_0^{2\pi} PQ^3 d\theta = 2(r_1^2+r_2^2)\int PQ d\theta - r_1^2 r_2^2 \int \frac{d\theta}{PQ}$$

$$(8)\qquad \begin{aligned} &= 8r_1(r_1^2+r_2^2)E - 4r_1 r_2^2 F \\ &= 4r_1^3[2(1+c'^2)E - c'^2 F] \\ &= (r_1+r_2)^3[4(1+k^2)H - \tfrac{1}{2}k'^2(5+3k^2)K], \end{aligned}$$

$$(9)\qquad 3\int \frac{\sin^2\theta d\theta}{PQ} = \frac{16}{r_1}\frac{(1+c'^2)E-2c'^2F}{c^4} = \frac{8}{r_1+r_2}\frac{(1+k^2)H-(1-k_2^2)K}{k^2},$$

$$(10)\quad 3\int PQ\cos\theta d\theta = 4r_1\frac{(1+c'^2)E-2c'^2F}{c^2} = 2(r_1+r_2)\frac{(1+k^2)H-(1-k^2)K}{k},$$

so that as is evident from (7) these last two integrals (9) and (10) are essentially the same as well as the integral (1) § 30; and this shows that G is also the potential of the tore at any point C' of its axis, the tore being homogeneous and of mass $2\pi n$.

Potential of an elliptic disc, gravitational and magnetic.

34. Begin now with the potential V of an elliptic disc

$$(1)\qquad \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{0}=1$$

of uniform surface density 1; then (CAYLEY, Proceedings London Mathematical Society, vol. 6, p. 42)

$$(2)\qquad V = 2ab\int_\lambda^\infty \sqrt{\left(1-\frac{x^2}{a^2+\epsilon}-\frac{y^2}{b^2+\epsilon}-\frac{z^2}{\epsilon}\right)}\frac{d\epsilon}{\sqrt{(a^2+\epsilon\cdot b^2+\epsilon\cdot\epsilon)}},$$

where λ is the positive root of

$$(3)\qquad 1-\frac{x^2}{a^2+\epsilon}-\frac{y^2}{b^2+\epsilon}-\frac{z^2}{\epsilon}=0,$$

μ and ν denoting the other two.

Next by differentiation of V with respect to z the magnetic potential Ω is obtained of the elliptic disc, magnetized normally with uniform unit intensity, as represented by the component attraction of the disc perpendicular to the plane, or by the solid or conical angle subtended at a point; so that

$$(4)\qquad \Omega = -\frac{dV}{dz} = \int_\lambda^\infty \frac{2abzd\epsilon}{\epsilon\sqrt{\left(1-\frac{x^2}{a^2+\epsilon}-\frac{y^2}{b^2+\epsilon}-\frac{z^2}{\epsilon}\right)}\sqrt{(a^2+\epsilon\cdot b^2+\epsilon\cdot\epsilon)}},$$

reducing to

$$(5)\qquad \Omega = \int_\lambda^\infty \frac{4abzd\epsilon}{\epsilon\sqrt{(4\cdot\epsilon-\lambda\cdot\epsilon-\mu\cdot\epsilon-\nu)}} = \int_\lambda^\infty \frac{4\sqrt{(\lambda\mu\nu)}}{\epsilon}\frac{d\epsilon}{\sqrt{E}},$$

$$E = 4\cdot\epsilon-\lambda\cdot\epsilon-\mu\cdot\epsilon-\nu.$$

This is proved by determining x, y, z in terms of λ, μ, ν, the elliptic coördinates, such that (x, y, z) is a point of intersection of the three confocals of (3) to the elliptic disc given in (1), when

$$(6)\qquad \epsilon = \lambda,\ \mu,\ \nu,$$

$$(7)\qquad \infty > \lambda > 0 > \mu > -b^2 > \nu > -a^2 > -\infty,$$

and then

$$(8)\quad x^2 = \frac{a^2+\lambda\cdot a^2+\mu\cdot a^2+\nu}{a^2-b^2\cdot a^2-0},\quad y^2 = \frac{b^2+\lambda\cdot b^2+\mu\cdot b^2+\nu}{b^2-a^2\cdot b^2-0},\quad z^2 = \frac{0+\lambda\cdot 0+\mu\cdot 0+\nu}{a^2-0\cdot b^2-0},$$

$$(9)\qquad 1-\frac{x^2}{a^2+\epsilon}-\frac{y^2}{b^2+\epsilon}-\frac{z^2}{\epsilon} = \frac{\epsilon-\lambda\cdot\epsilon-\mu\cdot\epsilon-\nu}{a^2+\epsilon\cdot b^2+\epsilon\cdot\epsilon}.$$

Interpreted geometrically, Ω is the solid or conical angle subtended by the elliptic disc (1), or the *apparent area*, as it may be called; and SCHWARZ has shown that the apparent area or *Scheinbare Grösse* of the ellipsoid or of any elliptic section of the tangent cone is given by an expression equally simple (Göttinger Nachrichten, 1885),

$$(10)\quad \Omega = \int_\lambda^\infty \frac{2\sqrt{(-4\cdot\lambda_0-\lambda\cdot\lambda_0-\mu\cdot\lambda_0-\nu)}\,d\epsilon}{(\epsilon-\lambda_0)\sqrt{(4\cdot\epsilon-\lambda\cdot\epsilon-\mu\cdot\epsilon-\nu)}} = 2\pi - \int_\nu^\mu \frac{2\sqrt{(-E_0)}}{\lambda_0-\epsilon}\frac{d\epsilon}{\sqrt{E}}$$

$$(E_0 = 4\cdot\lambda_0-\lambda\cdot\lambda_0-\mu\cdot\lambda_0-\nu),$$

where the ellipsoid is defined by

$$(11)\qquad \frac{x^2}{a^2+\lambda_0}+\frac{y^2}{b^2+\lambda_0}+\frac{z^2}{\lambda_0} = 1,$$

reducing to the elliptic disc as its focal ellipse when $\lambda_0 = 0$.

Put

$$(12)\quad \epsilon-\lambda = M^2(s-s_1),\qquad \epsilon-\mu = M^2(s-s_2),\qquad \epsilon-\nu = M^2(s-s_3)$$

and

$$\epsilon(\text{or } \epsilon-\lambda_0) = M^2(s-\sigma),$$

$$(13)\qquad E = 4\cdot\epsilon-\lambda\cdot\epsilon-\mu\cdot\epsilon-\nu = M^6 S,$$

$$(14)\qquad 4a^2b^2z^2 = 4\lambda\mu\nu = -E_0 = -M^6\Sigma;$$

and then, as in (3), § 5,

$$(15)\qquad \Omega = \int_{s_1}^\infty \frac{2\sqrt{-\Sigma}}{s-\sigma}\frac{ds}{\sqrt{S}} = 4A = 2\pi - 4B.$$

35. SCHWARZ's theorem is proved by considering the equation of the tangent cone from (x, y, z) to the ellipsoid defined by $\epsilon = \lambda_0$, which, when referred to its principal axes, the normals of the three confocals through (x, y, z) defined by $\epsilon = \lambda, \mu, \nu$ becomes (SALMON, *Solid Geometry*, § 173),

$$\frac{x^2}{\lambda - \lambda_0} + \frac{y^2}{\mu - \lambda_0} + \frac{z^2}{\nu - \lambda_0} = 0. \tag{1}$$

one of a family of cones confocal for different values of λ_0.

The conical angle Ω of this tangent cone, and the perimeter Φ of the sphero-conic on unit sphere of the reciprocal cone

$$(\lambda - \lambda_0)x^2 + (\mu - \lambda_0)y^2 + (\nu - \lambda_0)z^2 = 0 \tag{2}$$

are connected by the relation

$$\Omega + \Phi = 2\pi. \tag{3}$$

On the reciprocal cone replacing x, y, z by direction cosines l, m, n, we may take

$$l^2 = k(\mu - \nu)\left(\frac{1}{\lambda - \lambda_0} - \frac{1}{\epsilon - \lambda_0}\right), \tag{4}$$

$$m^2 = k(\nu - \lambda)\left(\frac{1}{\mu - \lambda_0} - \frac{1}{\epsilon - \lambda_0}\right),$$

$$n^2 = k(\lambda - \mu)\left(\frac{1}{\nu - \lambda_0} - \frac{1}{\epsilon - \lambda_0}\right),$$

subject to the condition

$$l^2 + m^2 + n^2 = k\left(\frac{\mu - \nu}{\lambda - \lambda_0} + \frac{\nu - \lambda}{\mu - \mu_0} + \frac{\lambda - \mu}{\nu - \nu_0}\right) = 1, \tag{5}$$

$$k = -\frac{\lambda - \lambda_0 \cdot \mu - \lambda_0 \cdot \nu - \lambda_0}{\mu - \nu \cdot \nu - \lambda \cdot \lambda - \mu}, \tag{6}$$

and now

$$(d\Phi)^2 = dl^2 + dm^2 + dn^2 = \frac{-E_0}{4(\epsilon - \lambda_0)^2}\frac{d\epsilon^2}{E}, \tag{7}$$

$$E = 4 \cdot \epsilon - \lambda \cdot \epsilon - \mu \cdot \epsilon - \nu, \qquad E_0 = 4 \cdot \lambda_0 - \lambda \cdot \lambda_0 - \mu \cdot \lambda_0 - \nu. \tag{8}$$

As a point travels round a quadrant of the sphero-conic, ϵ decreases from μ to ν, and

$$\lambda > \lambda_0 > 0 > \mu > \epsilon > \nu, \tag{9}$$

$$\Phi = 4\int_\nu^\mu \frac{\frac{1}{2}\sqrt{(-E_0)}}{\lambda_0 - \epsilon}\frac{d\epsilon}{\sqrt{E}}, \tag{10}$$

$$\Omega = 2\pi - \Phi = 4\int_\lambda^\infty \frac{\frac{1}{2}\sqrt{(-E_0)}}{\epsilon - \lambda_0}\frac{d\epsilon}{\sqrt{E}}. \tag{11}$$

in accordance with (5) § 5, Ω and Φ being $4A$ and $4B$, SCHWARZ's result.

36. In the notation of GAUSS (CAYLEY, Proceedings of the London Mathematical Society, vol. 6; HALPHEN, *Fonctions Elliptiques*, vol. 2, chap. 8)

$$G,\ G'',\ G' = \lambda,\ -\mu,\ -\nu, \tag{1}$$

and in the region $\mu > \epsilon > \nu$ we can put

$$G + G' \cos^2 T + G'' \sin^2 T = \lambda - \epsilon, \tag{2}$$

$$G' \cos^2 T + G'' \sin^2 T = \lambda_0 - \epsilon, \tag{3}$$

$$(G' - G'') \cos^2 T = \mu - \epsilon,\ (G' - G'') \sin^2 T = \epsilon - \nu, \tag{4}$$

$$dT = \frac{d\epsilon}{\sqrt{(4 \cdot \mu - \epsilon \cdot \epsilon - \nu)}}, \tag{5}$$

$$U = \int_0^T \frac{dT}{\sqrt{(G + G' \cos^2 T + G'' \sin^2 T)}} = \int_{-\nu}^{\epsilon} \frac{d\epsilon}{\sqrt{E}}, \tag{6}$$

$$U\sqrt{(\lambda - \nu)} = hK, \qquad T = \operatorname{am} hK. \tag{7}$$

In the region $\infty > \epsilon > \lambda > \lambda_0$, we should take

$$\sin^2 T = \frac{\lambda - \nu}{\epsilon - \nu}, \qquad \cos^2 T = \frac{\epsilon - \lambda}{\epsilon - \nu}, \qquad T = \operatorname{am} hK, \tag{8}$$

$$G + G' \cos^2 T + G'' \sin^2 T = \frac{\lambda - \nu \cdot \epsilon - \mu}{\epsilon - \nu}, \tag{9}$$

$$G' \cos^2 T + G'' \sin^2 T = \frac{(\lambda_0 - \nu \cdot \epsilon - \nu) - (\lambda - \nu \cdot \mu - \nu)}{\epsilon - \nu}, \tag{10}$$

equivalent to the substitution

$$\left(\epsilon - \nu,\ \frac{\lambda - \nu \cdot \mu - \nu}{\epsilon - \nu}\right). \tag{11}$$

37. The inequalities

$$\infty > \epsilon > \lambda > \lambda_0 > 0 > \mu > -b^2 > \nu > -a^2, \tag{1}$$

$$\infty > s > s_1 > \sigma > s_2 > s > s_3 > -\infty, \tag{2}$$

show that the form $A(fK')$ or $B(fK')$ is required; and from (3) § 5,

$$\Omega = 2\pi(1 - f) - 4K \operatorname{zn} fK', \tag{3}$$

equivalent to NAGAOKA's result for a circle expressed by Weierstrass functions; and here, as in (6), (7), (8), (9) § 5,

(4) $$\operatorname{zn} fK' = ZfK' = E\phi - fE,$$

(5) $$fK' = F(\phi, k'), \qquad \phi = \operatorname{am} fK',$$

so that LEGENDRE's Table IX can be employed for a numerical application,

(6) $$\sin^2\phi = \operatorname{sn}^2 fK' = \frac{s_1 - \sigma}{s_1 - s_2} = \frac{\lambda}{\lambda - \mu},$$

(7) $$\cos^2\phi = \operatorname{cn}^2 fK' = \frac{\sigma - s_2}{s_1 - s_2} = \frac{-\mu}{\lambda - \mu},$$

(8) $$\Delta^2\phi = \operatorname{dn}^2 fK' = \frac{\sigma - s_3}{s_1 - s_3} = \frac{-\nu}{\lambda - \nu},$$

(9) $$k^2 = \frac{s_2 - s_3}{s_1 - s_3} = \frac{\mu - \nu}{\lambda - \nu}, \qquad k'^2 = \frac{s_1 - s_2}{s_1 - s_3} = \frac{\lambda - \mu}{\lambda - \nu},$$

and the modular angle is then half the angle between the generating lines of the confocal hyperboloid of one sheet, or the focal lines of the tangent cone.

38. Thus for instance, if we take $f = \frac{1}{2}$,

(1) $$\Delta^2\phi = \frac{-\nu}{\lambda - \nu} = k = \sqrt{\frac{\mu - \nu}{\lambda - \nu}}, \qquad \frac{1}{\lambda} + \frac{1}{\mu} - \frac{1}{\nu} = 0, \qquad k = \frac{-\mu}{\lambda},$$
$$\mu = -k\lambda, \qquad v = -\frac{k\lambda}{1 - k},$$

and now

(2) $$\operatorname{zn} \tfrac{1}{2}K' = \tfrac{1}{2}(1 - k), \qquad \Omega = \pi - 2(1 - k)K.$$

At an infinite distance,

(3) $$\lambda = \infty, \quad \sin\phi = 1, \quad k = 0, \quad K = \tfrac{1}{2}\pi, \quad k' = 1, \quad K' = \infty, \quad E' = 1,$$

(4) $$\operatorname{zn} fK' = E\phi - fE' = \sin\phi - f = 1 - f,$$

so that

(5) $$\Omega = 2\pi(1 - f) - 2\pi(1 - f) = 0,$$

a verification.

Inside the focal ellipse, in its plane,

(6) $$\lambda = 0, \qquad f = 0, \qquad \Omega = 2\pi;$$

but in the plane outside the ellipse

(7) $$\mu = 0, \qquad f = 1, \qquad \operatorname{zn} K' = 0, \qquad \Omega = 0,$$

to serve as a verification.

Along the edge of the focal ellipse

$$\lambda = 0, \qquad \mu = 0, \qquad k' = 0, \qquad \operatorname{zn} fK' = 0, \tag{8}$$

$$\Omega = 2\pi(1 - f), \tag{9}$$

and two surfaces of constant Ω_1 and Ω_2 intersect along the edge at an angle

$$\tfrac{1}{2}(\Omega_2 - \Omega_1) = \pi(f_1 - f_2); \tag{10}$$

see MAXWELL, E. and M., § 487.

Along the focal hyperbola

$$\mu = \nu = -b^2, \qquad k = 0, \qquad K = \tfrac{1}{2}\pi, \tag{11}$$

$$k' = 1, \qquad E' = 1, \qquad E\phi = \sin\phi,$$

$$\operatorname{zn} fK' = \sin\phi - f,$$

so that

$$\Omega = 2\pi(1 - \sin\phi) = 2\pi\left(1 - \sqrt{\frac{\lambda}{\lambda - \mu}}\right), \tag{12}$$

and the conical angle is now bounded by a cone of revolution of vertical angle $\pi - 2\phi$ (CAYLEY, Proceedings of the London Mathematical Society, vol. 6).

On the axis perpendicular to the plane of the ellipse

$$x = 0, \qquad y = 0, \qquad z^2 = \lambda, \qquad b^2 + \mu = 0, \qquad a^2 + \nu = 0, \tag{13}$$

$$k^2 = \frac{a^2 - b^2}{a^2 + z^2}, \qquad k'^2 = \frac{b^2 + z^2}{a^2 + z^2},$$

$$\operatorname{sn}^2 fK' = \frac{z^2}{b^2 + z^2}, \qquad \operatorname{cn}^2 fK' = \frac{b^2}{b^2 + z^2}, \qquad \operatorname{dn}^2 fK' = \frac{a^2}{a^2 + z^2},$$

$$\operatorname{dn}^2(1 - f)K' = 1 - \frac{b^2}{a^2} = e^2.$$

39. The equipotential surface $\Omega = 2\pi f$, a constant, cuts the plane of the ellipse (or any other area whatever) along its edge at a constant angle πf; this is evident from simple geometrical considerations of the conical angle or apparent area of the ellipse as seen from a point close to the edge, when the apparent area on the unit sphere is cut out by two diametral planes, one parallel to the plane of the area, and the other drawn through the adjacent tangent of the area.

Magnetic interpretation of the potential of the elliptic disc.

40. In the potential of an elliptic disc, as given by CAYLEY,

$$V = 4ab\int_\lambda^\infty \left(1 - \frac{z^2}{\epsilon} - \frac{y^2}{b^2 + \epsilon} - \frac{x^2}{a^2 + \epsilon}\right)\frac{d\epsilon}{\sqrt{E}}, \tag{1}$$

when modified by (9) § 34, and employing (12) § 34, with

$$(2)\quad \epsilon = m^2(s-\sigma), \qquad b^2+\epsilon = m^2(s-\sigma'), \qquad a^2+\epsilon = m^2(s-\sigma''),$$

$$(3)\quad \infty > s > s_1 > \sigma > s_2 > \sigma' > s_3 > \sigma'' > -\infty,$$

$$(4)\quad \begin{aligned} \lambda &= m^2(s_1-\sigma), & b^2+\lambda &= m^2(s_1-\sigma'), & a^2+\lambda &= m^2(s_1-\sigma''),\\ \mu &= -m^2(\sigma-s_2), & b^2+\mu &= -m^2(s_2-\sigma'), & a^2+\mu &= m^2(s_2-\sigma''),\\ \nu &= -m^2(\sigma-s_1), & b^2+\nu &= -m^2(\sigma'-s_3), & a^2+\nu &= m^2(s_3-\sigma''), \end{aligned}$$

$$(5)\quad a^2b^2z^2 = \lambda\mu\nu = -\tfrac{1}{4}m^6\Sigma,$$

$$(6)\quad (a^2-b^2)b^2y^2 = b^2+\lambda\cdot b^2+\mu\cdot b^2+\nu = \tfrac{1}{4}m^6\Sigma',$$

$$(7)\quad (a^2-b^2)a^2x^2 = a^2+\lambda\cdot a^2+\mu\cdot a^2+\nu = -\tfrac{1}{4}m^6\Sigma'',$$

and the terms in V are

$$(8)\quad 4ab\int_\lambda^\infty \frac{d\epsilon}{\sqrt{E}} = \frac{4abK}{\sqrt{(\lambda-\nu)}},$$

$$(9)\quad \begin{aligned} 4abz^2\int\frac{d\epsilon}{\epsilon\sqrt{E}} &= 4z\int_{s_1}^\infty \frac{\frac{1}{2}\sqrt{(-\Sigma)}}{s-\sigma}\frac{ds}{\sqrt{S}}\\ &= 4zA(fK') = 4z\left[\tfrac{1}{2}\pi(1-f) - K\operatorname{zn}fK'\right], \end{aligned}$$

$$(10)\quad \begin{aligned} 4aby^2\int\frac{d\epsilon}{(b^2+\epsilon)\sqrt{E}} &= \frac{4ay}{\sqrt{(a^2-b^2)}}\int\frac{\frac{1}{2}\sqrt{(\Sigma')}}{s-\sigma'}\frac{ds}{\sqrt{S}}\\ &= \frac{4ay}{\sqrt{(a^2-b^2)}}E(f'K) = \frac{4ay}{\sqrt{(a^2-b^2)}}K\operatorname{zn}f'K, \end{aligned}$$

$$(11)\quad \begin{aligned} 4abx^2\int\frac{d\epsilon}{(a^2+\epsilon)\sqrt{E}} &= \frac{4bx}{\sqrt{(a^2-b^2)}}\int\frac{\frac{1}{2}\sqrt{(-\Sigma'')}}{s-\sigma''}\frac{ds}{\sqrt{S}}\\ &= \frac{4bx}{\sqrt{(a^2-b^2)}}C(f''K')\\ &= \frac{4bx}{\sqrt{(a^2-b^2)}}\left[K\operatorname{zs}f''K' - \tfrac{1}{2}\pi(1-f'')\right]. \end{aligned}$$

Here, as in (3) § 2,

$$(12)\quad f = \int_\sigma^{s_1} \div \int_{s_2}^{s_1}\frac{d\sigma}{\sqrt{(-\Sigma)}},\quad \operatorname{sn}^2fK' = \frac{\lambda}{\lambda-\mu},\quad \operatorname{cn}^2fK' = \frac{-\mu}{\lambda-\mu},\quad \operatorname{dn}^2fK' = \frac{-\nu}{\lambda-\nu},$$

$$(13)\quad f' = \int_{s_3}^{\sigma'} \div \int_{s_3}^{s_2}\frac{ds}{\sqrt{S}},\quad \operatorname{sn}^2f'K = \frac{-b^2-\nu}{\mu-\nu},\quad \operatorname{cn}^2f'K = \frac{b^2+\mu}{\mu-\nu},\quad \operatorname{dn}^2f'K = \frac{b^2+\lambda}{\lambda-\nu},$$

$$(14)\quad f'' = \int_{-\infty}^{\sigma''} \div \int_{-\infty}^{s_3}\frac{d\sigma}{\sqrt{(-\Sigma)}},\quad \operatorname{sn}^2f''K' = \frac{\lambda-\nu}{a^2+\lambda},\quad \operatorname{cn}^2f''K' = \frac{a^2+\nu}{a^2+\lambda},\quad \operatorname{dn}^2f''K' = \frac{a^2+\mu}{a^2+\lambda}.$$

We have seen already in (5) and (15) § 34, that

$$(15)\quad \Omega = -\frac{dV}{dz} = 4A(fK') = 2\pi(1-f) - 4K\,\mathrm{zn}\,fK',$$

is the magnetic potential of the elliptic disc with uniform normal magnetization parallel to Oz, or of the unit current round the elliptic ring; so also

$$(16)\quad \Omega' = -\frac{dV}{dy} = \frac{4a}{\sqrt{(a^2-b^2)}}E(f'K) = \frac{4a}{\sqrt{(a^2-b^2)}}K\,\mathrm{zn}\,f'K,$$

is the potential for uniform magnetization parallel to Oy the minor axis; and

$$(17)\quad \Omega'' = -\frac{dV}{dx} = \frac{4b}{\sqrt{(a^2-b^2)}}C(f''K') = \frac{4b}{\sqrt{(a^2-b^2)}}[K\,\mathrm{zs}\,f''K' - \tfrac{1}{2}\pi(1-f'')],$$

for uniform magnetization parallel to Ox the major axis: and Ω', Ω'' are the components F, G of the vector potential of the elliptic current.

For uniform magnetization of the elliptic disc with direction cosines l, m, n, the magnetic potential is

$$(18)\qquad n\Omega + m\Omega' + l\Omega''.$$

Induced magnetism of a hollow ellipsoid of soft iron.

41. We may cite here the expression of the induction of a uniform magnetic field on a hollow ellipsoid of soft iron, bounded by confocals, of the family

$$(1)\qquad \frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{\lambda} = 1,$$

λ_0 and λ_1 defining the outer and inner surface of the shell.

We require the three incomplete II. E. I.'s

$$(2)\qquad A, B, C = \int_\lambda^\infty \frac{d\epsilon}{(a^2+\epsilon,\, b^2+\epsilon,\, \epsilon)\sqrt{P}},$$

$$P = 4\cdot a^2 + \epsilon\cdot b^2 + \epsilon\cdot\epsilon, \qquad A+B+C = \frac{1}{\sqrt{(a^2+\lambda\cdot b^2+\lambda\cdot\lambda)}};$$

and with the fraction h defined as in § 2 by

$$(3)\qquad h = \int_\lambda^\infty \div \int_0^\infty \frac{d\epsilon}{\sqrt{P}},$$

we find from § 3,

$$A = \frac{1}{a(a^2 - b^2)} [h(K - H) - \operatorname{zn} hK], \tag{4}$$

$$B = \frac{a}{b^2(a^2 - b^2)} [h(H - k'^2 K) - \operatorname{zn}(1 - h)K], \tag{5}$$

$$C = \frac{1}{ab^2} [-hH + \operatorname{zs}(1 - h)K], \tag{6}$$

$$k^2 = 1 - \frac{b^2}{a^2}, \qquad k' = \frac{b}{a}. \tag{7}$$

If the external inducing magnetic field has a potential

$$V = Xx + Yy + Zz, \tag{8}$$

we can examine the components one at a time, so that if the component Xx induces a magnetic potential Ω in the iron of the shell, Ω_0 in the outside space, and Ω_1 in the cavity, the magnetic conditions are satisfied by putting

$$\Omega_1 = L_1 x, \qquad \Omega_0 = L_0 \frac{A}{A_0} x, \tag{9}$$

$$\Omega = L_0 \frac{A_1 - A}{A_1 - A_0} x + L_1 \frac{A - A_0}{A_1 - A_0} x, \tag{10}$$

where

$$\frac{A_1 - A}{A_1 - A_0} = \frac{(h_1 - h)(K - H) - (\operatorname{zn} h_1 K - \operatorname{zn} hK)}{(h_1 - h_0)(K - H) - (\operatorname{zn} h_1 K - \operatorname{zn} h_0 K)}, \cdots; \tag{11}$$

and then determining L_0 and L_1 from the condition, with magnetic permeability μ,

$$\mu \frac{d}{d\nu}(Xx + \Omega) = \frac{d}{d\nu_1}(Xx + \Omega_1) \text{ or } \frac{d}{d\nu_0}(Xx + \Omega_0) \tag{12}$$

in crossing the inner or outer surface, noticing that

$$x \frac{dA}{d\nu} = -\frac{x}{a^2 + \lambda} \frac{1}{\sqrt{P}} \frac{d\lambda}{d\nu} = -\frac{2px}{a^2 + \lambda} \frac{1}{\sqrt{P}} = -\frac{2l}{\sqrt{P}}. \tag{13}$$

Other cases for components of V such as xy, xyz, $\cdots$ are given in the Journal de Physique (1881); the hydrodynamical analogues can be developed by means of the same analysis.

Potential of a circular disc, gravitational and magnetic.

42. For a circular disc put $a^2 = b^2$,

$$V = 2a^2 \int_\lambda^\infty \sqrt{\left(1 - \frac{x^2+y^2}{a^2+\epsilon} - \frac{z^2}{\epsilon}\right)} \frac{d\epsilon}{(a^2+\epsilon)\sqrt{\epsilon}}; \tag{1}$$

ellipsoidal coördinates are now unsuitable, and the elliptic coördinates, v and w, must be employed, with the substitution

$$a^2 + \epsilon = a^2 \operatorname{ch}^2 u, \qquad \epsilon = a^2 \operatorname{sh}^2 u, \tag{2}$$

$$x^2 + y^2 = a^2 \operatorname{ch}^2 v \cos^2 w, \qquad z^2 = a^2 \operatorname{sh}^2 v \sin^2 w, \tag{3}$$

$$r_1,\ r_2 = a(\operatorname{ch} v \pm \cos w), \tag{4}$$

r_1, r_2 denoting the focal distances of P from the foci A, B in fig. A; and

$$\begin{aligned} 1 - \frac{x^2+y^2}{a^2+\epsilon} - \frac{z^2}{\epsilon} &= 1 - \frac{\operatorname{ch}^2 v \cos^2 w}{\operatorname{ch}^2 u} - \frac{\operatorname{sh}^2 v \sin^2 w}{\operatorname{sh}^2 u} \\ &= \frac{(\operatorname{ch}^2 u - \operatorname{ch}^2 v)(\operatorname{ch}^2 u - \cos^2 w)}{\operatorname{ch}^2 u \operatorname{sh}^2 u}, \end{aligned} \tag{5}$$

$$V = 4a \int_v^\infty \frac{\sqrt{(\operatorname{ch}^2 u - \operatorname{ch}^2 v \cdot \operatorname{ch}^2 u - \cos^2 w)}}{\operatorname{ch}^2 u \operatorname{sh} u} du, \tag{6}$$

$$\begin{aligned} \Omega = -\frac{dV}{dz} &= \int_\lambda^\infty \frac{2a^2 z d\epsilon}{\epsilon(a^2+\epsilon)\sqrt{\epsilon}\sqrt{\left(1 - \frac{x^2+y^2}{a^2+\epsilon} - \frac{z^2}{\epsilon}\right)}} \\ &= 4\int_v^\infty \frac{\operatorname{sh} v \sin w du}{\operatorname{sh} u \sqrt{(\operatorname{ch}^2 u - \operatorname{ch}^2 v \cdot \operatorname{ch}^2 u - \cos^2 w)}}. \end{aligned} \tag{7}$$

Next substitute

$$\operatorname{sh}^2 u = m^2(s - \sigma), \qquad \operatorname{ch}^2 u = m^2(s - s_3), \tag{8}$$

$$m^2(\sigma - s_3) = 1, \qquad \operatorname{ch}^2 u = \frac{s - s_3}{\sigma - s_3}, \qquad \operatorname{sh}^2 u = \frac{s - \sigma}{\sigma - s_3}, \tag{9}$$

$$\operatorname{ch}^2 u - \operatorname{ch}^2 v = \frac{s - s_1}{\sigma - s_3}, \qquad \operatorname{ch}^2 u - \cos^2 w = \frac{s - s_2}{\sigma - s_3}, \tag{10}$$

$$\operatorname{ch}^2 v = \frac{s_1 - s_3}{\sigma - s_3}, \qquad \operatorname{sh}^2 v = \frac{s_1 - \sigma}{\sigma - s_3}, \tag{11}$$

$$\cos^2 w = \frac{s_2 - s_3}{\sigma - s_3}, \qquad \sin^2 w = \frac{\sigma - s_2}{\sigma - s_3}, \tag{12}$$

$$k^2 = \frac{s_2 - s_3}{s_1 - s_3} = \frac{\cos^2 w}{\operatorname{ch}^2 v}, \qquad k = \frac{r_1 - r_2}{r_1 + r_2}, \tag{13}$$

as in MAXWELL's second expression for M, § 701.

The inequality sequence runs

$$\infty > s > s_1 > \sigma > s_2 > s_3 > -\infty, \tag{14}$$

so that, from § 2, dropping K',

$$\operatorname{ch} v = \frac{1}{\operatorname{dn} f} = \frac{\operatorname{dn}(1-f)}{k}, \qquad \cos w = \operatorname{dn}(1-f). \tag{15}$$

If then we write A for $\sqrt{(x^2+y^2)}$, $x = A\cos\phi$, $y = A\sin\phi$,

$$\frac{A}{a} = \operatorname{ch} v \cos w = \frac{\operatorname{dn}(1-f)}{\operatorname{dn} f} = \frac{k}{\operatorname{dn}^2 f} = \frac{\operatorname{dn}^2(1-f)}{k}, \tag{16}$$

$$\frac{z}{a} = \operatorname{sh} v \operatorname{sn} w = \frac{k'^2 \operatorname{sn} f \operatorname{cn} f}{\operatorname{dn}^2 f} = \frac{k'^2}{k} \operatorname{sn}(1-f)\operatorname{cn}(1-f), \tag{17}$$

$$\frac{r_1, r_2}{a} = \operatorname{ch} v \pm \cos w = \left(\frac{1}{k} \pm 1\right)\operatorname{dn}(1-f). \tag{18}$$

Since

$$\frac{\operatorname{sh} u\, du}{\sqrt{(\operatorname{ch}^2 u - \operatorname{ch}^2 v \cdot \operatorname{ch}^2 u - \cos^2 w)}} = \frac{\sqrt{(\sigma - s_3)}\, ds}{\sqrt{S}}, \tag{19}$$

equation (7), as before in (15) § 34, becomes transformed to

$$\Omega = 4\int_{s_1}^{\infty} \frac{\frac{1}{2}\sqrt{(-\Sigma)}}{s-\sigma} \frac{ds}{\sqrt{S}} = 4A(fK') = 2\pi(1-f) - 4K \operatorname{zn} fK'. \tag{20}$$

To make s oscillate between s_2 and s_3, substitute

$$\operatorname{ch}^2 u = \frac{\sigma - s_3}{s - s_3}, \quad \operatorname{sh}^2 u = \frac{\sigma - s}{s - s_3}, \quad \operatorname{ch}^2 v = \frac{\sigma - s_3}{s_2 - s_3}, \quad \cos^2 w = \frac{\sigma - s_3}{s_2 - s_3}. \tag{21}$$

43. The potential for uniform magnetization of the circular disc parallel to AB is, as before in (16), (17), § 40,

$$\Omega' = -\frac{dV}{dA} = \int_{\lambda}^{\infty} \frac{2a^2 A\, d\epsilon}{(a^2+\epsilon)^2 \sqrt{\epsilon}\sqrt{\left(1 - \frac{A^2}{a^2+\epsilon} - \frac{z^2}{\epsilon}\right)}} \tag{1}$$

$$= \int_{v}^{\infty} \frac{4 \operatorname{ch} v \cos w \operatorname{sh} u \operatorname{dn}}{\operatorname{ch}^2 u \sqrt{(\operatorname{ch}^2 u - \operatorname{ch}^2 v \cdot \operatorname{ch}^2 u - \cos^2 w)}}$$

$$= \frac{4}{\cos w}\int_{s_1}^{\infty} \frac{s_2 - s_3}{s - s_3} \frac{\sqrt{(s_1 - s_3)}\, ds}{\sqrt{S}} = 4\frac{K - H}{\cos w}.$$

From MAXWELL's second expression in § 701,

$$(2) \qquad M = 4\pi(r_1 + r_2)(K - H) = 8\pi a \operatorname{ch} v(K - H) = 2\pi A\Omega',$$

and this is evident when we notice in figure B that M is composed of elements

$$(3) \qquad \frac{\cos\theta}{PQ''} a d\theta - \frac{\cos\theta}{PQ} a d\theta,$$

which is the magnetic potential at P of a strip QQ'' parallel to AB, magnetized longitudinally; and these elements are integrated over the circle AQB, and multiplied afterwards by $2\pi A$ to obtain M for the two circles.

Replace A by x, then at any point $(x, 0, z)$

$$(4) \qquad M = 2\pi x\Omega' = -2\pi x \frac{dV}{dx}, \qquad 2\pi\Omega = -2\pi \frac{dV}{dz},$$

$$(5) \qquad 2\pi \frac{d\Omega}{dx} = -2\pi \frac{d^2V}{dx\,dz} = \frac{1}{x}\frac{dM}{dz},$$

$$(6) \qquad 2\pi \frac{d\Omega}{dz} = -2\pi \frac{d^2V}{dz^2} = 2\pi\left(\frac{d^2V}{dx^2} + \frac{1}{x}\frac{dV}{dx}\right) = -\frac{1}{x}\frac{dM}{dx},$$

and a magnetic line of force along which M is constant is the orthogonal trajectory of the equipotential surfaces $\Omega =$ constant; in fact M is the Stokes function of the magnetic potential, and

$$(7) \qquad \frac{d^2\Omega}{dz^2} + \frac{d^2\Omega}{dx^2} + \frac{1}{x}\frac{d\Omega}{dx} = 0,$$

$$(8) \qquad \frac{d^2M}{dz^2} + \frac{d^2M}{dx^2} - \frac{1}{x}\frac{dM}{dx} = 0,$$

or as they may be written

$$(9) \qquad \frac{d}{dz}\left(x\frac{d\Omega}{dz}\right) + \frac{d}{dx}\left(x\frac{d\Omega}{dx}\right) = 0,$$

$$(10) \qquad \frac{d}{dz}\left(\frac{1}{x}\frac{dM}{dz}\right) + \frac{d}{dx}\left(\frac{1}{x}\frac{dM}{dx}\right) = 0.$$

These are MAXWELL's results of § 703, obtained from independent physical considerations when x and z are replaced by his a and b; the symmetry round the axis z shows that the integration employed in MAXWELL's treatment can be effected, so that his result, where V and Ω are equivalent, may be written

$$(11) \qquad \frac{dM}{da} = \int_0^{2\pi} a\frac{dV}{dy}\,d\theta = 2\pi a\frac{dV}{dy},$$

$$(12) \qquad \frac{dM}{db} = -\int a\frac{dV}{dr}\,d\theta = -2\pi a\frac{dV}{dr}.$$

The potential of the circular disc is now written, either from (1) or (6), § 42,

$$
\begin{aligned}
V &= 4a\int_0^\infty \left(1 - \frac{\operatorname{ch}^2 v \cos^2 w}{\operatorname{ch}^2 u} - \frac{\operatorname{sh}^2 v \sin^2 w}{\operatorname{sh}^2 u}\right) \frac{\operatorname{sh} u du}{\sqrt{[(\operatorname{ch}^2 u - \operatorname{ch}^2 v)\cdot(\operatorname{ch} u - \cos^2 w)]}} \\
&= 4\int_{s_1}^\infty \left[a - A\frac{\sqrt{(s_1 - s_3\cdot s_2 - s_3)}}{s - s_3} - z\frac{\sqrt{(s_1 - \sigma\cdot\sigma - s_2)}}{s - \sigma}\right]\frac{\sqrt{(\sigma - s_3)}\,ds}{\sqrt{S}} \\
&= 4a\sqrt{\left(\frac{\sigma - s_3}{s_1 - s_3}\right)}K - 4A\sqrt{\left(\frac{\sigma - s_3}{s_2 - s_3}\right)}(K - H) - 4zA(fK') \\
&= \frac{4aK}{\operatorname{ch} v} - A\Omega' - z\Omega,
\end{aligned}
\tag{13}
$$

of which each term can be interpreted by means of the homogeneity relation

$$V = a\frac{dV}{da} + A\frac{dV}{dA} + z\frac{dV}{dz}, \tag{14}$$

since

$$\frac{dV}{dz} = -\Omega, \qquad \frac{dV}{dA} = -\Omega', \tag{15}$$

and therefore

$$\frac{dV}{da} = \frac{4K}{\operatorname{ch} v} = \frac{8aK}{r_1 + r_2}, \tag{16}$$

the potential at P of the circular ring round the rim of the disc.

Similarly in the case of the elliptic disc in § 40 the term in (8)

$$4ab\int\frac{d\epsilon}{\sqrt{E}} = \frac{4abK}{\sqrt{(\lambda - \nu)}} \tag{17}$$

may be interpreted as the potential at P of a ring round the ellipse cut off by a consecutive confocal ellipse, so that the thickness is inversely proportional to the perpendicular from the center on the tangent.

The Stokes function of a circular plate and ring.

44. Denote the Stokes function of the plate by N, using b again instead of z; then, as in §§ 31, 43,

$$\frac{dN}{db} = 2\pi A\frac{dV}{dA} = -2\pi A\Omega' = -4\pi(r_1 + r_2)(K - H), \tag{1}$$

$$
\begin{aligned}
\frac{dN}{dA} &= -2\pi A\frac{dV}{db} = 2\pi A\Omega \\
&= 2\pi A[2\pi(1 - f) - 4K\operatorname{zn} fK'] \\
&= 2\pi A\cdot 4A(fK'),
\end{aligned}
\tag{2}
$$

and by analogy

$$(3) \qquad \frac{dN}{da} = 2\pi a \cdot 4A(1-f)K' = 2\pi a\,[\,2\pi f - 4K\,\mathrm{zn}\,(1-f)K'\,],$$

a change of f into $1-f$, and A into a; and this term is the Stokes function of the circular ring, of which the potential function is given in (1), § 33: so that the Stokes function at P of the circle AQB is $2\pi a$ times the apparent area at P'.

Then by the homogeneity of the second degree in a, A, b, of N

$$(4) \qquad \begin{aligned} 2N &= a\frac{dN}{da} + A\frac{dN}{dA} + b\frac{dN}{db} \\ &= 8\pi a^2 A(1-f)K' + 8\pi A^2 \cdot AfK' - 4\pi b(r_1 + r_2)(K - H), \end{aligned}$$

$$(5) \qquad N = 4\pi(a^2 - A^2)B(fK') + 2\pi^2 A^2 - 2\pi b(r_1 + r_2)(K - H + K\,\mathrm{dn}^2 fK').$$

For points on the plate, $f = 0$, $N = 2\pi^2 A^2$: while $f = 1$ for points in its plane outside, and $N = 2\pi^2 a^2$; and $N - 2\pi^2 A^2$ may be taken as the Stokes function at P of this plate and a coaxial equal and opposite plate at the level of P.

Changing the superficial density from unity to $1/2\pi p$, we shall see in the sequel that

$$(6) \qquad \frac{2\pi^2 A^2 - N}{2\pi p} = M,$$

where M is the function in § 54 investigated by Viriamu Jones for a helical current, or the equivalent cylindrical current sheet; and the potential of the two end plates is the magnetic potential of the cylinder magnetized longitudinally, while the Stokes function $N - 2\pi^2 A^2$ is the magnetic potential for magnetization across the axis of the cylinder.

The various dissections of an integral.

45. To arrive geometrically at the expression of the conical angle Ω by means of the angles θ, ϕ, ψ, χ, ω defined previously on fig. A and B, we employ the idea given by Maxwell in E. and M. § 418, that the area Ω, cut out on unit sphere by a cone whose vertex is at the centre, is equal to 2π minus the length Φ of the curve traced out on the sphere by the reciprocal cone; this is a direct result and generalisation of the theorem that the sum of the area of a spherical triangle and the perimeter of the polar triangle is 2π.

The same idea is employed by Schwarz in his memoir *Die Scheinbare Grösse des Ellipsoides*, the apparent area and conical angle being the same thing, either of the ellipsoid, or of the tangent cone, or any elliptic section of it.

If the tangent plane PQY of the cone, shown in elevation and plan in figures A and B, turns through an angle $d\Phi$ while OQ turns through the small angle

$d\theta$, and if the normal line Pq of the cone cuts the plane AQB in q on YM produced (fig. B), then Q and q describe curves which are polar reciprocal with respect to M; so that QMy is perpendicular to qy the tangent at q; and the sector velocity of q round M

$$(1) \qquad \tfrac{1}{2}Mq^2\frac{d\theta}{dt} = \tfrac{1}{2}Pq^2\frac{d\Phi}{dt}\cos MyP,$$

$$(2) \qquad \frac{d\Phi}{d\theta} = \frac{Mq^2}{Pq^2}\frac{Py}{My} = \frac{PM^2}{PY^2}\cdot\frac{PQ}{PM} = \frac{1}{1-\dfrac{QY^2}{PQ^2}}\frac{PM}{PQ},$$

so that, in MAXWELL's notation, with $OM = A$,

$$(3) \qquad 2\pi - \Omega = \Phi = \int_0^{2\pi}\frac{1}{1-\dfrac{A^2\sin^2\theta}{PQ^2}}\cdot\frac{bd\theta}{PQ}.$$

Now from § 27

$$(4) \qquad \frac{A^2\sin^2\theta}{PQ^2} = \frac{4a^2A^2\sin^2\theta}{(r_1+r_2)^2CQ^2} = \left(\frac{2A}{r_1+r_2}\right)^2\sin^2\chi, \qquad \frac{bd\theta}{PQ} = \frac{2b}{r_1+r_2}\frac{d\chi}{\Delta\chi},$$

so that, in LEGENDRE's form of § 14,

$$(5) \qquad \Phi = \int_0^{\frac{1}{2}\pi}\frac{4\sqrt{\alpha}}{1+n\sin^2\chi}\frac{d\chi}{\Delta\chi},$$

$$(6) \quad n = -\left(\frac{2A}{r_1+r_2}\right)^2, \quad k = \frac{r_1-r_2}{r_1+r_2}, \quad \alpha = (1+n)\left(1+\frac{k^2}{n}\right) = \left(\frac{2b}{r_1+r_2}\right)^2.$$

To reduce Φ to our form B in § 5, we put

$$(7) \qquad \sin^2\chi = \frac{s-s_3}{s_2-s_3}, \qquad \frac{d\chi}{\Delta\chi} = \frac{\sqrt{(s_1-s_3)}\,ds}{\sqrt{S}},$$

and interpreted in fig. A, where $OM = A$, $OC = ak$, $OC' = a/k$,

$$(8) \quad \frac{\sigma-s_3}{s_2-s_3} = -\frac{1}{n} = \left(\frac{r_1+r_2}{2A}\right)^2, \quad \frac{\sigma-s_3}{s_1-s_3} = \mathrm{dn}^2 fK' = \left(\frac{r_1-r_2}{2A}\right)^2 = k\frac{r_1^2-r_2^2}{4A^2} = k\frac{a}{A} = \frac{OC}{OM},$$

$$(9) \qquad \frac{\sigma-s_2}{s_2-s_3} = \frac{(r_1+r)^2-4A^2}{4A^2}, \qquad \frac{s_1-\sigma}{s_1-s_3} = \frac{4A^2-(r_1-r_2)^2}{4A^2},$$

$$(10) \qquad \frac{s_1-\sigma\cdot\sigma-s_2}{s_1-s_3\cdot s_2-s_3} = \frac{b^2}{A^2},$$

so that, as before in § 34,

$$(11) \qquad \Phi = 4\int_{s_3}^{s_2}\frac{\frac{1}{2}\sqrt{(-\Sigma)}}{\sigma-s}\frac{ds}{\sqrt{S}} = 4B(fK'),$$

$$(12) \qquad \Omega = 2\pi - 4B(fK') = 4A(fK') = 4\int_{s_1}^{\infty}\frac{\frac{1}{2}\sqrt{(-\Sigma)}}{s-\sigma}\frac{ds}{\sqrt{S}}.$$

46. If $d\alpha$ denotes an element of area at any point Q inside the circle, due to any mode of dissection

$$V = \int \frac{d\alpha}{PQ}, \qquad \Omega = \int \frac{b\,d\alpha}{PQ^3}, \tag{1}$$

and the integrals which arise in the determination of these potential functions change in appearance with the method of dissection employed by the use of each system of variables; so it will save repetition in the sequel if we cite here some preliminary lemmas.

I. The potential at a point F on its axis of a flat ring-shaped disc, bounded internally and externally by concentric circles on diameter ab and AB, is

$$\frac{\text{area}}{\frac{1}{2}(FA + Fa)} = 2\pi(FA - Fa), \tag{2}$$

and thence also the same for any fraction of the ring cut out by two radii at an angle θ, replacing 2π.

II. The potential at P of the line AB in fig. A is

$$\int_{-a}^{a} \frac{dx}{PQ} = \log \frac{PA + PB + AB}{PA + PB - AB} = 2\,\text{th}^{-1} \frac{AB}{PA + PB} = 2\,\text{th}^{-1} \frac{PA - PB}{2\,OM}, \tag{3}$$

and when P is at F, the potential of AB is twice the potential of OA, and

$$2\,\text{th}^{-1} \frac{OA}{FA}. \tag{4}$$

In the elliptic coördinates of § 42 the potential at P assumes the various forms

$$\begin{aligned} \log \frac{\text{ch}\,v + 1}{\text{ch}\,v - 1} &= 2 \log \frac{\text{ch}\,v + 1}{\text{sh}\,v} = 2 \log \frac{\text{sh}\,v}{\text{ch}\,v - 1} = \int_v^\infty \frac{dv}{\text{sh}\,v} \\ &= 2\,\text{sh}^{-1} \frac{1}{\text{sh}\,v} = 2\,\text{ch}^{-1} \frac{\text{ch}\,v}{\text{sh}\,v} = 2\,\text{th}^{-1} \frac{1}{\text{ch}\,v} = \text{ch}^{-1} \frac{\text{ch}^2 v + 1}{\text{ch}^2 v - 1}, \text{ etc.} \end{aligned} \tag{5}$$

The Stokes function at P of the line AB is

$$2\pi(PA - PB) = 4\pi a \cos w, \tag{6}$$

an expression much simpler.

III. The conical angle of the ring as seen from F on its axis is

$$2\pi\left(\frac{FO}{Fa} - \frac{FO}{FA}\right) = \frac{FO \times \text{area}}{FA \cdot Fa \cdot \frac{1}{2}(FA + Fa)}, \tag{7}$$

and when the diameters ab, AB are nearly equal, this may be replaced by

$$\frac{FO \times \text{area}}{FA^3}. \tag{8}$$

So also for any strip cut out by two radii; and if the radii make a small angle $d\theta$, the same expression holds when the plane of the strip is tilted about its mean radius OA through any angle, provided FO is replaced by the perpendicular FM on the plane, and the expression becomes

$$\left(\frac{FM}{Fa}-\frac{FM}{FA}\right)d\theta. \tag{9}$$

IV. For a narrow parallel strip such as QQ', of breadth dy, and PR the perpendicular on it from P, the conical angle at P is

$$\frac{PM}{PR^2}(\cos PQR-\cos PQ'R)dy=\frac{PM\times\text{area}}{PR^2\cdot QQ'}\left(\frac{RQ}{PQ}-\frac{RQ'}{PQ'}\right). \tag{10}$$

47. Considering that $V+b\Omega$ is composed of complete I. and II. E. I.'s, the complete III. E. I. which arises in the determination of Ω will serve at the same time for V; and according to the method of dissection we shall find it depend on one of the five following forms, which we shall represent in the following notation, in which the III. E. I. is not restricted to be complete;

$$\Omega(MQ)=\int_0^\theta\frac{aA\cos\theta+a^2}{MQ^2}\frac{bd\theta}{PQ}=\int\frac{a\cdot MY}{MQ^2}\frac{bd\theta}{PQ}, \tag{1}$$

$$\Omega(PY)=\int\frac{PQ^2}{PY^2}\frac{bd\theta}{PQ}, \tag{2}$$

$$\Omega(PZ)=\int\frac{aA\cos\theta+A^2+b^2}{PZ^2}\frac{bd\theta}{PQ}, \tag{3}$$

where PZ is the perpendicular on OQ, and $PZ^2=A^2\sin^2\theta+b^2$;

$$\Omega(PN)=\int\frac{QN^2}{PN^2}\frac{bd\theta}{PQ}, \tag{4}$$

$$\Omega(PR)=\int\frac{a\cos\theta(a\cos\theta+A)}{PR^2}\frac{bd\theta}{PQ}=\int\frac{ON\cdot MN}{PR^2}\frac{bd\theta}{PQ}, \tag{5}$$

and these III. E. I.'s are connected in pairs by the ten equations following, as is verified by differentiation:

$$\begin{aligned}\Omega(PY)-\Omega(PZ)=I_1&=\sin^{-1}\frac{MZ\cdot QZ}{PY\cdot PZ}=\cos^{-1}\frac{PM\cdot PQ}{PY\cdot PZ}\\&=\tfrac{1}{2}\pi-\text{angle between planes } PQY,\ PQZ;\end{aligned} \tag{6}$$

$$\begin{aligned}\Omega(MQ)-\Omega(PZ)=I_2&=\sin^{-1}\frac{MZ\cdot PQ}{PZ\cdot MQ}=\cos^{-1}\frac{MN\cdot MP}{PZ\cdot MQ}\\&=\text{angle between } MPQ,\ QPZ;\end{aligned} \tag{7}$$

$$\Omega(MQ) - \Omega(PY) = I_3 = \sin^{-1}\frac{MZ \cdot MP}{PY \cdot MQ} = \cos^{-1}\frac{QZ \cdot PQ}{PY \cdot MQ} \tag{9}$$

$$= \text{angle between } PQT,\ PQY;$$

$$\Omega(MQ) + \Omega(PN) = I_4 = \sin^{-1}\frac{QN \cdot PM}{PN \cdot MQ} = \cos^{-1}\frac{MN \cdot PQ}{PN \cdot MQ} \tag{9}$$

$$= \text{angle between } PQT,\ QMR;$$

$$\Omega(MQ) + \Omega(PR) = I_5 = \sin^{-1}\frac{QN \cdot PQ}{PR \cdot MQ} = \cos^{-1}\frac{MN \cdot PM}{PR \cdot MQ} \tag{10}$$

$$= \text{angle between } MPQ,\ MQR;$$

$$\Omega(PR) - \Omega(PN) = I_6 = \sin^{-1}\frac{QN \cdot MN}{PR \cdot PN} = \cos^{-1}\frac{PM \cdot PQ}{PR \cdot PN} \tag{11}$$

$$= \tfrac{1}{2}\pi - \text{angle between } QPR,\ QPN;$$

$$\Omega(PR) + \Omega(PY) = I_7 = \sin^{-1} = \frac{\sin\theta(aA\cos\theta + a^2 + b^2)}{PR \cdot PY}$$

$$= \cos^{-1}\frac{PM\cos\theta \cdot PQ}{PR \cdot PY} \tag{12}$$

$$= \tfrac{1}{2}\pi - \text{angle between } QPR,\ QPY;$$

$$\Omega(PR) + \Omega(PZ) = I_8 = \sin^{-1}\frac{PM\sin\theta \cdot PQ}{PR \cdot PZ} = \cos^{-1}\frac{aA\sin^2\theta - b^2\cos\theta}{PR \cdot PZ} \tag{13}$$

$$= \text{angle between } QPZ,\ QPR;$$

$$\Omega(PY) - \Omega(PN) = I_9 = \sin^{-1}\frac{PM\sin\theta \cdot PQ}{PY \cdot PN}$$

$$= \cos^{-1}\frac{\sin\theta(aA\cos\theta + A^2 + b^2)}{PY \cdot PN} \tag{14}$$

$$= \text{angle between } QPN,\ QPY;$$

$$\Omega(PZ) + \Omega(PN) = I_{10} = \sin^{-1}\frac{\sin\theta(aA\cos\theta + A^2 + b^2)}{PZ \cdot PN}$$

$$= \cos^{-1}\frac{PM\cos\theta \cdot PQ}{PZ \cdot PN} \tag{15}$$

$$= \tfrac{1}{2}\pi - \text{angle between } QPN,\ QPZ.$$

In a complete III. E. I., when θ ranges from 0 to 2π, the I term disappears or is 2π, and

$$\begin{aligned}\Phi &= 2\pi - \Omega = \Omega(MQ) = \Omega(PY) = \Omega(PZ) \\ &= 2\pi - \Omega(PN) = 2\pi - \Omega(PR).\end{aligned} \tag{16}$$

We shall require in conjunction with $\Omega(PZ)$ the associated integral

$$\Omega_1(PZ) = \int \frac{\sqrt{(A^2+b^2)}(a + A\cos\theta)}{PZ^2}\frac{bd\theta}{PQ}, \tag{17}$$

and then

$$\Omega_1(PZ) + \Omega(PZ) = \int \frac{a + \sqrt{(A^2+b^2)}}{\sqrt{(A^2+b^2)} - A\cos\theta}\frac{bd\theta}{PQ}, \tag{18}$$

$$\Omega_1(PZ) - \Omega(PZ) = \int \frac{a - \sqrt{(A^2+b^2)}}{\sqrt{(A^2+b^2)} + A\cos\theta}\frac{bd\theta}{PQ}, \tag{19}$$

two III. E. I.'s, with parameter f_1F', f_2F', such that

$$\operatorname{sn} f_1F' = \frac{a - \sqrt{(A^2+b^2)}}{r_2}, \qquad \operatorname{sn} f_2F' = \frac{r_1}{a + \sqrt{(A^2+b^2)}}, \tag{20}$$

leading to a simple geometrical construction; and it is found that $2\pi - \Omega_1(PZ)$ is the conical angle subtended by the circle at P_1 on OP produced.

So too in conjunction with $\Omega(PR)$ we require

$$\Omega_1(PQ) = \int \frac{\sqrt{(a^2+b^2)}(A + a\cos\theta)}{PR^2}\frac{bd\theta}{PQ}, \tag{21}$$

$$\Omega_1(PR) + \Omega(PR) = \int \frac{A + a\cos\theta}{\sqrt{(a^2+b^2)} - a\cos\theta}\frac{bd\theta}{PQ}, \tag{22}$$

$$\Omega_1(PR) - \Omega(PR) = \int \frac{A + a\cos\theta}{\sqrt{(a^2+b^2)} + a\cos\theta}\frac{bd\theta}{PQ}, \tag{23}$$

and $\Omega_1(PR)$ is the conical angle at P_2 where P_1P_2 passes through B, or P_2P' is parallel to AB.

48. Thus with sector elements $\frac{1}{2}a^2d\theta$ for the determination of V, the potential dV at P of the element about OQ is given by

$$\begin{aligned}\frac{dV}{d\theta} &= \int_0^a \frac{rdr}{PQ} = \int \frac{r + A\cos\theta}{PQ}d\theta - \int \frac{A\cos\theta d\theta}{PQ} \\ &= PQ - PO - 2A\cos\theta \operatorname{th}^{-1}\frac{a}{PQ + PO},\end{aligned} \tag{1}$$

of which the first part is by Lemma I the potential of the strip $OQq'o$, between OQ, oq' radiating from Z, and the second is by Lemma II the potential of a shaving of uniform breadth $d\theta \cdot A\cos\theta$ and length OQ.

As required in integration by parts, the second part of (1) can be written

$$
\begin{aligned}
&2\frac{d}{d\theta}\left(A\sin\theta\,\mathrm{th}^{-1}\frac{a}{PQ+PO}\right)-\frac{a^2A^2\sin^2\theta}{PQ\cdot PO+aA\cos\theta+A^2+b^2}\frac{1}{PQ}\\
&=2\frac{d}{d\theta}\left(A\sin\theta\,\mathrm{th}^{-1}\frac{a}{PQ+PO}\right)-\frac{PQ\cdot PO-aA\cos\theta-A^2-b^2}{PZ^2}\frac{A^2\sin^2\theta}{PQ}\\
(3)\qquad &=2\frac{d}{d\theta}\left(A\sin\theta\,\mathrm{th}^{-1}\frac{a}{PQ+PO}\right)-PO\left(1-\frac{b^2}{PZ^2}\right)\\
&\qquad\qquad+\frac{aA\cos\theta+A^2+b^2}{PQ}-\frac{bd\Omega(PZ)}{d\theta};
\end{aligned}
$$

So also, if $d\Omega$ is the conical angle subtended by the sector element about OQ, the application of Lemmas III and IV shows that

$$
\begin{aligned}
&\frac{d\Omega}{d\theta}=\frac{PM}{PO}-\frac{PM}{PQ}-\frac{PM}{PZ^2}\left(\frac{QZ}{PQ}-\frac{OZ}{PO}\right)A\cos\theta\\
(3)\qquad &=b\frac{PO}{PZ^2}-\frac{d\Omega(PZ)}{d\theta}=b\frac{PO}{PZ^2}+\frac{dI_1}{d\theta}-\frac{d\Omega(PY)}{d\theta},
\end{aligned}
$$

so that

$$
(4)\qquad \frac{dV}{d\theta}+b\frac{d\Omega}{d\theta}=\frac{a^2+aA\cos\theta}{PQ}-2\frac{d}{d\theta}\left(A\sin\theta\,\mathrm{th}^{-1}\frac{a}{PQ+PO}\right)
$$

and thence the previous expressions in (13), § 43.

49. The dissection which leads directly to MINCHIN's integral in the Philosophical Magazine, February, 1894, is made by lines radiating from M.

Denoting the angle AMQ by η,

$$
(1)\qquad \tfrac{1}{2}MQ^2d\eta=\text{sector element about } MQ=\tfrac{1}{2}MY\cdot ad\theta=\tfrac{1}{2}(a^2+aA\cos\theta)d\theta,
$$

and by Lemma III,

$$
(2)\qquad d\Omega=\left(1-\frac{PM}{PQ}\right)d\eta,
$$

$$
(3)\qquad \frac{d\Omega}{d\theta}=\frac{d\eta}{d\theta}-\frac{a^2+aA\cos\theta}{MQ^2}\frac{b}{PQ}=\frac{d\eta}{d\theta}-\frac{d\Omega(MQ)}{d\theta},
$$

so that for any incomplete arc AQ

$$
(4)\qquad \Omega=\eta-\Omega(MQ).
$$

Now with the substitution of § 26 and in addition

$$
(5)\qquad MQ^2=m^2(\tau-t),
$$

so that

$$
(6)\qquad b^2=m^2(t_1-\tau),\quad (a\sim A)^2=m^2(\tau-t_2),\quad (a+A)^2=m^2(\tau-t_3),
$$

then when M is inside the circle

$$(7)\qquad a - A = m\sqrt{(\tau - t_2)}, \qquad a = \tfrac{1}{2}m\left[(\tau - t_2) + \sqrt{(\tau - t_3)}\right],$$

$$(8)\qquad aA\cos\theta = \tfrac{1}{4}m^2(t_2 + t_3 - 2t),$$

$$(9)\qquad aA\cos\theta + a^2 = \tfrac{1}{2}m^2\left[\tau - t + \sqrt{(\tau - t_2\cdot\tau - t_3)}\right].$$

Integrating round the circle, the angle η makes a complete revolution when M is inside, and

$$\Omega = 2\pi - \int_{t_3}^{t_2}\left[1 + \frac{\sqrt{(\tau - t_2\cdot\tau - t_3)}}{\tau - t}\right]\frac{\sqrt{(t_1 - \tau)}\,dt}{\tfrac{1}{2}\sqrt{T}}$$

$$(10)\qquad = 2\pi - 2\sqrt{\frac{t_1 - \tau}{t_1 - t_3}}\int\frac{\sqrt{(t_1 - t_3)}\,dt}{\sqrt{T}} - 2\int\frac{\tfrac{1}{2}\sqrt{(-T')}}{\tau - t}\frac{dt}{\sqrt{T}}$$

$$= 2\pi - 2Fc'\operatorname{sn}2fF' - 2B(2fF')$$

$$= 2\pi(1 - f) - 2F\operatorname{zn}2fF' - 2Fc'\operatorname{sn}2fF';$$

hence MINCHIN's result obtained directly.

If M is outside the circle, the angle η oscillates; but we must take

$$(11)\qquad a = \tfrac{1}{2}m\left[\sqrt{(\tau - t_3)} - \sqrt{(\tau - t_2)}\right],$$

and now

$$(12)\qquad \Omega = 2\int\left[\frac{\sqrt{(\tau - t_2\cdot\tau - t_3)}}{\tau - t} - 1\right]\frac{dt}{\sqrt{T}} = 2\pi f + 2F\operatorname{zn}2fF' - 2Fc'\operatorname{sn}fF',$$

equivalent to a change of f into $1 - f$, as in going from P to P'' so that (10) may be taken to hold for all cases.

As P travels round the circle CPC' the quantities c, k, F, E, K, H do not vary, and f may be taken as the independent variable.

Starting from C in the plane of the disc, where $f = 0$,

$$(13)\qquad \Omega = 2\pi, \qquad V = 4a\frac{E + c'F}{1 + c'} = 4aH, \qquad \frac{V}{AC} = 2(E + c'F).$$

At C' outside the disc in its plane, $f = 1$,

$$(14)\qquad \Omega = 0, \qquad V = 4a\frac{E - c'F}{1 - c'} = 4a\frac{H - k'^2K}{k}, \qquad \frac{V}{AC'} = 2(E - c'F).$$

As P travels to P'', f increases from f to $f'' = 1 - f$, so that

$$(15)\qquad \Omega'' = 2\pi f + 2F\operatorname{zn}2fF' - 2Fc'\operatorname{sn}2fF',$$

$$(16)\qquad \Omega + \Omega'' = 2\pi - 4Fc'\operatorname{sn}2fF' = 2\pi - 4F\sin\psi', \qquad \Omega - \Omega'' = 2\pi - 4B(2fF'),$$

and Ω'' is the apparent area of the coaxial circle through P seen from any point on the circle AB.

$$(17)\qquad \frac{V}{AP}+\frac{V''}{AP''}=4E-4F\sin^2\psi'-(\Omega+\Omega'')\sin\psi'=4E-2\pi\sin\psi',$$

which is MINCHIN's equation (23), p. 212, Philosophical Magazine, February, 1894.

When P reaches P', f has increased to $f'=1+f$,

$$(18)\qquad \Omega'=-2\pi f+2F\,\mathrm{zn}\,2fF'+2F\sin\psi',$$

$$(19)\qquad \Omega-\Omega'=2\pi,$$

but

$$(20)\qquad V'-V''=0.$$

At P''', f has reached $f'''=2-f$;

$$(21)\qquad \Omega-\Omega'''=4\pi,\qquad V-V'''=0,$$

so that V has no cyclic constant as P describes a curve linked with the circular edge; a cylic constant like the 4π for Ω would imply a non-conservative potential and the possibility of the creation of energy, called otherwise Perpetual Motion.

50. Dissected by QN into strips $ydx=a\sin^2\theta d\theta$, then by Lemma IV, § 46,

$$(1)\qquad d\Omega=\frac{bdx}{PN^2}\frac{a\sin\theta}{PQ}=\frac{a^2\sin^2\theta}{PN^2}\frac{bd\theta}{PQ},\qquad \frac{d\Omega}{d\theta}=\frac{d\Omega(PN)}{d\theta},$$

so that χ of § 27 is the appropriate coördinate angle, leading to E. F.'s of hK and fK', and

$$(2)\qquad \begin{aligned}\Omega&=\int_0^{2\pi}\frac{a^2\sin^2\theta}{\left(\frac{r_1+r_2}{2a}\right)^2CQ^2-a^2\sin^2\theta}\frac{bd\theta}{PQ}\\&=\int^{\frac{1}{2}\pi}\frac{4\sin^2\chi}{\left(\frac{r_1+r_2}{2a}\right)^2-\sin^2\chi}\frac{b}{r_1+r_2}\frac{d\chi}{\Delta\chi},\end{aligned}$$

which is reduced to the form A of § 5 by putting

$$(3)\qquad \sin^2\chi=\frac{s_1-s_3}{s-s_3},\qquad \left(\frac{r_1-r_2}{2a}\right)^2=\frac{s_1-s_3}{\sigma-s_3},$$

and this makes

$$(4)\qquad \Omega=4\int_{s_1}^{\infty}\frac{\frac{1}{2}\sqrt{(-\Sigma)}}{s-\sigma}\frac{ds}{\sqrt{S}}=4A(fK')=2\pi(1-f)-4K\,\mathrm{zn}\,fK'.$$

With the variable y of § 29

$$(5)\qquad 1-y=\frac{PN^2}{PQ^2},\qquad \frac{y}{1-y}=\frac{a^2\sin^2\theta}{PN^2},$$

and denoting the value of Y for $y=1$ by Y',

$$(6)\qquad Y'=4(y_1-1)(y_2-1)=-4\frac{b^2}{A^2}=-4\frac{b^2}{a^2}y_1y_2,$$

so that from (11) § 29

$$(7)\qquad \begin{aligned}\Omega&=\int_0^{y_2}\frac{2y}{1-y}\frac{b}{a}\frac{\sqrt{(y_1y_2)}\,dy}{\sqrt{Y}}=\int\left(\frac{1}{1-y}-1\right)\frac{2\sqrt{(-Y')}\,dy}{\sqrt{Y}}\\&=4B(f'K')-2K\sqrt{\frac{-Y'}{y_1}},\end{aligned}$$

in which

$$(8)\qquad \mathrm{dn}^2 f'K'=\frac{\sigma'-s_3}{s_1-s_3}=\frac{1}{y_1}=\left(\frac{2A}{r_1+r_2}\right)^2,\qquad \mathrm{dn}^2 fK'=\left(\frac{r_1-r_2}{2A}\right)^2,$$

from (8) § 45,

$$(9)\qquad \mathrm{dn}^2 f'K'\,\mathrm{dn}^2 fK'=k^2,\qquad f+f'=1,$$

$$(10)\qquad \begin{aligned}\Omega&=2\pi(1-f)+4K\,\mathrm{zn}\,(1-f)K'-4K\frac{2b}{r_1+r_2}\\&=2\pi(1-f)-4K\,\mathrm{zn}\,fK',\end{aligned}$$

as before in (4).

So also by Lemma II

$$(11)\qquad dV=dx_2^1\log\frac{PQ+a\sin\theta}{PQ-a\sin\theta}=a\sin\theta d\theta\,\mathrm{th}^{-1}\frac{a\sin\theta}{PQ},$$

and as before in § 48 we may transform this into

$$(12)\quad \frac{dV}{d\theta}=-\frac{d}{d\theta}\left[(a\cos\theta+A)\,\mathrm{th}^{-1}\frac{a\sin\theta}{PQ}\right]+\frac{a^2+aA\cos\theta}{PQ}-b\frac{d\Omega(PN)}{d\theta},$$

so that, as before in (4) § 48, over the circle,

$$(13)\qquad V+b\Omega=\int_0^{2\pi}(a+A\cos\theta)\frac{ad\theta}{PQ}=a\frac{dV}{da}+A\frac{dV}{dA}.$$

To connect with the E. F.'s of hF and $2fF'$, by means of $\Omega(MQ)$,

$$(14)\qquad \frac{d\Omega}{d\theta}=\frac{d\Omega(PN)}{d\theta}=\frac{dI_4}{d\theta}-\frac{d\Omega(MQ)}{d\theta},$$

$$(15)\qquad \Omega=2\pi-\Omega(MQ),$$

as in (4) § 49.

51. Dissecting by lines QQ'' parallel to AB into strips

$$QQ''dy = 2a^2 \cos^2 \theta d\theta, \tag{1}$$

of which by Lemma IV

$$d\Omega = \frac{bdy}{PR^2}\left(\frac{QR}{PQ} \pm \frac{RQ''}{PQ''}\right) = \left(\frac{a^2\cos^2\theta + aA\cos\theta}{PQ} + \frac{a\cos^2\theta - aA\cos\theta}{PQ''}\right)\frac{bd\theta}{PR^2}, \tag{2}$$

we may take

$$\frac{d\Omega}{d\theta} = \frac{a^2\cos^2\theta + aA\cos\theta}{PR^2}\,\frac{b}{PQ} = \frac{d\Omega(PR)}{d\theta}, \tag{3}$$

and then integrate round the circle from 0 to 2π.

So also from Lemma II, with the same dissection,

$$dV = dy \log\frac{PQ + PQ'' + 2a\cos\theta}{PQ + PQ'' - 2a\cos\theta} = 2dy\,\mathrm{th}^{-1}\frac{2a\cos\theta}{PQ + PQ} = 2dy\,\mathrm{th}^{-1}\frac{PQ - PQ''}{2A}, \tag{4}$$

$$\begin{aligned}\frac{dV}{d\theta} - 2\frac{d}{d\theta}\left(y\,\mathrm{th}^{-1}\frac{PQ - PQ''}{2A}\right) &= \frac{4a^2A^2\sin^2\theta}{4A^2 - (PQ - PQ'')^2}\left(\frac{1}{PQ} + \frac{1}{PQ''}\right)\\ &= \tfrac{1}{2}a^2\sin^2\theta\,\frac{PQ\cdot PQ'' - A^2 + a^2 - b^2}{PR}\left(\frac{1}{PQ} + \frac{1}{PQ''}\right)\\ &= (PR^2 - b^2)\left(\frac{a^2 + aA\cos\theta + b^2}{PR^2\cdot PQ} + \frac{a^2 - aA\cos\theta + b^2}{PQ^2\cdot PR'}\right),\end{aligned} \tag{5}$$

and here again, in integrating round the circle we can take

$$V = \int_0^{2\pi}(PR^2 - b^2)\,\frac{a^2 + aA\cos\theta + b^2}{PR^2}\,\frac{d\theta}{PQ}, \tag{6}$$

and $V + b\Omega$ is the same as before in (13) § 50.

52. Finally there is the integration for V and Ω required when the circle is dissected into concentric rings; and now dV/dr is the potential of a circle of radius r, and

$$\frac{dV}{da} = 4F\frac{a}{r_1} = 8K\frac{a}{r_1 + r_2}, \tag{1}$$

$$V = \int_0^a 4F\frac{a}{r_1}\,da = \int 8K\frac{a}{r_1 + r_2}\,da, \tag{2}$$

of which the result is known already by the other dissections, although the integration appears intractable at first sight.

So also, from (3) § 31

$$\frac{d\Omega}{da} = -\frac{d^2V}{da\,db} = \frac{4ab}{r_1^3}\,\frac{E}{c'^2} = \frac{32ab}{(r_1 + r_2)^3}\,\frac{2H - k'^2K}{k'^4}, \tag{3}$$

leading to an integral of appearance still more intractable.

53. Treating the potential U of the solid cylinder in the preceding manner, as given by a function homogeneous in the second degree of a, A, b,

$$(1) \qquad 2U = a\frac{dU}{da} + A\frac{dU}{dA} + b\frac{dU}{db},$$

the first term is a times the potential of the curved surface, or skin potential as it may be called of the cylinder, so that

$$(2) \qquad \frac{1}{a}\frac{dU}{da} = \int_0^{2\pi} \text{th}^{-1}\frac{b}{PQ}\,d\theta = \int_{t_3}^{t_2} 2\text{th}^{-1}\sqrt{\frac{t_1-\tau}{t_1-t}}\cdot\frac{dt}{\sqrt{(t_2-t\cdot t-t_3)}},$$

not an elliptic integral; but dU/dA and dU/db are given in §§ 44, 54 as the potential of the cylinder due to transverse and longitudinal magnetization.

We shall apply the same method in § 64 to Mr. Coleridge FARR's problem (Proceedings of the Royal Society, November, 1898), to determine the electro-magnetic force of a cylindrical coil of finite thickness, where a similar intractable integral is encountered, in conjunction with elliptic integrals, which we consider tractable.

A graphical representation can be given on the Mercator chart of the integral dW/ada in (2) by denoting the angle MQP by ϕ, so that

$$(3) \qquad \text{th}^{-1}\frac{b}{PQ} = \text{th}^{-1}\sin\phi = \log\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi\right);$$

then if ϕ, θ are taken as latitude and longitude, and the curve drawn on the Mercator chart, the area will represent the integral (2).

Or else $\text{th}^{-1}(b/PQ)$ can be expanded in odd powers of $b/PQ = b/(r_1\Delta\omega)$, and then the integral of each term such as

$$(4) \qquad D_n = \int_0^{\frac{1}{2}\pi} \frac{d\omega}{(\Delta\omega)^{2\pi+1}}$$

can be expressed by means of F and E, by means of the recurring relation

$$(5) \qquad (2n-1)c'^2 D_n - (2n-2)(1+c'^2)/D_{n-1} + (2n-3)D_{n-2} = 0,$$

starting with $D_0 = F$, $D_1 = E/c'^2$.

Mutual induction of a circle and coaxial helix.

54. The III. E. I. arises in the calculation of M between a helix and a coaxial ring; this has been carried out by Viriamu JONES in the Philosophical Magazine, January, 1889; Philosophical Transactions (1891); Proceedings of the Royal Society, December, 1897; consult also WEBSTER, *Electricity and Magnetism*, p. 457; MASCART and JOUBERT, *Electricity* (translated by ATKINSON); and GRAY, *Absolute Measurements in Electricity*, volume II, p. 313.

It results from the form of the integral that M is the same as between the ring and a uniform current sheet in a coaxial cylinder replacing the helix, in which the linear helical current γ_h and the superficial current γ across unit height of the cylinder are connected with p the pitch of the helix by the relation

$$\gamma_h = 2\pi p\gamma. \tag{1}$$

For if N denotes the number of windings and Θ the total angle of the helix of height b,

$$N\gamma_h = b\gamma, \qquad \text{where} \qquad \Theta = \frac{b}{p}, \qquad N = \frac{\Theta}{2\pi} = \frac{b}{2\pi p}. \tag{2}$$

The potential of this current sheet has been worked out by MINCHIN in the Philosophical Magazine, February, 1894, as being the same as the potential of a cylinder magnetized longitudinally, or of two end plates of the cylinder, positive and negative; the expression for the potential of a plate has been cited already in § 43 above.

We notice now that M is the Stokes function of the two end plates, or of the cylinder magnetized longitudinally, and that $M\gamma_h/2\pi A$ is the magnetic potential in the plane $x0z$ at a distance A from the axis of the cylinder magnetized transversely parallel to this plane with intensity γ; and denoting this magnetic potential by $\Omega'\gamma$, and by W the potential of the solid cylinder,

$$\frac{M\gamma_h}{2\pi A} = \Omega'\gamma, \qquad \text{or} \qquad Mp = A\Omega'; \tag{3}$$

and

$$A\Omega' = -A\frac{dU}{dx} = \int_0^b\int_0^{2\pi} \frac{-aA\cos\theta\, d\theta\, dz}{\sqrt{(\alpha^2+z^2)}} = \int_0^{2\pi} -aA\cos\theta\, d\theta\, \text{th}^{-1}\frac{b}{PQ}, \tag{4}$$

the first integral expression given by Viriamu JONES, employing his notation of α^2 and α^2+z^2 for MQ^2 and PQ^2, but using z instead of x, and changing his θ into $\pi-\theta$, to agree with the preceding treatment.

The result for Mp is the same whether $2a$ denotes the diameter of the helix and $2A$ of the ring, or vice-versa as with Viriamu JONES, being symmetrical in a and A.

Integration by parts enables him to replace (4) by

$$A\Omega' = \int_0^{2\pi} \frac{a^2A^2b\sin^2\theta\, d\theta}{\alpha^2\sqrt{(\alpha^2+b^2)}} = \tfrac{1}{2}\int(-aA\cos\theta + a^2)\frac{b\,d\theta}{PQ} - \tfrac{1}{2}(a^2-A^2)\,\Omega(MQ), \tag{5}$$

integrals discussed already in §§ 33, 49; and (5) is the equivalent of the double integral

$$A\Omega' = \int_0^b\int_0^{2\pi} \frac{a^2A^2\sin^2\theta\, d\theta\, dz}{(\alpha^2+z^2)^{\frac{3}{2}}} = \int_0^b\int_0^{\pi} \frac{2a^2\, d\theta\, dz}{PQ^3}, \tag{6}$$

showing that the magnetic potential Ω' at P of the cylinder magnetized parallel to OM is the same as that of the surface magnetized circumferentially with intensity $a\sin\theta$.

It is the second form (5) which is reducible immediately to the standard elliptic form, and by the substitution in §§ 26, 49, writing

$$MQ^2 = \alpha^2 = A^2 + 2Aa\cos\theta + a^2 = m^2(\tau - t), \tag{7}$$

$$PQ^2 = \alpha^2 + b^2 = m^2(t_1 - t), \tag{8}$$

$$b^2 = m^2(t_1 - \tau), \qquad (a \sim A)^2 = m^2(\tau - t_2), \qquad (a + A)^2 = m^2(\tau - t_3), \tag{9}$$

and thence from (5)

$$A\Omega' = \tfrac{1}{2}b\int_{t_3}^{t_2} \frac{m^4(t_2 - t)(t - t_3)}{m^2(\tau - t)} \frac{dt}{\frac{\sqrt{(t_2 - t\cdot t - t_3)}}{m\sqrt{(t_1 - t)}}} \tag{10}$$

$$= mb\int \frac{(t_2 - t)(t - t_3)\,dt}{(\tau - t)\sqrt{T}},$$

$$pM = A\Omega' = mb\int\left(t - t_3 + \tau - t_2 - \frac{\tau - t_2\cdot\tau - t_3}{\tau - t}\right)\frac{dt}{\sqrt{T}}$$

$$= r_1 b\left[\int \frac{t - t_3}{\sqrt{(t_1 - t_3)}}\frac{dt}{\sqrt{T}} + \frac{\tau - t_2}{t_1 - t_3}\int\frac{\sqrt{(t_1 - t_3)}\,dt}{\sqrt{T}}\right. \tag{11}$$

$$\left. - \sqrt{\frac{\tau - t_2\cdot\tau - t_3}{t_1 - \tau\cdot t_1 - t_3}}\int \frac{\frac{1}{2}\sqrt{(-T')}}{\tau - t}\frac{dt}{\sqrt{T}}\right].$$

Hence, on fig. A,

$$\sqrt{\frac{t_1 - \tau}{t_1 - t_2}} = \frac{b}{r_2} = \sin\psi, \qquad \sqrt{\frac{\tau - t_2}{t_1 - t_2}} = \frac{a - A}{r_2} = \cos\psi, \tag{12}$$

$$\sqrt{\frac{\tau - t_3}{t_1 - t_3}} = \frac{a + A}{r_1} = \cos\psi' = \Delta\psi, \qquad \psi = \operatorname{am} 2fF',$$

so that

$$pM = r_1 b\left[F - E + Fc'^2\cos^2\psi - \cot\psi\cos\psi' B(2fF')\right]. \tag{13}$$

$$\frac{M}{r_1\Theta} = F - E - \cot\psi\cos\psi' A(1 - 2f)F', \tag{14}$$

the equivalent of Viriamu JONES's result in (6) p. 198, Proceedings of the Royal Society, 1898.

We shall replace his angle β by ψ_1 to avoid confusion with MAXWELL's modular angle β, and then

$$\psi_1 = \operatorname{am}(1 - 2f)F', \quad \sin\psi_1' = \frac{\tan\psi'}{\tan\psi} = \frac{a - A}{a + A}\frac{r_1}{r_2}, \quad \cos\psi_1' = \frac{\sqrt{(aA)}}{\frac{1}{2}(a + A)}, \tag{15}$$

so that the angle ψ_1' is constructed on fig. A by drawing the circle on the diameter AM cutting the axis OF in G, and then drawing OP_1 perpendicular to the radius GO' of this circle, then the angle OAP_1 or OGO' is ψ_1'.

55. With the notation of § 47, equation (5) § 54, is written

$$(1)\quad 2A\Omega' + (a^2 - A^2)\Omega(MQ) = \int (a^2 - aA\cos\theta)\frac{bd\theta}{PQ}.$$

$$(2)\quad 2A\Omega' + (a^2 - A^2)\Omega(PY) = \int (a^2 - aA\cos\theta)\frac{bd\theta}{PQ} - (a^2 - A^2)I_3;$$

and in a complete revolution of θ from 0 to 2π,

$$(3)\quad \begin{aligned}\Omega(MQ) &= \int_0^\pi \frac{MQ^2 + a^2 - A^2}{MQ^2}\frac{bd\theta}{PQ} = b\int\frac{d\theta}{PQ} + \int\frac{a^2 - A^2}{MQ^2}\frac{bd\theta}{PQ}\\ &= 2F\frac{b}{r_1} + 2B(2fF') = 2Fc'\operatorname{sn} 2fF' + 2B(2fF')\end{aligned}$$

equivalent to the result in (10) § 49.

Also I_3 vanishes in a complete revolution of θ, and from (16) § 47,

$$\Omega(PY) = \Phi = 4B(fK'),$$

so that

$$(4)\quad \begin{aligned}2A\Omega' + 4(a^2 - A^2)B(fK') &= \int_0^{2\pi} (a^2 - aA\cos\theta)\frac{bd\theta}{PQ}\\ &= \frac{8a^2bK}{r_1 + r_2} + 2b(r_1 + r_2)(K - H),\end{aligned}$$

in which

$$(5)\quad \frac{2a}{r_1 + r_2} = \frac{1}{\operatorname{ch} v} = \operatorname{dn} fK',$$

$$(6)\quad pM = A\Omega' = b(r_1 + r_2)K\operatorname{dn}^2 fK' + b(r_1 + r_2)(K - H) - 2(a^2 - A^2)B(fK'),$$

in the form given in (5) § 44, the quadric transformation of (13) § 54.

This again, by means of (2), (3), § 17, can be transformed into

$$(7)\quad \frac{M}{(r_1 + r_2)\Theta} = K - H + \frac{\operatorname{cn} 2fK'}{\operatorname{sn} 2fK'}C(1 - 2f)K'.$$

In the indeterminate case of $2f = 1$ and $C(1 - 2f)K' = \infty$, we have $A = a$; but returning to the integral in (10) § 54, and putting $\tau = t_2$,

$$(8)\quad pM = A\Omega' = \frac{mb}{a}\int\frac{(t - t_3)dt}{\sqrt{T}} = bm\sqrt{(t_1 - t_3)}(F - E) = r_1b(F - E),$$

$$(9)\quad \frac{M}{\Theta} = r_1(F - E) = (r_1 + r_2)(K - H + kK),$$

and this is the coefficient of self induction of the helix on a final turn of itself.

Numerical calculation of the mutual induction.

56. The numerical calculation of M for given dimensions a, A, b now requires the five operations following, in the numerical tables of F. E., vol. II, LEGENDRE, and it is useful to check the calculation on fig. A, drawn to scale.

I. Calculate ψ and ψ' from

$$(1) \qquad \tan\psi = \frac{b}{a-A}, \qquad \tan\psi' = \frac{b}{a+A};$$

and then

$$r_1 = (a+A)\sec\psi', \qquad r_2 = (a-A)\sec\psi.$$

II. Calculate ψ_1 and the modular angle γ from

$$(2) \qquad \sin\psi_1 = \frac{\cos\psi}{\cos\psi'}, \qquad \cos\gamma = \frac{\sin\psi'}{\sin\psi}.$$

III. Calculate f by LEGENDRE's Table IX from

$$(3) \qquad 2f = \frac{F(\psi, c')}{F(\frac{1}{2}\pi, c')}, \qquad 1-2f = \frac{F(\psi_1, c')}{F(\frac{1}{2}\pi, c')}.$$

IV. Calculate $\operatorname{zn}(1-2f)F'$ by Table IX from

$$(4) \qquad \operatorname{zn}(1-2f)F' = E(\beta', c') - (1-2f)E(\tfrac{1}{2}\pi, c').$$

V. Look out

$$(5) \qquad F = F(\tfrac{1}{2}\pi, c), \qquad E = E(\tfrac{1}{2}\pi, c).$$

When f is one of the rational fractions of (1) § 4, we can express c, c', $\operatorname{sn} 2fF'$, $\operatorname{zn} 2fF'$, $\cdots$, as algebraical functions of a parameter, leaving F and E as the only transcendents, and these are tabulated by LEGENDRE in his Table I to an extra degree of accuracy.

In the numerical application of Viriamu JONES, Philosophical Transactions (1891), Proceedings of the Royal Society (1897),

$$2a = 21.02673, \qquad 2A = 13.01997, \qquad 2b = 5.02480 \text{ (inches)},$$

$$a - A = 4.00338, \qquad a + A = 17.02335, \qquad b = 2.51240;$$

and working with four figure logarithms we find $\psi = 32°\,3'$, $\psi' = 8°\,24'$, $\gamma = 74°\,1'$, $\psi_1 = 58°\,58'$; or to the nearest degree, $\gamma = 74°$, $\psi_1 = 59°$, thus avoiding proportional parts in Legendre's Table IX: $F = 2.70807$, $E = 1.08443$, and $F' = 1.60198$, $E(\psi_1, c') = 1.04123$, $1-2f = 0.6500$, $f = 0.1750$, $\pi f = 0.54985$, $E' = 1.54052$, $(1-2f)E' = 1.00134$, $E\psi_1 = 1.01847$, $\operatorname{zn}(1-2f)F' = 0.01713$, $F\operatorname{zn}(1-2f)F' = 0.04638$, $A(1-2f)F' = 0.50347$, $\cot\psi\cos\psi' A(1-2f)F' = 0.7952$, $F - E = 1.6236$, $M/r_1\Theta = 0.8284$, $\Theta = 201\pi$, $r_1 = 17.21$, $M = 9001$ inches, as against 9028 inches, given by Viriamu JONES.

This rough calculation is retained for the purpose of showing the term requiring extra calculated figures, compared with the parts where fewer figures are retained for the same degree of accuracy; thus of the three chief terms of $M/r_1\Theta$, the first $F-E=1.6236$ is about double the second $\pi f \cot\psi \cos\psi' = 0.8638$, while the remainder $\cot\psi \cos\psi' F \operatorname{zn}(1-2f)F' = 0.0734$.

The modular angle must thus be determined with extra accuracy for the determination of $F-E$; and in this case raising the modular angle by one minute to $74°\,1'$ will make $F-E=1.6258$, and raise M to 9024.

A diminution of 0.1 per cent. in f in the second term will raise M about 0.05 per cent. to about 9028.5, without affecting the last term appreciably.

Variation of the induction due to change in the dimensions.

57. Viriamu JONES calculates the effect of small error of measurement dA, da, db in A, a, b in giving a change dM in M, such that

$$(1)\qquad dM=\left(A\frac{\partial M}{\partial A}\right)\frac{dA}{A}+\left(a\frac{\partial M}{\partial a}\right)\frac{da}{a}+\left(b\frac{\partial M}{\partial b}\right)\frac{db}{b},$$

so that 1 per cent. change in A, a, b gives a change in M of

$$(2)\qquad \frac{A}{100}\frac{\partial M}{\partial A},\qquad \frac{a}{100}\frac{\partial M}{\partial a},\qquad \frac{b}{100}\frac{\partial M}{\partial b};$$

and their sum is $M/100$, because M is a homogeneous function of the variables A, a, b, and of the first degree, the total helical angle Θ or number of windings $\Theta \div 2\pi$ being kept constant. But if the pitch p of the helix is kept constant, M is of the second degree in A, a, b.

Differentiating the double integral expression for M in (4) § 54,

$$(3)\qquad M=\frac{\Theta}{b}\int_0^b\int_0^{2\pi}-\frac{Aa\cos\theta\,d\theta\,dz}{\sqrt{(\alpha^2+z^2)}},$$

and supposing the limits of θ to be 0 and 2π in the sequel, we have

$$(4)\qquad \frac{\partial M}{\partial A}=\frac{\Theta}{b}\int\int\frac{-a\cos\theta\,d\theta\,dz}{\sqrt{(\alpha^2+z)}}+\frac{\Theta}{b}\int\int\frac{(A+a\cos\theta)Aa\cos\theta\,d\theta\,dz}{(\alpha^2+z^2)^{\frac{3}{2}}},$$

$$(5)\qquad A\frac{\partial M}{\partial A}=M+\Theta\int\frac{(A^2+Aa\cos\theta)Aa\cos\theta\,d\theta}{\alpha^2\sqrt{(\alpha^2+b^2)}}=\Theta A^2\int\frac{a^2+aA\cos\theta}{MQ^2}\frac{d\theta}{PQ},$$

and similarly

$$(6)\qquad a\frac{\partial M}{\partial a}=M+\Theta\int\frac{(Aa\cos\theta+a^2)Aa\cos\theta\,d\theta}{\alpha^2\sqrt{(\alpha^2+b^2)}}=\Theta a^2\int\frac{aA\cos\theta+A^2}{MQ^2}\frac{d\theta}{PQ},$$

so that

$$(7)\qquad a\frac{dM}{da}+A\frac{dM}{dA}=2M-\Theta\int\frac{-aA\cos\theta d\theta}{PQ},$$

while, keeping Θ constant,

$$(8)\qquad bM=\Theta\int_0^b\int\frac{-aA\cos\theta d\theta dz}{\sqrt{(a^2+z^2)}},$$

$$(9)\qquad b\frac{dM}{db}+M=\Theta\int\frac{-aA\cos\theta d\theta}{PQ},$$

and then

$$(10)\qquad a\frac{dM}{da}+A\frac{dM}{dA}+b\frac{dM}{db}=M,$$

a verification of the homogeneity. Also

$$(11)\qquad \frac{1}{A}\frac{dM}{dA}+\frac{1}{a}\frac{dM}{du}=\Theta\int\frac{d\theta}{PQ}=4F\frac{\Theta}{r_1}=4F\frac{\sin\psi'}{p},$$

$$(12)\qquad \frac{1}{A}\frac{dM}{dA}-\frac{1}{a}\frac{dM}{da}=\Theta\int\frac{a^2-A^2}{MQ^2}\frac{d\theta}{PQ}=\frac{4B(2fF')}{p},$$

$$(13)\qquad \frac{1}{A}\frac{dM}{dA}=\frac{2}{p}[Fc'\operatorname{sn}2fF'+B(2fF')]=\frac{4B(fK')}{p},$$

$$(14)\qquad \frac{1}{a}\frac{dM}{du}=\frac{2}{p}[Fc'\operatorname{sn}2fF'-B(2fF')]=-\frac{4A(1-f)K'}{p},$$

and we see now that the conditions are satisfied that this M of Viriamu Jones is the Stokes function of Minchin's pair of end circular plates of density $\gamma=\pm 1/2\pi p$, corresponding to a helical current $\gamma_h=1$, and

$$(15)\qquad 2M=a\frac{dM}{da}+A\frac{dM}{dA}+b\frac{dM}{db}+M,$$

$$(16)\qquad pM=-2a^2\cdot A(1-f)K'+2A^2\cdot B(fK')+b(r_1+r_2)(K-H),$$

in agreement with (6) § 55 above.

58. Thus the magnetic components for the helical current γ_h, axial and transversal, are, for magnetic potential Ω,

$$(1)\qquad \frac{d\Omega}{db}=-\frac{\gamma_h}{2\pi A}\frac{dM}{dA}=-4\gamma B(fK')=-\gamma(2\pi f+4K\operatorname{zn}fK'),$$

and, now keeping p constant in (8) § 57,

$$(2)\qquad \begin{aligned}\frac{d\Omega}{dA}&=\frac{\gamma_h}{2\pi A}\frac{dM}{db}=\frac{\gamma}{A}\int_0^{2\pi}\frac{-aA\cos\theta d\theta}{PQ}\\&=4\gamma\frac{r_1+r_2}{2A}(K-H)=4\gamma\operatorname{dn}fK'\frac{K-H}{k}=4\gamma\frac{K-H}{\operatorname{dn}(1-f)K'}.\end{aligned}$$

With $b = \infty$ we must take $t_1 = \infty$, and the integrals in (10) § 54 reduce to

$$(3) \qquad pM = A\Omega' = m^2 \int_{t_3}^{t_2} \frac{t_2 - t \cdot t - t_3}{\tau - t} \frac{dt}{\sqrt{(4 \cdot t_2 - t \cdot t - t_3)}}$$

$$(4) \qquad \begin{aligned} &= m^2 \int \left(t - t_3 + \tau - t_2 \frac{\tau - t_2 \cdot \tau - t_3}{\tau - t} \right) \frac{dt}{\sqrt{(4 \cdot t_2 - t \cdot t - t_3)}} \\ &= m^2 \left[\frac{t_2 - t_3}{2} + \tau - t_2 - \sqrt{(\tau - t_2 \cdot \tau - t_3)} \right] \tfrac{1}{2}\pi \end{aligned}$$

with

$$(5) \quad 4aA = m^2(t_2 - t_3), \quad (a \sim A)^2 = m^2(\tau - t_2), \quad (a + A)^2 = m^2(\tau - t_3),$$

so that

$$(6) \qquad pM = \pi A^2 (A < a) \qquad \text{or} \qquad \pi a^2 (A > a).$$

Otherwise, with $b = \infty$ in (5) § 54,

$$(7) \qquad pM = A\Omega' = \int_0^{2\pi} \frac{a^2 A^2 \sin^2 \theta d\theta}{a^2 + 2aA \cos \theta + A^2} = \pi A^2 \text{ or } \pi a^2,$$

employing the substitution of § 26.

Thus for the infinite helix, in the plane of the end AB,

$$(8) \qquad \frac{d\Omega}{db} = -\frac{\gamma_h}{p} = -2\pi\gamma \qquad \text{or} \qquad O,$$

and at M or M',

$$(9) \qquad \frac{d\Omega}{dA} = 4\gamma \frac{K_1 - H_1}{k_1} \qquad \text{or} \qquad 4\gamma(K_1 - H_1),$$

$$(10) \qquad k_1 = \frac{OM}{OA} \qquad \text{or} \qquad \frac{OA}{OM'}.$$

Thence by subtraction, for a helix extending from b to infinity,

$$(11) \qquad \frac{d\Omega}{db} = 4\gamma A (fK'),$$

$$(12) \qquad \frac{d\Omega}{dA} = 4\gamma \left[\frac{K - H}{\operatorname{dn}(1 - f)K'} - \frac{K_1 - H_1}{k_1} \right],$$

with $2f \lesseqgtr 1$, according as $A \lesseqgtr a$.

Potential energy of two coaxial helices.

59. There remains still the evaluation of the potential energy of two coaxial helical currents or of their equivalent cylindrical current sheets, given by V. JONES in § 13, p. 202, Proceedings of the Royal Society, 1897, and this is the same as the mutual potential energy of the two pairs of equivalent end plates. Viriamu JONES shows that this energy depends on the integral

$$F(z) = \frac{Aa}{p}\int_0^{2\pi} \cos\theta d\theta f(z), \tag{1}$$

$$f(z) = z\log\left[\sqrt{\left(1+\frac{z^2}{\alpha^2}\right)}+\frac{z}{\alpha}\right]-\sqrt{(\alpha^2+z^2)}, \quad f'(z)=\left[\log\sqrt{\left(1+\frac{z^2}{\alpha^2}\right)}+\frac{z}{\alpha}\right], \tag{2}$$

$$F(z) = Mz + \int_0^{2\pi} \frac{-Aa\cos\theta}{p}\, d\theta \sqrt{(\alpha^2+z^2)}, \tag{3}$$

and the second term in $F(z)$,

$$\begin{aligned} H(z) &= \frac{m^3}{p}\int_{t_3}^{t_2} (2t-t_2-t_3)(t_1-t)\frac{dt}{\sqrt{T}} \\ &= \frac{m^3}{p}\int\left((t_2-t\cdot t-t_3)-\sqrt{T}\frac{d}{dt}\sqrt{T}\right)\frac{dt}{\sqrt{T}}, \end{aligned} \tag{4}$$

the same integral as for G in § 32.

But it is the force between the two helices which alone is required, and this is given by $dF(z)/dz$, or by the change in $F(z)$ due to a small relative axial displacement of the helices, and as this is equivalent to the removal of a circular element from one end to the other of a helix, this force depends only on the difference of the values of M between one helix and the two circular ends of the other helix; and this is calculated by the preceding analysis, which gives the value of M as a function, say $M(b)$, of b the height of one helix say of radius A, when the ring of radius a is in the plane of one base of the helix.

When the circle is at a distance z from the mean section of a helix of height h

$$M = M(z+\tfrac{1}{2}h) - M(z-\tfrac{1}{2}h); \tag{5}$$

and since $M(-z) = -M(z)$, this makes $M = 2M(\tfrac{1}{2}h)$, in the mean section.

The hydrodynamical analogy is complete, M being the Stokes function, and $V(z+\tfrac{1}{2}h) - V(z-\tfrac{1}{2}h)$ the velocity function, $V(z)$ denoting the potential of a circular plate, due to a uniform distribution of source or sink over the plate, thereby producing a streaming motion circulating through the helix, a line of flow having $M =$ constant: thereby we gain a physical conception of the lines of magnetic force inside and outside a solenoid.

MAXWELL's figure XVIII can be utilised by the method of superposition for drawing these lines when the ring is made to expand into a cylinder by progressive axial displacement.

Exploration of the electromagnetic field.

60. The exploration of the electromagnetic field of the ring current is given by MAXWELL, E. and M. § 702, and in fig. XVIII by drawing the twelve surfaces $\Omega = 2\pi f$, taking

$$f = (0, 1, 2, 3, \cdots, 11, 12)\tfrac{1}{12}; \tag{1}$$

equivalent to direction at the ring on a clock face at B, for every hour and half hour from XII to VI, and again from VI to XII.

Any surface $\Omega = 2\pi f$ will cross the axis Oz at a point H such that

$$\sin\psi = \sin OBH = \frac{OH}{HB} = 1 - \frac{2\pi f}{2\pi} = 1 - f, \tag{2}$$

so that H is determined geometrically by drawing a circle center A and radius $(1-f)AB$, cutting the circle on diameter AB in H', and joining BH' cutting Oz in H.

Thence at O, where $f = 1$,

$$\frac{d\Omega}{db} = \frac{2\pi(1-f)}{a(1-f)} = \frac{2\pi}{a},$$

the galvanometer constant for the ring.

Practically we need only draw the surfaces from $f = 0$ to $\frac{1}{2}$, or between XII and III o'clock, as the remainder from III to VI can be inferred by the theorems in (25) that $OM \cdot OM' = OB^2$ for points P, P' on a line through B.

In another method of exploration we can utilize the analytical results given in Philosophical Transactions, 1904, and determine the value of Ω, V, and M in §§ 34, 42, 43, 44, 54, when $2f$ is one of the simplest rational fractions, such as

$$2f = 1, \frac{2}{3}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{3}{5}, \frac{1}{6}, \cdots. \tag{3}$$

In this method it is simple to work with the elliptic functions to modulus c' and argument $2fF'$ and to use the quadric transformation to obtain the result to argument fK', if required.

A curve along which $2f$ is a rational fraction is an algebraical curve, passing through A at an angle $f\pi$ with Ox, and having Oz as an asymptote.

We proceed to trace a few of the simplest, and for this purpose it is useful to employ a stereographic chart with poles at A and B, and center of projection the antipodes of O, in which case, as in § 25,

$$k = \tan\tfrac{1}{2}\lambda = \operatorname{th}\alpha, \qquad c' = \tan(45^\circ - \tfrac{1}{2}\lambda) = e^{-2\alpha}, \tag{4}$$

λ denoting the latitude, the angle FOL on fig. A.

Exploring the values of Ω, V, M at the principal divisions of the clock face:

XII o'clock, $f=0$, $b=0$, $\psi=0$, $x=ak=a\dfrac{1-c'}{1+c'}$,

$$\Omega=2\pi,\qquad V=16a\frac{E+c'F}{1+c'},\qquad M=0.$$

VI o'clock, $f=1$, $b=0$, $\psi=\pi$, $x=\dfrac{a}{k}=a\dfrac{1+c'}{1-c'}$,

$$\Omega=0,\qquad V=16a\frac{E-c'F}{1-c'},\qquad M=0.$$

III o'clock, $2f=1$, $\psi=\frac{1}{2}\pi$, $x=a$, $z=2a\dfrac{c'}{c}$, $\sin\psi'=c'$, $\psi=2\,\mathrm{am}\,\frac{1}{2}K'-\frac{1}{2}\pi$,

$$\Omega=\pi-2Fc',\quad V=4a\frac{H}{c}-\pi z=z\left(\frac{2H}{c'}-\pi\right),\quad \frac{M}{\Theta}=2a\frac{F-E}{c}=z\frac{F-E}{c'},$$

on reference to (9) § 55.

II o'clock, $2f=\frac{2}{3}$; and (see Philosophical Transactions, 1904, page 261) in region B, $3>p>1$,

$$c^2=\frac{(p-1)^3(p+3)}{16p},\qquad c'^2=\frac{(p+1)^3(-p+3)}{16p},$$

$$\mathrm{zn}\,\tfrac{2}{3}F'=\frac{3-p}{6}\sqrt{p},\qquad \mathrm{zn}\,\tfrac{2}{3}F''=\frac{3-p}{6}\sqrt{p},$$

$$\cos\psi=\mathrm{cn}\,\tfrac{2}{3}F'=\frac{p-1}{p+1},\qquad \cos\psi'=\mathrm{dn}\,\tfrac{2}{3}F'=\frac{p-1}{2},\qquad \sec\psi-\sec\psi'=1,$$

along the curve, analogous to

$$\psi+\psi',\ \text{or}\ \tan\psi+\tan\psi',\ \text{or}\ \cos\psi+\cos\psi',\ \cdots\ \text{a constant},$$

a circle, or parabola, or magnetic line of force, $\cdots$ in biangular coördinates ψ, ψ'.

I o'clock: $2f=\frac{1}{3}$, $\sin\Psi=\mathrm{sn}\,\frac{1}{3}F'=\dfrac{2}{p+1}$,

$$\sin\Psi'=c'\,\mathrm{sn}\,\tfrac{1}{3}F'=\sqrt{\frac{p+1\cdot-p+3}{4p}},\qquad \mathrm{zn}\,\tfrac{1}{3}F'=\frac{9-p^2}{12\sqrt{p}}.$$

I. 30, $2f=\frac{1}{2}$, $\psi=\mathrm{am}\,\frac{1}{2}F'$, $\sin\psi=\sqrt{\dfrac{1}{1+c}}$, $\sin\psi'=\sqrt{(1-c)}$,

$$\tan^2\psi-\tan^2\psi'=1,\qquad \frac{z^2}{(a-x)^2}-\frac{z^2}{(a+x)^2}=1,$$

$$\mathrm{zn}\,\tfrac{1}{2}F'=\tfrac{1}{2}(1-c),\qquad \Omega=\tfrac{3}{2}\pi-F(1-c)-2F\sqrt{(1-c)}.$$

XII. 45, or II. 15, $2f = \frac{1}{4}$ or $\frac{3}{4}$ (Philosophical Transactions, page 478),

$$1 > p > \sqrt{2} - 1, \qquad \sqrt{c} = \tfrac{1}{2}\left(\frac{1}{p} - p\right),$$

$$\cos^2\psi' = \mathrm{dn}^2 \tfrac{1}{4}F' = \frac{(1+p)^3(1-p)}{4p}, \qquad \sin^2\psi' = \frac{(1+p^2)(-1+2p+p^2)}{4p},$$

$$\tan^2\psi = \frac{4p^3}{(1+p)(1-p)^3}, \qquad \mathrm{zn}\,\tfrac{1}{4}F' = \frac{(1+2p+3p^2)(-1+2p+p^2)}{16p^2},$$

and a change of p into $-1/p$ will change $2f$ from $\frac{1}{4}$ to $\frac{3}{4}$.

XII. 36, I. 12, I. 48, II. 24; $2f = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$, and the results in Philosophical Transactions, page 264, to be consulted.

XII. 30, II. 30, $2f = \frac{1}{6}, \frac{5}{6}$; consult page 283.

61. In an apparatus constructed by Professor Ayrton and Viriamu Jones,

$$2A = 8, \qquad 2a = 13 \text{ inches};$$

and then for the height b we shall find in correspondence with $2f$,

$2f$	0,	$\frac{1}{4}$,	$\frac{1}{3}$,	$\frac{1}{2}$,	$\frac{2}{3}$,	$\frac{3}{4}$,	1
b	0,	1.36,	1.47,	2.57,	4.59,	9.79,	∞.

For if

(1) $$2f = \tfrac{1}{2}, \qquad b = \frac{a^2 - A^2}{2\sqrt{(aA)}} = 2.574,$$

(2) $$2f = \tfrac{2}{3}, \qquad \left(\frac{a-A}{a+A}\right)^2 = c'^2\,\mathrm{sn}^2(1-2f)F' = \frac{(p+1)(-p+3)}{4p},$$

$$p = 2.8322, \qquad \frac{b}{a-A} = \mathrm{tn}\,\tfrac{2}{3}F' = \frac{2\sqrt{p}}{p-1}, \qquad b = 4.593,$$

(3) $$2f = \tfrac{1}{3}, \qquad \frac{2\sqrt{aA}}{a+A} = \mathrm{dn}\,\tfrac{2}{3}F' = \frac{p-1}{2}, \qquad p = 2.9428,$$

$$\frac{b}{a-A} = \mathrm{tn}\,\tfrac{1}{3}F' = \frac{2}{\sqrt{(p-1\cdot p+3)}}, \qquad b = 1.471.$$

(4) $$2f = \tfrac{3}{4}, \qquad \mathrm{dn}^2\,\tfrac{1}{4}F' = \frac{4aA}{(a+A)^2} = \frac{416}{441} = \frac{(1+p)^3(1-p)}{4p},$$

a quartic for p having a root $p = 0.43$ and making

$$c = \tfrac{1}{4}\left(\frac{1}{p} - p\right)^2 = 0.719, \qquad c' = 0.695,$$

$$\cos^2\psi' = \mathrm{dn}^2\,\tfrac{3}{4}F' = \tfrac{441}{416}c^2 = 0.547 \qquad (\psi' = 43^\circ),$$

$$b = (a + A)\tan\Psi' = 9.79.$$

Similarly for $2f = \frac{1}{4}$, $b = 1.364$.

62. To determine the proportions of the apparatus when the modular angle γ is raised to 75°, and we take $2f = \frac{1}{3}$, as in fig. A, we put

$$(1) \qquad c^2 = \frac{(p-1)^3(p+3)}{16p} = \sin^2 75^\circ = \frac{2+\sqrt{3}}{4},$$

of which one root is $p = -\sqrt{3}$, but this root leads to

$$(2) \qquad \operatorname{sn} \tfrac{1}{3}F = \sqrt{3} - 1, \qquad \operatorname{dn} \tfrac{1}{3}F = \tfrac{1}{2}\sqrt{2}.$$

But with the other real root,

$$(3) \qquad p = \frac{(\sqrt[3]{2}+1)^2}{\sqrt{3}} = \left(\frac{\sqrt[3]{2}+1}{\sqrt[3]{2}-1}\right)^{\frac{1}{2}} = 2.948661,$$

$$(4) \qquad \csc\psi = \operatorname{sn} \tfrac{1}{3}F' = \tfrac{1}{2}(p+1) = 1.97433 \qquad (\psi = 30^\circ 26'),$$

$$(5) \qquad \sin\psi' = c' \sin\psi = \sqrt{\frac{p+1.-p+3}{4p}} = 0.1319 \qquad (\psi' = 7^\circ 34'),$$

$$(6) \qquad \sin\psi_1 = \operatorname{sn} \tfrac{2}{3}F' = \frac{2\sqrt{p}}{p+1} \qquad (\psi_1 = 60^\circ 26'),$$

$$(7) \qquad \cos\psi_1' = \operatorname{dn} \tfrac{2}{3}F' = \tfrac{1}{2}(p-1) = 0.97433 \qquad (\psi_1' = 13^\circ 13'),$$

$$(8) \qquad r_1 = (a+A)\sec\psi' = (a+A)\operatorname{dn} \tfrac{1}{3}F' = (a+A)\sqrt{\frac{2p}{p-1\cdot p+3}},$$

$$(9) \qquad \operatorname{zn}(1-2f)F' = \operatorname{zn} \tfrac{2}{3}F' = \tfrac{1}{6}(3p^{\frac{1}{2}} - p^{\frac{3}{2}}),$$

$$(10) \qquad \frac{M}{r_1\Theta} = F - E - \frac{p-1\cdot p+3}{4\sqrt{p}}\left(\frac{\pi}{6} - F\frac{3p^{\frac{1}{2}} - p^{\frac{3}{2}}}{6}\right),$$

$$(11) \qquad F = 2.768063145, \qquad E = 1.076405113, \qquad F - E = 1.691658032.$$

These angles are contrasted with those of the apparatus of Viriamu JONES in the table following.

$2f$	$\frac{1}{3} = 0.33$	0.35
$\sin^{-1} c = \gamma$	75°	74° 1′
$\sin^{-1} k = \beta$	36° 4′	34° 36′
ψ	30° 26′	32° 3′
ψ'	7° 34′	8° 24′
ψ_1	60° 26′	58° 58′
ψ_1'	13° 3′	13° 36′
A/a	0.63196	0.61921

63. If the windings of the helix are not complete, or if the current is considered through a fraction of the circular ring, the theta functions are required or the pseudo-elliptic algebraical III. E. I. can be employed for a rational value of the fraction f, the simplest cases of which will be found worked out in the Philosophical Transactions, 1904.

Lord RAYLEIGH has shown, in a Report to the British Association (1899), that all error in extreme cases of this assumption as between two coaxial helices is eliminated by taking an average for different symmetrical positions.

One object of the present memoir is to see if by a slight change in one of the dimensions of a helix, say the height b, designed for the experimental determination of the ampere, it would be possible to make $2f$ one of the rational fractions given above, and so simplify and check the numerical work.

Or we may construct the apparatus so as to make the angle γ in fig. A or β in fig. B an integral number of degrees, so that the value of F and E or K and H can be taken out of LEGENDRE's Table I and IX without use of proportional parts, and utilise all the decimal figures.

Potential and Stokes function of a solid coil.

64. To solve Coleridge FARR's problem for a solid cylindrical coil, we have to determine the potential W and Stokes function P of a series of coaxial helical currents or equivalent cylindrical current sheets filling up a solid cylinder, integrating with respect to the radius the expression for a cylindrical sheet.

It is simpler to determine W for a coil stretching in fig. A from the plane AB to infinity on the right, as this is equivalent to integrating over a single circular plate AB of superficial density $\gamma = \gamma_h/2\pi p$ (qualified as shown in (8) by the factor $a - r$, r referring to any internal radius OQ'), the influence of the other end plate at infinity being insensible.

In § 48 we have found for the potential V of the plate AB of unit density

$$(1) \qquad \frac{dV}{d\theta} = PQ - PO - A\cos\theta \cdot 2\text{th}^{-1}\frac{a}{PQ + PO},$$

the equivalent of $d^2W/da\,d\theta$, and integrating by parts with respect to a,

$$(2) \qquad \frac{dW}{d\theta} = \int_0^a (PQ - PO)\,da - A\cos\theta \int 2\text{th}^{-1}\frac{a}{PQ + PO}\,da$$

$$= (\tfrac{1}{2}a + \tfrac{3}{2}A\cos\theta)\,PQ - (a + \tfrac{3}{2}A\cos\theta)\,PO$$

$$- 2\text{th}^{-1}\frac{a}{PQ + PO}(\tfrac{1}{4}A^2 - \tfrac{1}{2}b^2 + aA\cos\theta + \tfrac{3}{4}A^2\cos 2\theta).$$

Integrating again with respect to θ from 0 to 2π, the term

$$I = \int 2\text{th}^{-1} \frac{a}{PQ + PO} d\theta \tag{3}$$

is not an elliptic integral, and so considered intractable, like the integral (2) in § 53, to be denoted by I_1; but

$$\begin{aligned} \int aA \cos\theta \cdot 2\text{th}^{-1} \frac{a}{PQ+PO} d\theta &= a \int \left(1 - \frac{b^2}{PZ^2}\right)(aA\cos\theta + A^2 + b^2) \frac{d\theta}{PQ} \\ &\quad - a\sqrt{(A^2+b^2)} \int \left(1 - \frac{b^2}{PZ^2}\right) d\theta \end{aligned} \tag{4}$$

$$= \int (aA\cos\theta + A^2 + b^2) a \frac{d\theta}{PQ} + ab[(2\pi - \Omega(PZ)] - 2\pi a\sqrt{(A^2+b^2)}.$$

$$\begin{aligned} \int A^2 \cos 2\theta \cdot 2\text{th}^{-1} \frac{a}{PQ + PO} d\theta &= -aA^2 \int \frac{\sin^2\theta d\theta}{PQ} \\ &\quad + (A^2 + b^2) \int (a + A\cos\theta) \frac{d\theta}{PQ} - b\sqrt{(A^2+b^2)}\,\Omega_1(PZ), \end{aligned} \tag{5}$$

so that W is composed of

$$(\tfrac{1}{2}b^2 - \tfrac{1}{4}A^2) I, \qquad -ab[2\pi - \Omega(PZ)], \qquad \tfrac{3}{4}b\sqrt{(A^2+b)}\,\Omega_1(PZ), \tag{6}$$

and I. and II. E. I.'s, amounting to employing the theorems of § 33,

$$\begin{aligned} 2W &= \frac{aF}{2r_1}(a^2 - A^2 - 13b^2) + \frac{3ar_1 E}{2}, \\ &= (b^2 - \tfrac{1}{2}A^2) I - 2ab[2\pi - \Omega(PZ)] + \tfrac{3}{2}b\sqrt{(A^2+b^2)}\,\Omega_1(PZ) \\ &\quad + \frac{aF}{r_1}(a^2 - A^2 - 13b^2) + 3ar_1 E. \end{aligned} \tag{7}$$

The potential W may also be considered due to the circular plate AB, supposing the density at any interior radius $OQ' = r$ is $a - r$; and then, expressed as a double integral

$$W = \int\int \frac{(a-r) r\, dr\, d\theta}{PQ'}, \tag{8}$$

$$\frac{dW}{da} = \int\int \frac{r\, dr\, d\theta}{PQ'} = \text{potential of the end plate of uniform density} \tag{9}$$

$$= \int (a^2 + aA\cos\theta) \frac{d\theta}{PQ} - b[2\pi - \Omega(PZ)],$$

$$\frac{dW}{dA} = \int\int \frac{-(a-r)(r\cos\theta + A) r\, dr\, d\theta}{PQ^3}, \tag{10}$$

(11) $$\frac{dW}{db} = \int\int \frac{-(a-r)rdr\cdot bd\theta}{PQ^3};$$

and

(12) $$a\frac{dW}{da} + A\frac{dW}{dA} + b\frac{dW}{db} = 2W,$$

the homogeneity condition of W, of two dimensions in length.

65. Other double integrals required in the calculation are

(1) $$2\pi - \Omega(PZ) = \int\int \frac{rdr\cdot bd\theta}{PQ^3},$$

the apparent area or normal attraction of the disc AB;

(2) $$\Omega_1(PZ) = \int\int \frac{\sqrt{(A^2+b^2)}\,dr\cdot bd\theta}{PQ^3},$$

(3) $$I = \int\int \frac{dr\,d\theta}{PQ},$$

the intractable integral (3) § 64 which is the potential of the disc AB, with density r^{-1};

$$\int\int \frac{r\cos\theta\,dr\,d\theta}{PQ^3} = \int\int \frac{(r + A\cos\theta)\cos\theta - A\cos^2\theta}{PQ^3}\,drd\theta$$

(4) $$= \int\left(\frac{1}{PO} - \frac{1}{PQ}\right)\cos\theta d\theta - \int \frac{A\cos^2\theta}{PZ^2}\left(\frac{a + A\cos\theta}{PQ} - \frac{A\cos\theta}{PO}\right)d\theta$$

$$= \frac{a}{A}\int\frac{d\theta}{PQ} - \frac{\sqrt{(A^2+b^2)}}{Ab}\Omega_1(PZ) + \sqrt{(A^2+b^2)}\int\frac{\cos\theta d\theta}{PZ^2},$$

from (17) § 47, and the third integral vanishes;

(5) $$\int\int \frac{r^2\,dr\,d\theta}{PQ^3} = I - 2a\int\frac{d\theta}{PQ} + \frac{\sqrt{(A^2+b^2)}}{b}\Omega_1(PZ),$$

(6) $$\frac{1}{b}\frac{dW}{db} = I - 2a\int\frac{d\theta}{PQ} + \frac{\sqrt{(A^2+b^2)}}{b}\Omega_1(PZ) - \frac{a}{b}[2\pi - \Omega(PZ)],$$

(7) $$A\frac{dW}{dA} = 2W - a\frac{dW}{du} - b\frac{dW}{db}$$

$$= -\tfrac{1}{2}A^2 I + \tfrac{1}{2}b\sqrt{(A^2+b^2)}\,\Omega_1(PZ) - \frac{aF}{r_1}(a^2 - A^2 + 3b^2) + ar_1 E,$$

(8) $$\frac{dP}{db} = -2\pi A\frac{dW}{dA}, \qquad \frac{dP}{dA} = 2\pi A\frac{dW}{db}.$$

66. The Stokes function P can be found by an integration of Viriamu JONES's M, but an infinite constant comes in when the coil stretches from b to ∞; so returning to a coil extending from 0 to b, or from AB to the parallel plane through P,

$$\frac{d^2P}{da\,d\theta}=\frac{dpM}{d\theta}=\frac{a^2A^2\sin^2\theta}{MQ^2}\frac{b}{PQ}, \tag{1}$$

and since

$$a^2=MQ^2-2A\cos\theta(a+A\cos\theta)+A^2\cos 2\theta, \tag{2}$$

$$\begin{aligned}\frac{dP}{d\theta}&=A^2\sin^2\theta\int\frac{bda}{PQ}-2A^2\sin^2\theta\cos\theta\int\frac{a+A\cos\theta}{MQ^2}\frac{bda}{PQ}\\&\qquad+A^2\sin^2\theta\cos 2\theta\int\frac{1}{MQ^2}\frac{bda}{PQ}\\&=A^2b(1-\cos 2\theta)\,\mathrm{th}^{-1}\frac{a}{PQ+PO}+2A^3\sin^2\theta\cos\theta\left(\mathrm{th}^{-1}\frac{b}{PQ}-\mathrm{th}^{-1}\frac{b}{PO}\right)\\&\qquad-A^3\sin\theta\cos 2\theta\left[I_2-\tan^{-1}\left(\frac{PO}{b}\tan\theta\right)\right],\end{aligned} \tag{3}$$

where I_2 is given in (7) § 47, and the integration of this term by parts with respect to θ will depend on $\Omega(PZ)$ and $\Omega_1(PZ)$; and P is thus composed of $\frac{1}{2}A^2bI$ and I., II., and III. E. I.'s, which can be evaluated in the preceding manner.

The intractable integral

$$I=\int 2\mathrm{th}^{-1}\frac{a}{PQ+PO}d\theta=\int\mathrm{th}^{-1}\frac{a+A\cos\theta}{PQ}d\theta-\int\mathrm{th}^{-1}\frac{A\cos\theta}{PO}d\theta, \tag{4}$$

of which the second integral vanishes.

Putting $(a+A\cos\theta)/PQ=\sin\phi'$, so that ϕ' is the angle between PO and PZ, and drawing the curve on the Mercator chart connecting latitude ϕ' and longitude θ, the area will represent I, as before in § 53 for I_1.

For a flat coil in the plane AB,

$$P=\int\int\frac{-2\pi Ar\cos\theta\,drd\theta}{PQ'}, \tag{5}$$

$$\frac{dP}{d\theta}=-2\pi A\cos\theta(PQ-PO)+\pi A^2(1+\cos 2\theta)\,\mathrm{th}^{-1}\frac{a}{PQ+PO}, \tag{6}$$

so that

$$P=\tfrac{1}{2}\pi A^2I \text{ and I., II., and III. E. I.'s.} \tag{7}$$

Legendre's discussion of the oblique cone.

67. The method of discussion by LEGENDRE of the surface of the oblique cone in his *Fonctions elliptiques* I, p. 329, may be resumed here, because his Φ, the angle of the sector of the developed surface, and Ω the conical angle of opening of the reciprocal cone are connected by the relation

$$\Phi + \Omega = 2\pi, \tag{1}$$

so that one calculation implies the other.

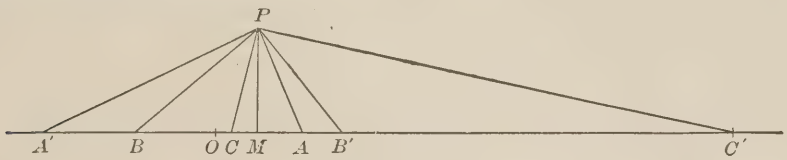

FIG. C.

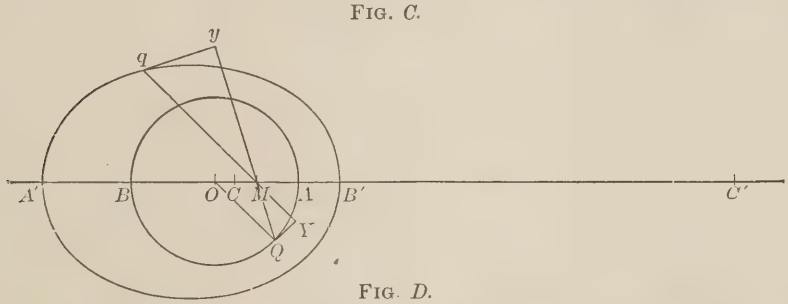

FIG. D.

Adopting LEGENDRE's notation for his oblique cone on a circular base, figures C and D, and considering it as the reciprocal of the cone discussed previously in figs. A and B, then PM the perpendicular from the vertex P on the plane of the circular base AB making an angle λ with PC the bisector of the angle APB, and the edges PA', PB' of the reciprocal cone making an angle λ' with PC, and putting (LEGENDRE)

$$AO = OB = r, \qquad OM = f, \qquad MP = h, \qquad PA = \alpha, \qquad PB = \alpha', \tag{2}$$

$$\sin PAC = \sin(\lambda' + \lambda) = \frac{h}{\alpha}, \qquad \cos(\lambda' + \lambda) = \frac{r - f}{\alpha}, \tag{3}$$

$$\sin PBC = \sin(\lambda' - \lambda) = \frac{h}{\alpha}, \qquad \cos(\lambda' - \lambda) = \frac{r + f}{\alpha'}, \tag{4}$$

$$\cos 2\lambda = \frac{r^2 - f^2 + h^2}{\alpha\alpha'}, \qquad \sin 2\lambda = \frac{2fh}{\alpha\alpha'}, \tag{5}$$

$$\cos 2\lambda' = \frac{r^2 - f^2 - h^2}{\alpha\alpha'}, \qquad \sin 2\lambda' = \frac{2rh}{\alpha\alpha'}, \tag{6}$$

$$\cos\lambda \sin\lambda' = \frac{h(\alpha' + \alpha)}{2\alpha\alpha'}, \qquad \sin\lambda \cos\lambda' = \frac{h(\alpha' - \alpha)}{2\alpha\alpha'}, \tag{7}$$

$$\frac{\tan\lambda}{\tan\lambda'} = \frac{\alpha' - \alpha}{\alpha' + \alpha} = \cos\theta, \tag{8}$$

in LEGENDRE's notation.

If Pq is the generating line of the reciprocal cone perpendicular to the plane PQY, making an angle β with PM, and c denotes $\cos AOQ$,

$$\cot\beta = \frac{MY}{PM} = \frac{r - fc}{h} = \frac{\sin 2\lambda' - c \operatorname{am} 2\lambda}{\cos 2\lambda - \cos\lambda'}, \tag{9}$$

$$\begin{aligned} \cos\beta\cos 2\lambda' + \sin\beta\sin 2\lambda' &= \cos(2\lambda' - \beta) \\ &= \cos\beta\cos 2\lambda + c\sin\beta\sin 2\lambda' = \cos\beta', \end{aligned} \tag{10}$$

if β' denotes the angle between PQ' and PM' the perpendicular on the other circular section of the cone in fig. D; so that

$$\beta + \beta' = 2\lambda', \tag{11}$$

or the sum of the angles which PQ' makes with PM, PM' is constant, and these are the focal lines of the reciprocal cone in fig. A; hence the fundamental property of a sphero-conic.

68. Legendre denotes the angle of the sector APQ when developed by Φ, and the angle AOQ by ω, so that

$$PQ^2 d\Phi = PY \cdot r d\omega, \tag{1}$$

$$\Phi = \int \frac{PY^2}{PQ^2} \cdot \frac{r d\omega}{PY}, \tag{2}$$

and, writing c for $\cos\omega$,

$$PQ^2 = r^2 - 2rfc + f^2 + h^2, \qquad PY^2 = (r - fc)^2 + h^2. \tag{3}$$

The angle QPY is the ψ of Legendre, and to reduce Φ to our form, put

$$m^2(\sigma - s) = \frac{PQ^2}{PY^2} = \sec^2\psi = \frac{r^2 - 2rfc + f^2 + h^2}{(r - fc)^2 + h^2}, \tag{4}$$

$$m^2(s_2 - s) = \frac{QY^2}{PY^2} = \tan^2\psi = \frac{f^2(1 - c^2)}{(r - fc)^2 + h^2}, \tag{5}$$

$$m^2(\sigma - s_2) = 1. \tag{6}$$

Then, as before in § 29,

$$m^2(s_1 - s) = \frac{\alpha\alpha'}{h^2}\cos^2\lambda\left(\frac{(\alpha' + \alpha) - (\alpha' - \alpha)c}{2PY}\right)^2, \tag{7}$$

$$m^2(s - s_3) = \frac{\alpha\alpha'}{h^2}\sin^2\lambda\left(\frac{(\alpha' + \alpha)c - (\alpha' - \alpha)}{2PY}\right)^2. \tag{8}$$

Putting $s = s_2$, $c = \pm 1$,

$$\begin{aligned} m^2(s_1 - s_2) &= \frac{\alpha\alpha'}{h^2}\cos^2\lambda, \qquad m^2(s_2 - s_3) = \frac{\alpha\alpha'}{h^2}\sin^2\lambda, \\ m^2(s_1 - s_3) &= \frac{\alpha\alpha'}{h^2}, \qquad m^4(s_1 - s_2)(s_2 - s_3) = \frac{f^2}{h^2}, \end{aligned} \tag{9}$$

$$\sin^2\lambda = \frac{s_2 - s_3}{s_1 - s_3} = k^2, \tag{10}$$

so that λ is the modular angle,

$$m^2(s_1 - \sigma) = \frac{\alpha\alpha'}{h^2}\cos^2\lambda - 1 = \frac{\alpha\alpha'}{h^2}\cos^2\lambda', \tag{11}$$

$$m^2(\sigma - s_3) = \frac{\alpha\alpha'}{h^2}\sin^2\lambda + 1 = \frac{\alpha\alpha'}{h^2}\sin^2\lambda, \tag{12}$$

$$\tfrac{1}{2}m^3\sqrt{(-\Sigma)} = \frac{r}{h}. \tag{13}$$

Again from (11) § 67,

$$\begin{aligned}\tan(\beta - \lambda') &= \tan(\lambda' - \beta') = \tan\tfrac{1}{2}(\beta - \beta') \\ &= \tan\lambda\,\frac{c\cos\lambda\sin\lambda' - \sin\lambda\cos\lambda'}{\cos\lambda\sin\lambda' - c\sin\lambda\cos\lambda'} = \tan\lambda\,\frac{(\alpha' + \alpha)c - (\alpha' - \alpha)}{\alpha' + \alpha - (\alpha' - \alpha)c} = \sqrt{\frac{s - s_3}{s_1 - s}},\end{aligned} \tag{14}$$

so that as before in § 35,

$$d\beta = \frac{fhdc}{(r - fc)^2 + h^2} = d\tan^{-1}\sqrt{\frac{s - s_3}{s_1 - s}} = \frac{ds}{\sqrt{(4\cdot s_1 - s\cdot s - s_3)}}, \tag{15}$$

$$\frac{rd\omega}{PY} = \frac{-rdc}{PY\sqrt{(1 - c^2)}} = \frac{-frdc}{PY^2\cdot m\sqrt{(s - s_2)}} = -\frac{r}{h}\frac{ds}{m\sqrt{S}}. \tag{16}$$

$$\Phi = \int_{s_3}^{s_2}\frac{\sigma - s_2}{\sigma - s}\frac{r}{h}\frac{ds}{\sqrt{S}} = \int\frac{\frac{1}{2}\sqrt{(-\Sigma)}}{\sigma - s}\frac{ds}{\sqrt{s}}. \tag{17}$$

We notice now that in LEGENDRE's notation, with

$$\cos\theta = \frac{\alpha' - \alpha}{\alpha' + \alpha}, \qquad \text{and} \qquad \tan\tfrac{1}{2}\omega = \tan\tfrac{1}{2}\theta\tan\tfrac{1}{2}\phi, \tag{18}$$

equivalent geometrically to a change from the excentric anomaly ω to the true anomaly ϕ in an orbit of excentricity $\cos\theta$,

$$\frac{(\alpha' + \alpha)c - (\alpha' - \alpha)}{\alpha' + \alpha - (\alpha' - \alpha)c} = \frac{\cos\omega - \cos\theta}{1 - \cos\omega\cos\theta} = \cos\phi, \tag{19}$$

$$\sqrt{\frac{s - s_3}{s_1 - s}} = \tan\lambda\cos\phi, \qquad \sqrt{\frac{s - s_3}{s_2 - s_3}} = \frac{\cos\phi}{\Delta\phi} \tag{20}$$

and the elliptic functions here are a quarter-period out of phase with the preceding in § 26, and

$$\phi = \operatorname{am}(1 - h)K. \tag{21}$$

Also

$$\frac{\operatorname{cn}^2 fK'}{\operatorname{dn}^2 fK'} = \frac{\sigma - s_2}{\sigma - s_3}\frac{s_1 - s_3}{s_1 - s_2} = \frac{h^2}{\alpha\alpha'}\frac{1}{\sin^2\lambda'\cos^2\lambda} = \frac{4\alpha\alpha'}{(\alpha' + \alpha)^2} = \sin^2\theta, \tag{22}$$

so that

$$\theta = \operatorname{am}(1 - f)K'. \tag{23}$$

Stereographic projection of confocal sphero-conics.

69. Describe the unit sphere with center P, cutting the focal lines PM, PM' in A, B and PQ' in C, so that in the spherical triangle ABC (fig. E)

(1) $$AB = c = 2\lambda, \qquad BC + CA = a + b = \beta + \beta' = 2\lambda',$$

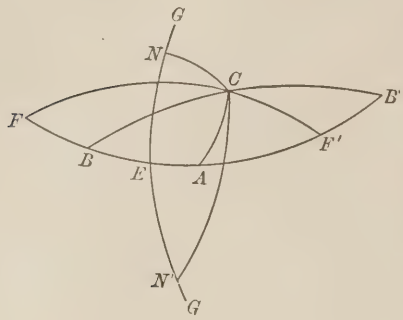

FIG. E.

and the locus of C is a sphero-conic for constant $a + b$ or λ' or f, the orthogonal confocal sphero-conics having $a - b$ or h constant.

Projecting these confocal sphero-conics stereographically on the tangent plane at F, the middle point of $A'B$, where the spherical triangle $A'BC$ is colunar with ABC, and putting $CF = \theta$, $CFB = \psi$, we can put, according to BURNSIDE, Messenger of Mathematics, vol. 20 (1891), p. 60,

(2) $$\tan\tfrac{1}{2}\theta e^{\psi i} = \sqrt{c'}\,\mathrm{sn}\,(v + ui, c') \qquad [c' = \tan\tfrac{1}{4}(\pi - c)],$$

and obtain the orthogonal system of curves given by HOLZMÜLLER in *Isogonale Verwandtschaften.*

Here $FA = 90 + \frac{1}{2}c$, $FB = 90 - \frac{1}{2}c$,

(3) $$\begin{aligned} \cos a &= \cos\theta \sin\tfrac{1}{2}c + \sin\theta\cos\tfrac{1}{2}c\cos\phi\cos b \\ &= -\cos\theta\sin\tfrac{1}{2}c + \sin\theta\cos\tfrac{1}{2}c\cos\phi. \end{aligned}$$

(4) $$2\cos\theta\sin\tfrac{1}{2}c = \cos a - \cos b, \qquad \cos\theta = \frac{\sin\frac{1}{2}(b-a)\sin\frac{1}{2}(b+a)}{\sin\frac{1}{2}a}.$$

But, writing w, w' for $v \pm ui$

(5) $$\tan\tfrac{1}{2}\theta = c'\,\mathrm{sn}\,w\,\mathrm{sn}\,w', \qquad \cos\theta = \frac{1 - c'\,\mathrm{sn}\,w\,\mathrm{sn}\,w}{1 + c'\,\mathrm{sn}\,w\,\mathrm{sn}\,w'}.$$

(6) $$\cos\theta = \frac{1 - c'\,\mathrm{sn}^2 v}{1 + c'\,\mathrm{sn}^2 v}\cdot\frac{1 + c'\,\mathrm{sn}^2 ui}{1 - c'\,\mathrm{sn}^2 ui},$$

so that replacing v and u by fF' and hF, and employing the quadric transformation,

(7) $$\cos\theta = \frac{\mathrm{dn}\,fK'}{\mathrm{dn}\,2hKi},$$

(8) $$\sin\tfrac{1}{2}(b + a) = \mathrm{dn}\,fK', \qquad \cos\tfrac{1}{2}(b + a) = K'\,\mathrm{sn}\,fK',$$

(9) $$\frac{\sin\frac{1}{2}(b-a)}{\sin\frac{1}{2}c} = \frac{1}{\mathrm{dn}\,2Ki} = \frac{\mathrm{cn}\,2hK}{\mathrm{dn}\,2hK} = \mathrm{sn}\,(1 - 2h)\,K,$$

(10) $$\sin\tfrac{1}{2}(b - a) = K\,\mathrm{sn}\,(1 - 2h)\,K,$$

(11) $$\cos\tfrac{1}{2}(b - a) = \mathrm{dn}\,(1 - 2h)\,K,$$

(12) $$k = \sin\tfrac{1}{2}c, \qquad k' = \cos\tfrac{1}{2}c.$$

Thus $h = \frac{1}{2}$ makes $a = b$, and the stereographic projection of the great circle GH is a circle, the coalescence of two branches of a quartic curve in HOLZMÜLLER's diagram.

From NAPIER's formula

$$(13)\qquad \frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)} = \frac{\tan\frac{1}{2}(a+b)}{\tan\frac{1}{2}c} = \frac{\operatorname{dn} fK'}{\kappa \operatorname{sn} fK'} = \frac{1}{\operatorname{cn}(1-f)K'}$$

we deduce

$$(14)\qquad \tan\tfrac{1}{2}A \tan\tfrac{1}{2}B = \frac{1-\operatorname{cn}(1-f)K'}{1+\operatorname{cn}(1-f)K'};$$

and from

$$(15)\qquad \frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)} = \frac{\tan\frac{1}{2}(a-b)}{\tan\frac{1}{2}c} = \frac{k'\operatorname{sn}(2h-1)K}{\operatorname{dn}(2h-1)K} = -\operatorname{cn} 2hK$$

we deduce

$$(16)\qquad \tan\tfrac{1}{2}A \div \tan\tfrac{1}{2}B = \frac{1-\operatorname{cn} 2hK}{1+\operatorname{cn} 2hK},$$

$$(17)\qquad \tan^2\tfrac{1}{2}A = \frac{1-\operatorname{cn}(1-f)}{1+\operatorname{cn}(1-f)}\cdot\frac{1-\operatorname{cn} 2h}{1+\operatorname{cn} 2h},$$

$$(18)\qquad \operatorname{cn} A = \frac{\operatorname{cn}(1-f)+\operatorname{cn} 2h}{1+\operatorname{cn}(1+f)\operatorname{cn} 2h}, \qquad \sin A = \frac{\operatorname{sn}(1-f)\operatorname{sn} 2h}{1+\operatorname{cn}(1-f)\operatorname{cn} 2h},$$

$$(19)\qquad \tan^2\tfrac{1}{2}B = \frac{1-\operatorname{cn}(1-f)}{1+\operatorname{cn}(1-f)}\cdot\frac{1+\operatorname{cn} 2h}{1-\operatorname{cn} 2h},$$

$$(20)\qquad \cos B = \frac{\operatorname{cn}(1-f)-\operatorname{cn} 2h}{1-\operatorname{cn}(1-f)\operatorname{cn} 2h}, \qquad \sin B = \frac{\operatorname{sn}(1-f)\operatorname{sn} 2h}{1-\operatorname{cn}(1-f)\operatorname{cn} 2h},$$

and so on.

70. If N, N' are the poles of the circular sections of the cone, or the foci of the sphero-conics of the reciprocal cone,

$$(1)\qquad \sin EN = \frac{\tan\frac{1}{2}c}{\tan\frac{1}{2}(a+b)} = \cos\theta = \operatorname{cn}(1-f)K',$$

and we find by Spherical Trigonometry

$$(2)\qquad \cos CN,\ CN' = \cos CE \cos EN \pm \sin CE \sin EN \sin CEA$$

and

$$(3)\qquad \cos CE = \frac{\cos\frac{1}{2}(a+b)\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}c} = \operatorname{sn} f \operatorname{dn}(1-2h),$$

$$(4)\qquad \cos EN = \frac{\sqrt{\sin s \sin(s-c)}}{\cos\frac{1}{2}c \sin\frac{1}{2}(a+b)} = \operatorname{sn}(1-f),$$

$$(5)\qquad \sin CE \sin CEA = \frac{\sqrt{\{\sin s \sin(s-a)\sin(s-b)\sin(s-c)\}}}{\sin\frac{1}{2}c \cos\frac{1}{2}c} = \operatorname{cn} f \operatorname{cn}(1-2h),$$

so that

$$(6)\qquad \cos CN,\ CN' = \frac{\operatorname{sn} f \operatorname{cn} f}{\operatorname{dn} f}\left[\operatorname{dn}(1-2h) \pm k \operatorname{cn}(1-2h)\right] = \operatorname{sn} 2fF' \operatorname{dn}(\tfrac{1}{2}\pm h)F,$$

and

$$\cos CN \cos CN' = c' \operatorname{sn}^2 2fF', \text{ a constant,} \tag{7}$$

a fundamental property of the sphero-conic.

71. If dA denotes the element of plane area cut out by consecutive quartic curves,

$$dA = c' \operatorname{cn} w \operatorname{dn} w \operatorname{cn} w' \operatorname{dn} w' \cdot F \cdot dh \cdot F' \cdot df \tag{8}$$

and if dw denotes the corresponding conical angle,

$$dw = 4 \cos \Psi \tfrac{1}{2}\theta dA = 4 \frac{c' \operatorname{cn} w \operatorname{dn} w \operatorname{cn} w' \operatorname{dn} w'}{(1 + c' \operatorname{sn} w \operatorname{sn} w')^2} FF' df dh, \tag{9}$$

and putting $w, w' = \frac{1}{2}fK' \pm hKi$, this becomes by the quadric transformation,

$$\begin{aligned} dK &= 4 \frac{\dfrac{1-k}{1+k} \dfrac{(\operatorname{dn}^4 w - k^2)(\operatorname{dn}^4 w' - k^2)}{(1-k^2)^2 \operatorname{dn}^2 w \operatorname{dn}^2 w'}}{\left(1 + \dfrac{k'^2 \operatorname{sn} w \operatorname{cn} w \operatorname{sn} w' \operatorname{cn} w'}{\operatorname{dn} w \operatorname{dn} w'}\right)^2} \tfrac{1}{2}(1+k)^2 KK' df dh \\ &= \frac{2}{1-k^2} \frac{(\operatorname{dn}^4 w - k^2)(\operatorname{dn}^4 w' - k^2)}{(\operatorname{dn} w \operatorname{dn} w' + k'^2 \operatorname{sn} w \operatorname{cn} w \operatorname{sn} w' \operatorname{cn} w')^2} KK' df dh \\ &= 2 \frac{\operatorname{dn}^2(w+w') \operatorname{dn}^2(w-w') - k^2}{\operatorname{dn}^2(w-w')} KK' df dh \\ &= 2\left(\operatorname{dn}^2 fK' - \frac{k^2}{\operatorname{dn}^2 2hKi}\right) KK' df dh, \end{aligned} \tag{10}$$

in which the variables are separated.

Performing the integrations between limits such that h ranges from 0 to 1, and f from f to 1 and doubling for the whole conical angle Ω,

$$\iint \operatorname{dn}^2 fK' KK' df dh = K[(1-f)H' - \operatorname{zn} fK'], \tag{11}$$

$$\begin{aligned} \iint \frac{k^2}{\operatorname{dn}^2 2hKi} KK' df dh &= \iint [1 - \operatorname{dn}^2(1-2h)K]\, KK' df dh \\ &= (1-f)K' \int_0^1 [1 - \operatorname{dn}^2(1-2h)K]\, K dh \\ &= (1-f)K'(K - H), \end{aligned} \tag{12}$$

so that

$$\Omega = 4(1-f)(KH' + K'H - KK') - 4K \operatorname{zn} fK' = 2\pi(1-f) - 4K \operatorname{zn} fK', \tag{13}$$

as before, for the complete cone; but (10) shows that the apparent area of any curvilinear quadrilateral, bounded by two pairs of HOLZMÜLLER's curves can be expressed in the same manner by the I. and II. E. I.

Looking back over the calculation we conclude that the Stokes function has the advantage in simplicity over the potential function, except for the case of the circular ring of § 44.

1 STAPLE INN, LONDON, W. C.,
January 14, 1907.

NOTES AND ERRATA: VOLUME 8

L. P. EISENHART: *Applicable surfaces with asymptotic lines* ···.

Pp. 113–134. Since the publication of this memoir, the author is informed by Professor BIANCHI that he obtained the fundamental theorem (p. 122) in a note, "*Sulle coppie di superficie applicabili con assignata rappresentazione sferica*," published in the *Rendiconti della R. Accademia dei Lincei*, volume 13 (1904), pp. 147–161. The methods are essentially the same.

W. R. LONGLEY: *A class of periodic orbits* ···.

P. 165, l. 23. *For* $(m < \mu)$ *read* $(m < \bar{\mu})$.

P. 166, l. 7. " $\dfrac{\sqrt{1-(\bar{e}+e)^2}}{[1-(\bar{e}+e)]^2}$ " $\dfrac{\sqrt{1-(\bar{e}+e)^2}}{(1+\alpha)^{\frac{3}{2}}[1-(\bar{e}+e)]^2}$.

L. E. DICKSON: *Invariants of binary forms* ···.

P. 219, l. 6. *For* $p < 2$ *read* $p > 2$.

P. 223, formula (60). " $3n$ " 3^n.

C. N. MOORE: *On the introduction of convergence factors* ···.

P. 305, l. 1 up. *For* $\phi(0) = 0$ *read* $\lim_{x=+0} \phi(x) = \phi(0) = 1$.

P. 306, l. 1. Condition (iii) should read: $\phi''(x)$ exists for $x > 0$ and $\phi''(x) \geqq 0$ $(0 < x^a \leqq c)$.

" l. 5 and l. 6. *Replace* the sentence beginning in l. 5 by the following: Condition (a) follows for $x = 0$ from (ii) and for $x > 0$ from the fact that the second derivative of $\phi(x)$ exists for all values $x > 0$.

" l. 2 up. *For* $0 \leqq x \leqq c$ *read* $0 < x \leqq c$.

P. 307, l. 3. *After* Condition (ii) " follows from (c) and the continuity of $\phi(x)$ for $x = 0$.

P. 308, l. 3 up. *Replace* this line by the following: If for any value of α, (d) or (d') holds for all values of n, (e) is unnecessary.

A similar change must be made in the corresponding footnote for Theorem V, and may be made, and thus

secure greater generality, in the corresponding footnote for Theorems I and IV.

P. 318, l. 3. In the parenthesis at the right hand end of the line, *for* α *read* α'.

P. 323, l. 10 up.	*For*	$0 < \alpha < \frac{c}{m}$	*read*	$0 < \alpha$.
P. 327, l. 3.	"	$\phi(\alpha, x)$	"	$\lvert\phi(\alpha, x)\rvert$.
" l. 1 up	"	$0 < \alpha < \frac{c}{m}$	"	$0 < \alpha$.

G. A. Bliss, *A new form of the simplest problem* $\cdots$.

P. 411, l. 3 up. The function $E(x, y, \sigma, \tau)$ should be defined as the negative of the expression given.

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